Gamma Distribution and its Relatives

(number of rare events)

* **Poisson process:** Suppose that events occur at random anywhere in the time interval $[0,t]$, and these events are independent, 'rare', and 'isolated'.

Then, the number of events in $[0,t]$ $X(t)$ turns out to have approximately a Poisson distribution

$$P(X(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

with parameter λt (where λ, the rate of the event process, is the expected number of events occurring in 1 unit of time).

* **Exponential density:** (waiting time until the first rare event)

Let X_1 be the waiting time until the first event by starting from $t=0$.

Clearly if X_1 is greater than t if and only if $X(t)=0$; that is

$(t>0)$

$$X_1 > t \overset{\text{iff}}{\iff} X(t)=0.$$

Then the cdf of X_1 is

$$F_1(t) = P(X_1 \le t) = 1 - P(X_1 > t) = 1 - P(X(t)=0)$$

$$= 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} = 1 - e^{-\lambda t}$$

($f_1(t)$ with $\lambda=1$)

The pdf of X_1 is

$$f_1(t) = \frac{d}{dt}\left[F_1(t)\right] = F_1'(t) = \lambda e^{-\lambda t}, \quad t>0,$$

thus it is exponential.

* Remark: The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events.

* Gamma density: (waiting time until the αth event occurs.) ~~we have~~

Let X_α be the waiting time from $t=0$ until the moment when the αth event occurs. We have
($1<\alpha$, integer)

$$X_\alpha > t \quad \text{iff} \quad X(t) < \alpha$$

Hence the cdf of X_α is

$$\begin{aligned} F_\alpha(t) &= 1 - P(X_\alpha > t) \\ &= 1 - P(X(t) < \alpha) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \end{aligned}$$

Thus the pdf of X_α is

$$f_\alpha(t) = \frac{d}{dt}\left[F_\alpha(t)\right] = F_\alpha'(t) = (\lambda t)^{\alpha-1}\lambda e^{-\lambda t}/(\alpha-1)!$$

This is known as the <u>gamma density</u>, with parameter λ and α.

* <u>Gamma function</u>: For $\alpha > 0$, the <u>gamma</u> function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

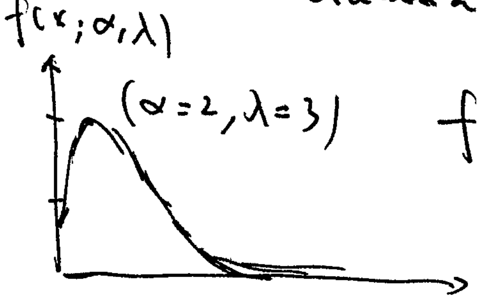
* <u>Properties</u> of the gamma function

1/. For $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ (integration by parts)

2/ for any positive integer n, $\Gamma(n) = (n-1)!$

3/ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

* Gamma distribution.



$$f(x;\alpha,\lambda) = \begin{cases} (\lambda x)^{\alpha-1}\lambda e^{-\lambda x}/\Gamma(\alpha), & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

* <u>Remark</u>: The exponential pdf is a special case of the general gamma pdf in which $\alpha = 1$.

* <u>Proposition</u>: The mgf of a gamma rv is

$$M(t) = \frac{1}{(1 - t/\lambda)^\alpha}$$

proof:

$$M(t) = \int_0^\infty e^{tx} f(x;\alpha,\lambda)\,dx$$

$$= \int_0^\infty e^{tx} \frac{(\lambda x)^{\alpha-1} \lambda e^{-\lambda x}}{\Gamma(\alpha)}\,dx$$

$$= \int_0^\infty \frac{(\lambda x)^{\alpha-1} \lambda e^{-x(-t+\lambda)}}{\Gamma(\alpha)}\,dx$$

$$= \frac{1}{(1 - t/\lambda)^\alpha} \underbrace{\int_0^\infty \frac{[\lambda(1-\frac{t}{\lambda})x]^{\alpha-1} [\lambda(1-\frac{t}{\lambda})] e^{-\lambda(1-\frac{t}{\lambda})x}}{\Gamma(\alpha)}}_{}$$

$$= \frac{1}{(1 - t/\lambda)^\alpha}$$

a gamma pdf

$f(x; \alpha, \lambda(1-\frac{t}{\lambda}))$

$\Rightarrow \int_0^\infty f(x)\,dx = 1$

<u>proposition</u>: The mean and variance of a gamma rv $\underline{X}$ with pdf $f(x;\alpha,\lambda)$ are

$$E(\underline{X}) = \alpha/\lambda, \quad V(\underline{X}) = \alpha/\lambda^2.$$

<u>proposition</u>:

$$F(x;\alpha,\lambda) = F(\lambda x;\alpha,1) = F(\lambda x;\alpha)$$

where $F(y;\alpha)$ is called incomplete gamma function.

$y = \lambda u$

proof.

$$F(x;\alpha,\lambda) = P(\underline{X} \le x) = \int_0^{\lambda x} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy$$

$$= \int_0^{x} \frac{(\lambda u)^{\alpha-1}\lambda e^{-\lambda u}}{\Gamma(\alpha)}\,du$$

$$= F(\lambda x;\alpha).$$

<u>Remark</u>: $F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy$ is the cdf of a <u>standard</u> gamma rv with $f(x;\alpha,\lambda=1)$.
(in which $\lambda = 1$)