Other Continuous Distributions (Skewed Distributions)

\+ Gamma distribution:

$$f(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \qquad x \geq 0$$

\- $\beta = 1$, ✓ the standard pdf

$$f(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \qquad x \geq 0$$

where the gamma function $\Gamma(\alpha)$ is defined for $\alpha > 0$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx \quad \rightarrow \text{Tables for values}$$

∘ $\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}$.

✓ the cdf

$$F(x;\alpha) = \int_0^{x} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y}\, dy \quad \rightarrow \text{Table for values}$$

is called the incomplete gamma function

\- the mgf is

$$M_X(t) = \frac{1}{(1-\beta t)^{\alpha}}$$

\- the mean and variance:

$$E(X) = \mu = \alpha\beta, \quad V(X) = \sigma^2 = \alpha\beta^2$$

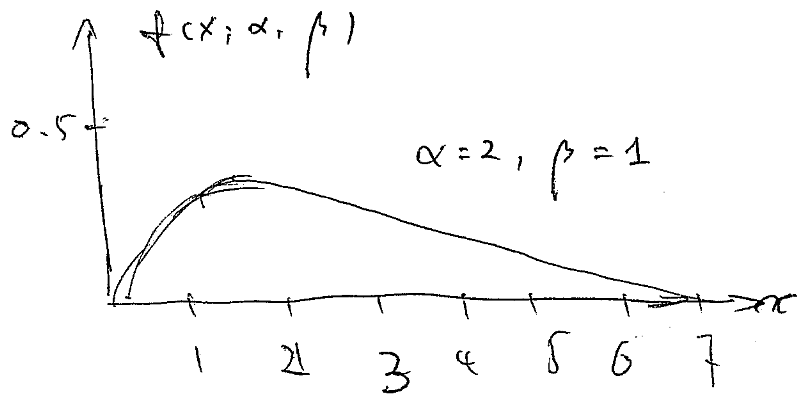
\- $\alpha = 1$: → the exponential distribution

$$f(x;\beta) = \frac{1}{\beta} x^{-x/\beta}, \qquad x \geq 0$$

the cdf $F(x;\beta) = 1 - e^{-x/\beta}, \quad x \geq 0$

* Application

- Gamma distribution: 1°/. the reaction time $\underline{X}$ (second) of a randomly selected individual to a certain stimulus ~~has~~ is supposed to have a standard gamma distribution with $\alpha = 2$. $E(\underline{X}) = \alpha$, $V(\underline{X}) = 2$



2°/. the survival time $\underline{X}$ in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation is supposed to have a gamma distribution with $\alpha = 8$ and $\beta = 15$.

$$\begin{cases} E(x) = 8 \times 15 = 120 \text{ weeks} \\ V(\underline{X}) = 8 \times (15)^2 = 1800 = \sigma^2 \\ \Rightarrow \sigma_x = 42.43 \text{ weeks.} \end{cases}$$

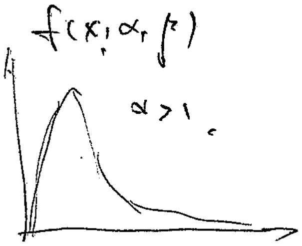
- Exponential distribution ($\alpha = 1$): 1°/ the ~~response time~~ number of days $\underline{X}$ between successive calls to a 24-hour "suicide hotline".

* Chi-squared distribution ($\alpha = \nu/2,\ \beta = 2$)

$$f(x;\nu) = \frac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\, x^{\frac{\nu}{2}-1} e^{-x/2}, \quad x \ge 0$$

=> it is the basis for a number of procedures in statistical inference, and is intimately related to normal distributions.

* Weibull distribution:



$$f(x;\alpha,\beta) = \frac{\alpha}{\beta^{\alpha}}\, x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x \ge 0$$

- $\alpha = 1$, => it reduces to exponential distribution.
- the mean and variance

$$E(X) = \mu = \beta\,\Gamma\left(1+\frac{1}{\alpha}\right), \quad \sigma^2 = V(X) = \beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2\right\}$$

- the cdf:

$$F(x;\alpha,\beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x \ge 0.$$

- Application: better fits to observed data.

1/ model engine emission of various pollutants

2/ model the corrosion weight loss for a magnesium alloy plate immersed in an inhibited aqueous solution of $MgBr_2$.

* Lognormal Distribution.

Def: A nonnegative rv X is said to have a lognormal distribution if the rv

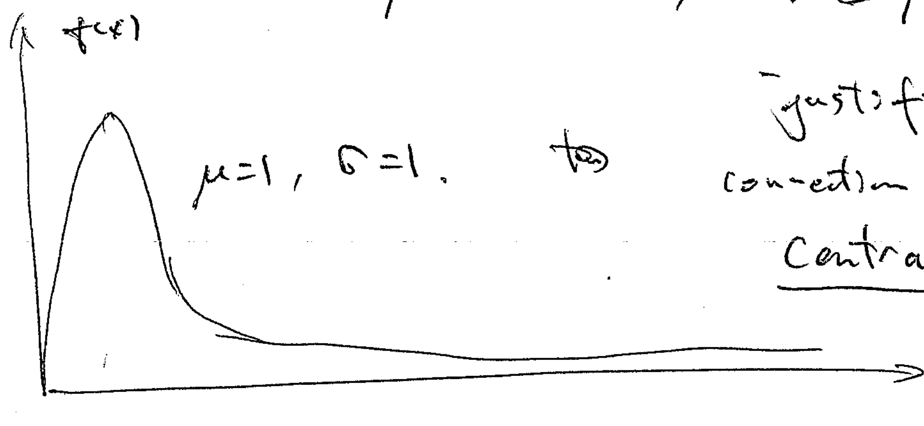
$$Y = \ln(X)$$

has a normal distribution (with parameter μ and σ^2). The resulting pdf of X is

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{-[\ln(x)-\mu]^2/(2\sigma^2)}, \quad x \geq 0.$$

- The mean and variance:

$$E(X) = e^{\mu + \frac{\sigma^2}{2}}, \quad V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$$



to justification in connection with the Central Limit Theorem

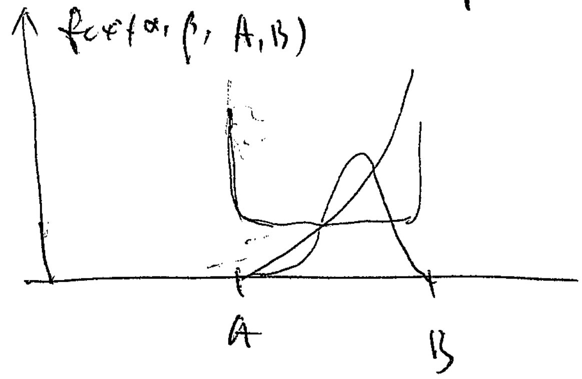
- the cdf of X can be expressed in terms of the cdf of a standard normal rv Z.
$\Phi(z)$

$$F(x; \mu, \sigma) = P(X \leq x) = P[\ln(X) \leq \ln(x)]$$

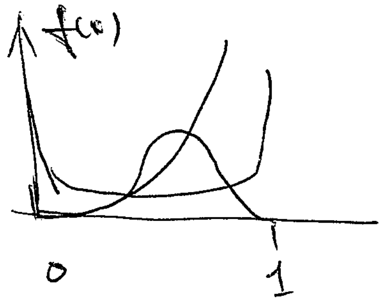
$$= P\left(Z \leq \frac{\ln(x)-\mu}{\sigma}\right) = \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right).$$

▷ Beta distribution

$$f(x;\alpha,\beta,A,B)=\frac{1}{B-A}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\left(\frac{x-A}{B-A}\right)^{\alpha-1}\left(\frac{B-x}{B-A}\right)^{\beta-1},\quad x\in[A,B]$$



→ $A=0,\ B=1$: standard beta distribution



› The mean and variance

$$E(X)=\mu=A+(B-A)\frac{\alpha}{\alpha+\beta}$$

$$V(X)=\sigma^2=\frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- it provides positive density only for X in an interval of finite length.