Solutions to 18 of the problems in sections 3.9, 4.1, 4.2 and 4.3.

3.9 #7: Use differentials to estimate the value of the indicated expression. Then compare your estimate with the result given by a calculator.

$$(33)^{3/5}.$$

Note: we know $(32)^{3/5} = \left[(32)^{1/5}\right]^3 = 2^3 = 8.$

Set $f(x) = x^{3/5}$. From differential approximation (the same thing as tangent line approximation)

$$f(33) \approx f(32) + f'(32)\cdot(33-32)$$

$$\boxed{\begin{aligned} f'(x) &= \tfrac{3}{5}x^{-2/5} \\ f'(32) &= \tfrac{3}{5}(32)^{-2/5} \\ &= \tfrac{3}{5}\,2^{-2} = \tfrac{3}{20} \end{aligned}}$$

$$= 8 + \frac{3}{20}\cdot(1)$$

$$= \frac{163}{20}.$$

Note: $f(32) = 8$ from above

3.9 #11: Use a differential to estimate the value of the expression. (Remember to convert to radian measure.) Compare your estimate with the result given by a calculator.

$$\tan 28°.$$

$28°$ is close to $30°$; and we know $\tan(30°) = \sqrt{3}$

Convert to radians: $30° = \pi/6$

$$28° = 28 \cdot \frac{\pi}{180} = \frac{7\pi}{45}$$

Set $f(x) = \tan(x)$

$$f\left(\frac{7\pi}{45}\right) \approx f(\pi/6) + f'(\pi/6)\left(\frac{7\pi}{45} - \frac{\pi}{6}\right)$$

$f'(x) = \sec^2(x)$

$f'(\pi/6) = \left(\frac{2}{\sqrt{3}}\right)^2$

$= \frac{4}{3}$

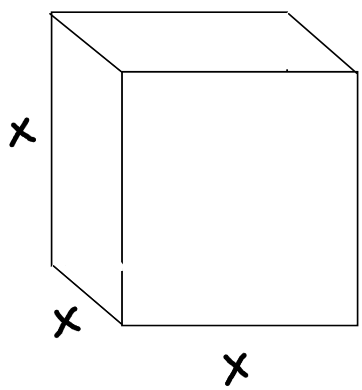
$$= \sqrt{3} + \frac{4}{3}\left(\frac{28\pi - 30\pi}{180}\right)$$

$$= \sqrt{3} - \frac{2\pi}{135}.$$

3.9 #17: A box is to be constructed in the form of a cube to hold 1000 cubic feet. Use a differential to estimate how accurately the inner edge must be made so that the volume will be correct to within 3 cubic feet.

$x = 10$ ft

$|dV|$ no more than $3\text{ ft}^3$




$$V = x^3$$

$$dV = 3x^2 \cdot h$$

change in edge length

$\therefore$

$$-3 \le 3 \cdot 10^2 \cdot h \le 3$$

$$-\frac{1}{100} \le h \le \frac{1}{100}$$

$\therefore$ inner edge needs to be within $\frac{1}{100}$ ft of 10 ft.

3.9 #31: In Exercises 27–32, use the Newton-Raphson method to estimate a root of the equation $f(x) = 0$ starting at the indicated value of $x$: (a) Express $x_{n+1}$ in terms of $x_n$. (b) Give $x_4$ rounded off to five decimal places and evaluate $f$ at that approximation.

$$f(x) = \cos x - x; \quad x_1 = 1.$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$$

$$= x_n + \frac{\cos(x_n) - x_n}{\sin(x_n) + 1}$$

$$\boxed{x_1 = 1}$$

$$x_2 = 1 + \frac{\cos(1) - 1}{\sin(1) + 1}$$

$x_2 := 0.7503638679$

$$x_3 = x_2 + \frac{\cos(x_2) - x_2}{\sin(x_2) + 1}$$

$x_3 := 0.7391128909$

$$x_4 = x_3 + \frac{\cos(x_3) - x_3}{\sin(x_3) + 1}$$

$x_4 := 0.7390851334$

4.1 #2: Show that $f$ satisfies the conditions of Rolle's theorem on the indicated interval and find all numbers $c$ on the interval for which $f'(c) = 0$.

$$f(x) = x^4 - 2x^2 - 8; \quad [-2, 2].$$

$f$ is a polynomial, so it is both continuous and differentiable on the given interval.

Rolle's theorem is the mean value theorem in the special case when $f(a) = f(b)$. Consequently,

$$f'(c) = \frac{f(b) - f(a)}{b-a} = 0.$$

$f(2) = 16 - 2 \cdot 4 - 8 = 0$

$f(-2) = 16 - 2 \cdot 4 - 8 = 0$

Find $c$ between $-2$ and $2$ so that $f'(c) = 0$.

$f'(x) = 4x^3 - 4x$

$\therefore$ $f'(c) = 0$ iff $4c^3 - 4c = 0$

$4c(c^2 - 1) = 0$

$\boxed{c = 0, 1, -1}$

All of these values are between $-2$ and $2$, and they give all values where $f'(c) = 0$.

4.1 #7: Verify that $f$ satisfies the conditions of the mean-value theorem on the indicated interval and find all numbers $c$ that satisfy the conclusion of the theorem.

$$f(x) = x^3; \quad [1, 3].$$

$f$ is a polynomial, so it is both continuous and differentiable on the given interval.

Find c between 1 and 3 so that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\boxed{f'(x) = 3x^2}$$

$$3c^2 = \frac{27 - 1}{2} = 13$$

$$c^2 = \frac{13}{3} \Rightarrow c = \cancel{\pm}\sqrt{\frac{13}{3}}$$

We omit $-\sqrt{13/3}$ since it does not lie between 1 and 3.

$$\therefore c = \sqrt{13/3}$$

4.1 #9: Verify that $f$ satisfies the conditions of the mean-value theorem on the indicated interval and find all numbers $c$ that satisfy the conclusion of the theorem.

$$f(x) = \sqrt{1-x^2}; \quad [0,\ 1].$$

$f$ is continuous on [0,1] and differentiable on (0,1). In fact, $x = 0$ is the only place where $f$ is not differentiable.

Find c between 0 and 1 so that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\boxed{f'(x) = \frac{-x}{\sqrt{1-x^2}}}$$

$$\frac{-c}{\sqrt{1-c^2}} = \frac{0 - 1}{1} = -1$$

$$\sqrt{1-c^2} = c$$

$$1 - c^2 = c^2$$

$$2c^2 = 1 \Rightarrow c = \cancel{\pm}\frac{1}{\sqrt{2}}$$

We omit $c = -\frac{1}{\sqrt{2}}$ since it does not lie btwn 0 and 1.

$$\boxed{\therefore\ c = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}}$$

4.2 #6: Find the intervals on which $f$ increases and the intervals on which $f$ decreases.

$$f(x) = x(x+1)(x+2).$$

polynomial. Domain: All $x$. Differentiable everywhere.

$$= x^3 + 3x^2 + 2x$$

continuous for all $x$.

$$f'(x) = 3x^2 + 6x + 2$$

$$f'(x) = 0 \iff x = \frac{-6 \pm \sqrt{36 - 4\cdot 3\cdot 2}}{6}$$

$$= \frac{-6 \pm \sqrt{12}}{6}$$

$$= -1 \pm \frac{\sqrt{3}}{3}$$

Slope chart

| $f'(x)$ | $+ + + +$ | $0$ | $- - - - - -$ | $0$ | $+ + +$ |
|---|---|---|---|---|---|
| $x$ | $-2$ | $-1-\frac{\sqrt{3}}{3}$ | $-1$ | $-1+\frac{\sqrt{3}}{3}$ | $0$ |

$f'(-2) = 12 - 12 + 2 > 0$

$f'(-1) = 3 - 6 + 2 < 0$

$f'(0) = 2 > 0$

Intervals of increase: $\left(-\infty, -1-\frac{\sqrt{3}}{3}\right]$ and $\left[-1+\frac{\sqrt{3}}{3}, \infty\right)$

Intervals of decrease: $\left[-1-\frac{\sqrt{3}}{3}, -1+\frac{\sqrt{3}}{3}\right]$.

4.2 #10: Find the intervals on which $f$ increases and the intervals on which $f$ decreases.

$$f(x) = \frac{x}{1+x^2}.$$

rational function. Denom. is never 0. ∴ Domain: all $x$. Differentiable everywhere.

$$f'(x) = \frac{(1+x^2)\cdot 1 - x \cdot 2x}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$ ← continuous for all $x$.

Note: Denominator is always positive.

$f'(x) = 0$ iff $1-x^2 = 0$

$x = \pm 1$.

Slope chart

| $f'(x)$ | $- - -$ | $- -$ | $0$ | $+ +$ | $+ +$ | $0$ | $- - -$ | $- -$ |
|---|---|---|---|---|---|---|---|---|
| $x$ | | $-2$ | $-1$ | | $0$ | $1$ | | $2$ |

$f'(-2) < 0$

$f'(0) > 0$

$f'(2) < 0$

Intervals of increase: $[-1, 1]$

Intervals of decrease: $(-\infty, -1]$ and $[1, \infty)$.

4.2 #16: Find the intervals on which $f$ increases and the intervals on which $f$ decreases.

$$f(x) = x^2 + \frac{16}{x^2}. = x^2 + 16x^{-2}$$

Domain: All $x$ except $x = 0$.

$$f'(x) = 2x - 32x^{-3}$$

(defined and continuous at all $x$ except $0$)

$$f'(x) = 0 \iff 2x^4 - 32 = 0$$

$$x = \pm 2.$$

Slope chart:

| $f'(x)$ | $---$ | $0$ | $++++$ | DNE | $----$ | $0$ | $+++$ |
|---|---|---|---|---|---|---|---|
| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |

$f'(-3) < 0$ $\qquad$ $f'(1) < 0$

$f'(-1) > 0$ $\qquad$ $f'(3) > 0$

Intervals of increase: $[-2, 0)$ and $[2, \infty)$

Intervals of decrease: $(-\infty, -2]$ and $(0, 2]$.

4.2 #23: Find the intervals on which $f$ increases and the intervals on which $f$ decreases.

$$\underbrace{f(x) = \sqrt{3}\,x - \cos 2x}_{\text{continuous}}, \quad \underbrace{0 \le x \le \pi}.$$

we are only interested in this interval.

$f'(x) = \sqrt{3} + 2\sin(2x).$

**Note:** $0 \le x \le \pi \iff 0 \le 2x \le 2\pi$

$f'(x) = 0 \iff -\frac{\sqrt{3}}{2} = \sin(2x)$

$\therefore\ 2x = \frac{4\pi}{3}$ or

$2x = \frac{5\pi}{3}$

i.e. $x = \frac{2\pi}{3}$ or $x = \frac{5\pi}{6}$

slope chart:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | ++ + | + + | 0 | - - - | 0 | + + | |
| $x$ | $0$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | |

$f'(\pi/2) = \sqrt{3} > 0$

$f'(3\pi/4) = \sqrt{3} - 2 < 0$

$f'(\pi) = 3 > 0$

Intervals of increase: $\left[0, \frac{2\pi}{3}\right]$ and $\left[\frac{5\pi}{6}, \pi\right]$

Intervals of decrease: $\left[\frac{2\pi}{3}, \frac{5\pi}{6}\right]$.

4.2 #27: find $f$ given the following information.

$$f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1 \quad \text{for all } x, \quad f(0) = 5.$$

Note: $\frac{d}{dx} x^5 = 5x^4$, $\frac{d}{dx} x^4 = 4x^3$, $\frac{d}{dx} x^3 = 3x^2$

$\frac{d}{dx} x^2 = 2x$, $\frac{d}{dx} x = 1$

$\therefore \quad f(x) = x^5 + x^4 + x^3 + x^2 + x + C$

Since $f(0) = 5$

$5 = 0^5 + 0^4 + 0^3 + 0^2 + 0 + C$

$\Rightarrow C = 5$

$$\boxed{f(x) = x^5 + x^4 + x^3 + x^2 + x + 5}$$

4.3 #4: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = x^2 - \frac{3}{x^2}. = x^2 - 3x^{-2}$$

Critical values are values in the domain where $f'(x) = 0$ or $f'(x)$ d.n.e.

Domain: All $x$ except $x=0$.

$$f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$$

$f'(x)$ exists at all $x$ except $x=0$, and $x=0$ is not in the domain of $f$.

$\therefore$ Critical values will only occur where $f'(x) = 0$

$$f'(x) = 0 \text{ iff } 2x + \frac{6}{x^3} = 0$$

$$2x^4 + 6 = 0$$

No solutions!

$\therefore$ no critical values.

$\therefore$ no local extreme values

4.3 #5: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = x^2(1-x). \quad = x^2 - x^3$$

polynomial.

$$f'(x) = 2x - 3x^2$$

exists everywhere.

$\therefore$ critical values occur where $f'(x) = 0$

$$2x - 3x^2 = 0$$

$$x(2 - 3x) = 0$$

$$x = 0, \quad x = \frac{2}{3}$$

$$f''(x) = 2 - 6x$$

Using the 2nd deriv. test:

$f''(0) = 2 > 0 \Rightarrow x = 0$ is place where $f$ has a local min.

$f''(\frac{2}{3}) = 2 - 4 < 0 \Rightarrow x = \frac{2}{3}$ is a place where $f$ has a local max.

$$f(0) = 0$$

$$f(2/3) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27}$$

4.3 #9: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = \frac{2}{x(x+1)}. \leftarrow \text{rational function.}$$

<u>Domain</u> : All $x$ except $x=0$, $x=-1$.

$$f'(x) = \frac{(x^2+1)\cdot 0 - 2(2x+1)}{(x^2+x)^2}$$

$$= \frac{-4x-2}{(x^2+x)^2}$$

$f'(x)$ exists at all $x$ except $x=0$ and $x=-1$, and these are not in the domain.

$\therefore$ C.V.s only where $f'(x)=0$.

$$-4x-2=0 \iff x=-\tfrac{1}{2}$$

only Crit. val.

Slope chart

| | Asym | | | | Asym |
|---|---|---|---|---|---|
| $f'(x)$ | P | + + | 0 | − − − | P |
| $x$ | $-1$ | $-\frac{3}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $0$ |

$f'(-\tfrac{1}{4}) < 0$

$f'(-\tfrac{3}{4}) > 0$ $\therefore$ $f$ has a local max at $x=-\tfrac{1}{2}$.

$$f(-\tfrac{1}{2}) = \frac{2}{-\frac{1}{2}(-\frac{1}{2}+1)} = -8$$

4.3 #13: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = (1 - 2x)(x - 1)^3.$$

polynomial

$$f'(x) = (1-2x)\cdot 3(x-1)^2 + (x-1)^3\cdot(-2)$$

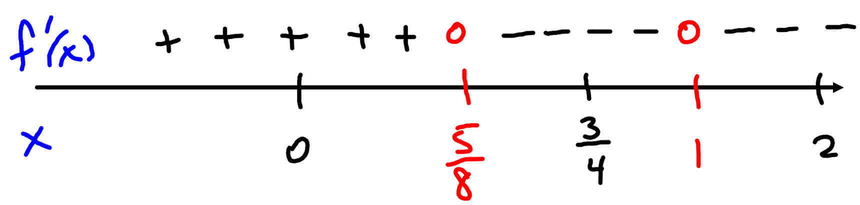
$$= [3(1-2x) - 2(x-1)](x-1)^2$$

$$= (5-8x)(x-1)^2$$

$f'(x)$ exists for all $x$ so critical values only occur at places where $f'(x) = 0$.

$f'(x) = 0$ iff $x = \frac{5}{8}$ or $x = 1$.

slope chart




$f'(0) = 5 > 0$

$f'(3/4) < 0$

$f'(2) < 0$

Shape:

$f$ has a local max at $x = 5/8$

$x = 1$ gives neither a local max nor local min.

$$f(5/8) = -\frac{1}{4}\left(-\frac{3}{8}\right)^3 = \frac{27}{2048}$$

4.3 #17: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = x^2\sqrt[3]{2+x} = x^2(2+x)^{1/3}$$

Domain: all $x$.

$$f'(x) = 2x(2+x)^{1/3} + x^2 \cdot \frac{1}{3}(2+x)^{-2/3}$$

exists at all $x$ except $x=-2$.

$\therefore$ C.V. at $x=-2$.

$f'(x)=0$

$$\frac{6x(2+x)+x^2}{(2+x)^{2/3}} = 0$$

$(2+x)^{2/3}$ ← positive except at $x=-2$

$$12x+7x^2=0$$

$$x(12+7x)=0$$

$$x=0, \quad x=-\frac{12}{7}$$

$\therefore$ C.V. $x=-2$, $x=0$, $x=-\frac{12}{7}$.

$f'(x)$: $+ + +$ DNE $+ +$ $0$ $- - - -$ $0$ $+ + +$

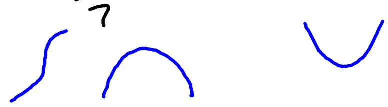
$x$: $-3$, $-2$, $-\frac{13}{7}$, $-\frac{12}{7}$, $-\frac{1}{2}$, $0$, $1$

$f'(-3)>0$

$f'(-\frac{13}{7})>0$

$f'(-\frac{1}{2})<0$

$f'(1)>0$



$\therefore$ $x=-2$ is neither a local max nor local min.

$x=-\frac{12}{7}$ is a place where $f$ has a local max

$x=0$ is a place where $f$ has a local min.

$f(-\frac{12}{7})=$ you do it

$f(0)=$ you do it

4.3 #19: Find the critical numbers of $f$ and the local extreme values.

$$f(x) = |x - 3| + |2x + 1|.$$ ← Domain: all $x$.

$f$ diff except at $x = 3$, $x = -\frac{1}{2}$ $\Big\}$ ∴ C.V. at $x = 3, -\frac{1}{2}$.

$$f(x) = \begin{cases} -(x-3) - (2x+1), & x < -\frac{1}{2} \\ -(x-3) + (2x+1), & -\frac{1}{2} < x < 3 \\ x - 3 + 2x + 1, & x > 3 \end{cases}$$

$$= \begin{cases} 2 - 3x, & x < -\frac{1}{2} \\ 4 + x, & -\frac{1}{2} < x < 3 \\ -2 + 3x, & x > 3 \end{cases}$$

$$\therefore f'(x) = \begin{cases} -3, & x < -\frac{1}{2} \\ 1, & -\frac{1}{2} < x < 3 \\ 3, & x > 3 \end{cases}$$

∴ only C.V. are at $x = -\frac{1}{2}$ and $x = 3$

Slope chart

| | | | | | |
|---|---|---|---|---|---|
| $f'(x)$ | $- - -$ | DNE | $+ + +$ | DNE | $+ + +$ |
| $x$ | | $-\frac{1}{2}$ | | $3$ | |

Shape:

$f$ has a local min at $x = -\frac{1}{2}$

$$f\left(-\tfrac{1}{2}\right) = \frac{7}{2}.$$