EVEN MORE
Review Problems for Test 4

Example:

$[a, b]$

Give the average value of $f(x) = x^2 - 3x$ on the interval $[-1, 2]$.

Average Value $\quad \frac{1}{b-a} \int_a^b f(x)\,dx$

$$= \frac{1}{3} \int_{-1}^{2} (x^2 - 3x)\,dx$$

$$= \frac{1}{3} \left(\frac{x^3}{3} - \frac{3}{2}x^2 \right) \Big|_{-1}^{2}$$

$$= \frac{1}{3} \left[\left(\frac{8}{3} - 6 \right) - \left(\frac{-1}{3} - \frac{3}{2} \right) \right]$$

$$= \frac{1}{3} \left[3 - 6 + \frac{3}{2} \right] = -\frac{1}{2}.$$

Example:

Give the number of values of c that satisfy the conclusion of the mean value theorem for integrals for the function $f(x) = x^2 - 3x$ on the interval $[-1, 2]$.

i.e. determine the # of values c with $-1 < c < 2$ and

$$f(c) = \frac{1}{2 - -1} \int_{-1}^{2} f(x)\,dx$$

$$f(c) = -\frac{1}{2} \iff c^2 - 3c = -\frac{1}{2}$$

$$c^2 - 3c + \frac{1}{2} = 0$$

$$c = \frac{3 \pm \sqrt{7}}{2}$$

Note: $\frac{3+\sqrt{7}}{2} > 2$

only $c = \frac{3-\sqrt{7}}{2}$ lies btwn -1 and 2.

Example:

$$\frac{d}{dx}\int_{-3x-1}^{2}\cos\left(\sqrt{t}\right)dt = -\frac{d}{dx}\int_{2}^{-3x-1}\cos\left(\sqrt{t}\right)dt$$

$$= -\cos\left(\sqrt{-3x-1}\right)\cdot(-3)$$

$$= 3\cos\left(\sqrt{-3x-1}\right).$$

Example:

$g'(x) = f(x)$, $g(4) = -1$, $g(-1) = 3$, $f(4) = 1$, $f(-1) = 2$.

Give $\int_{-1}^{4} (4f(x) - 3f'(x))\,dx$. $= \left(4g(x) - 3f(x)\right)\Big|_{-1}^{4}$

we need an anti-derivative for this.

Here it is: $4g(x) - 3f(x)$

b/c $g'(x) = f(x)$

$= \left[\left(4g(4) - 3f(4)\right) - \left(4g(-1) - 3f(-1)\right)\right]$

$= \left(4\cdot(-1) - 3(1)\right) - \left(4\cdot 3 - 3\cdot 2\right)$

$= -13.$

Example: $2x^2 - \cos(x) = \int_0^{2x} f(t)\,dt$. Find $f(x)$.

Differentiate wrt x.

$$4x + \sin(x) = f(2x)\cdot 2$$

$$4x + \sin(x) = 2f(2x)$$

$$2x + \tfrac{1}{2}\sin(x) = f(2x)$$

dummy variable.

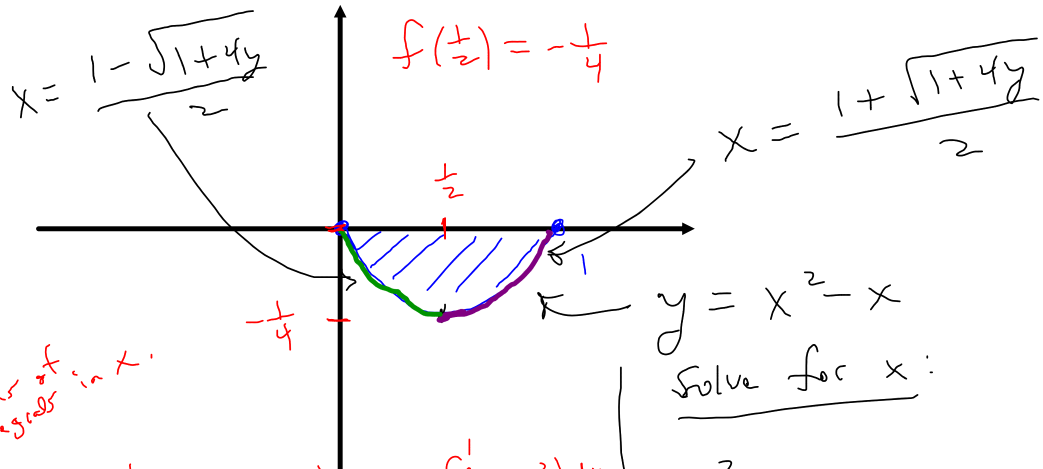
$$2u + \tfrac{1}{2}\sin(u) = f(2u)$$

Rename: $u = \frac{x}{2}$

$$2\,\frac{x}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) = f(x)$$

i.e. $\boxed{x + \frac{1}{2}\sin\left(\frac{x}{2}\right) = f(x)}$

Example: Give formulas for the area between the graph of $f(x) = x^2 - x$ and the x-axis in terms of integral(s) in x, and also in terms of integral(s) in y.



In terms of integrals in x.

$$\text{Area} = \int_0^1 \left(0 - (x^2 - x)\right) dx = \int_0^1 (x - x^2)\, dx = \cdots = \frac{1}{6}.$$

Solve for x:

$$x^2 - x - y = 0.$$

$$x = \frac{1 \pm \sqrt{1+4y}}{2}$$

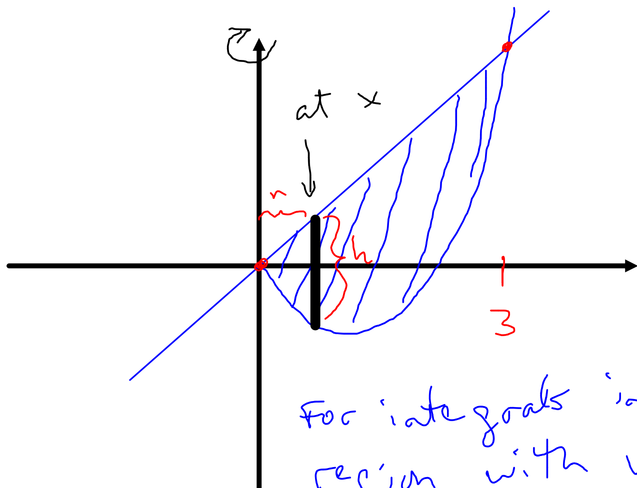
In terms of integrals in y

$$\text{Area} = \int_{-\frac{1}{4}}^{0} \left(\frac{1 + \sqrt{1+4y}}{2} - \frac{1 - \sqrt{1+4y}}{2} \right) dy$$

$$= \int_{-\frac{1}{4}}^{0} \sqrt{1+4y}\; dy$$

$$= \cdots = \frac{1}{6}.$$

Example: Rotate the region bounded between the graphs of $f(x) = x^2 - x$ and $g(x) = 2x$ around the y-axis. Give folumas for the volume generated in terms of integral(s) in x, and also in terms of integral(s) in y.



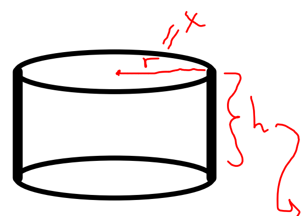
$x^2 - x = 2x$

$x^2 - 3x = 0$

$x = 0, \ x = 3$

For integrals in x, fill the region with vertical line segments.

Rotating one at x gives



shell or tube.

$2x - (x^2 - x)$

$= 3x - x^2$

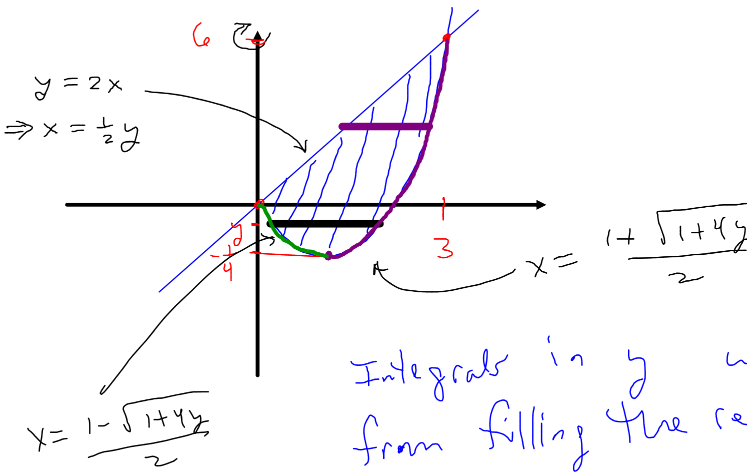
Surface area $= 2\pi r h$

$= 2\pi x(3x - x^2)$

Thickness $= dx$

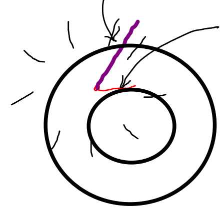
$$\boxed{\text{Volume} = \int_0^3 2\pi x(3x - x^2)\,dx}$$

← integral in x.



Integrals in y will result from filling the region with horizontal line segments.

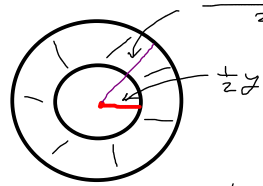
2 pieces: $-\frac{1}{4} \le y \le 0$



$$\text{Face area} = \pi\left(\frac{1+\sqrt{1+4y}}{2}\right)^2 - \pi\left(\frac{1-\sqrt{1+4y}}{2}\right)^2$$

$$\text{Thickness} = dy$$

$0 \le y \le 6$



$$\text{Face area} = \pi\left(\frac{1+\sqrt{1+4y}}{2}\right)^2 - \pi\left(\frac{1}{2}y\right)^2$$

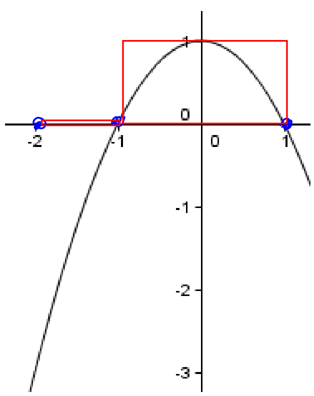
$$\text{Thickness} = dy$$

$$\boxed{\text{Volume} = \int_{-\frac{1}{4}}^{0}\left(\pi\left(\frac{1+\sqrt{1+4y}}{2}\right)^2 - \pi\left(\frac{1-\sqrt{1+4y}}{2}\right)^2\right)dy + \int_{0}^{6}\left(\pi\left(\frac{1+\sqrt{1+4y}}{2}\right)^2 - \pi\left(\frac{1}{2}y\right)^2\right)dy}$$

Example:

Give the upper Riemann sum of $f(x) = 1 - x^2$ on $[-2,1]$ with respect to the partition $P = \{-2,-1,1\}$.

(**Note:** On the exam, you will be asked to sketch the rectangles associated with the Riemann sum, as well as give a value.)

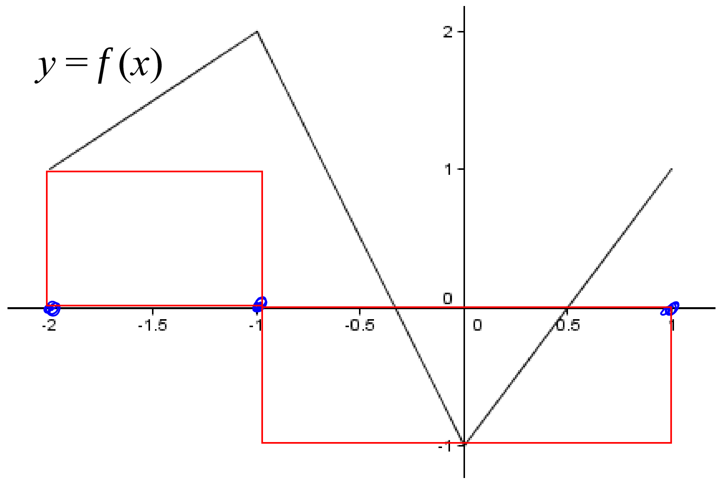


$$U_f = f(-1)\cdot 1 + f(0)\cdot 2$$

$$= 0 + 2 = 2$$

Example:

Give the lower Riemann sum of the function shown below on $[-2,1]$, with respect to the partition $P=\{-2,-1,1\}$.



(**Note:** On the exam, you will be asked to sketch the rectangles associated with the Riemann sum, as well as give a value.)

$$L_f = f(-2)\cdot 1 + f(0)\cdot 2$$

$$= 1 + (-1)\cdot 2$$

$$= -1$$

Examples: $\int \left(2\cos(3x) - 4\sin(2x)\right) dx =$

$$= \frac{2}{3}\sin(3x) + 2\cos(2x) + C$$

$$\frac{1}{6}\int_0^1 \frac{6x}{\sqrt{3x^2+1}}\,dx = \frac{1}{6}\int_{\boxed{1}}^{\boxed{4}} u^{-1/2}\,du$$

$u = 3x^2 + 1$

$du = 6x\,dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 4$

$$= \frac{1}{6}\cdot 2u^{1/2}\Big|_1^4$$

$$= \frac{1}{3}(4 - 2)$$

$$= \frac{2}{3}$$

Example: Suppose $F''(x) = x^2 - \sqrt{x} + 1$, $F(0) = -1$, $F'(0) = 2$.
Give $F(x)$.

$$F'(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + x + C_1$$

<u>Note</u>: $F'(0) = 2$

$$\Rightarrow$$

$$2 = C_1$$

$\therefore$

$$F'(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + x + 2$$

$\Rightarrow$

$$F(x) = \frac{1}{12}x^4 - \frac{4}{15}x^{5/2} + \frac{1}{2}x^2 + C_2$$

<u>Note</u>: $F(0) = -1$

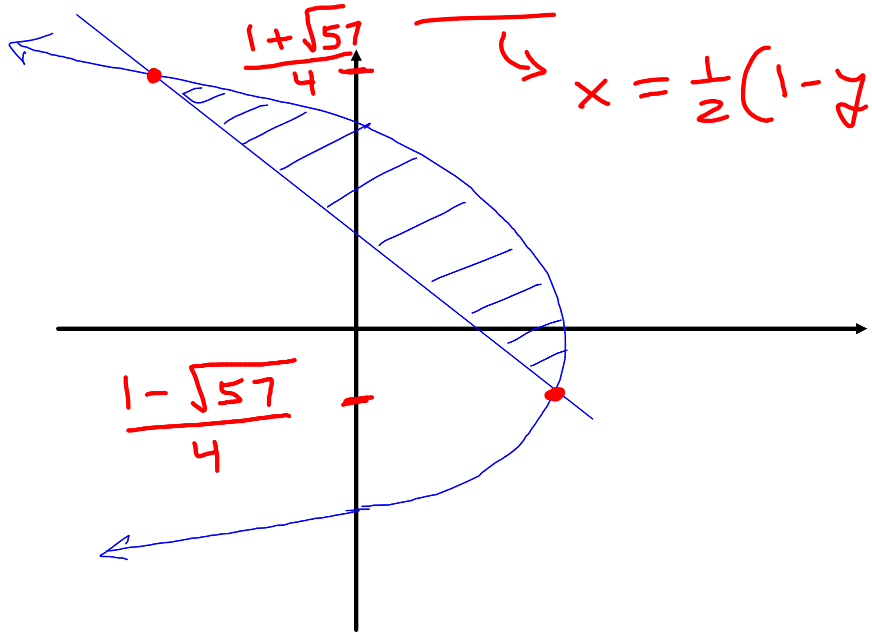
$$\Rightarrow -1 = C_2$$

$\Rightarrow$

$$F(x) = \frac{1}{12}x^4 - \frac{4}{15}x^{5/2} + \frac{1}{2}x^2 - 1.$$

Example: Give a formula for the area of the region bounded by the curves $2x + y = 1$ and $x = 4 - y^2$.

$x = \frac{1}{2}(1-y)$



$$\frac{1}{2}(1-y) = 4 - y^2$$

$$1 - y = 8 - 2y^2$$

$$2y^2 - y - 7 = 0$$

$$y = \frac{1 \pm \sqrt{1+56}}{4}$$

$$\text{Area} = \int_{\frac{1-\sqrt{57}}{4}}^{\frac{1+\sqrt{57}}{4}} \left((4-y^2) - \frac{1}{2}(1-y) \right) dy$$