Another Review for Test 3!!!!

Differentials: Formula - Uses

change in x

$$df = f'(x_0)\,h$$

The differential of f at x_0 with increment h.

approximates change in f.

$$f(x_0+h) - f(x_0) \approx f'(x_0)h$$

$$f(x_0+h) \approx f(x_0) + f'(x_0)h$$

Use differentials to estimate $\sin(\pi/60)$.

Use $f(x)=\sin(x)$.

Note: $\frac{\pi}{60}$ is close to 0 and we know $\sin(0)$.

$$f\left(\frac{\pi}{60}\right) \approx f(0) + f'(0)\,\frac{\pi}{60}$$

x_0+h $\quad x_0 \quad x_0 \quad h$

$f(0)=0$

$f'(x)=\cos(x)$

$f'(0)=1$

$$\approx 0 + 1\cdot \pi/60$$

$$\therefore \sin\left(\frac{\pi}{60}\right) \approx \frac{\pi}{60}.$$

Use differentials to estimate $(26)^{1/3}$.

$f(x)=x^{1/3}$ $\quad f'(x)=\frac{1}{3}x^{-2/3}$

Note: 26 is close to 27, and we know $(27)^{1/3}=3$.

$$(26)^{1/3}=f(26)\approx f(27)+f'(27)(-1)$$

x_0+h $\quad x_0 \quad x_0 \quad h$

$$\boxed{\frac{1}{3(27)^{2/3}}=\frac{1}{27}}$$

$$\approx 3+\frac{1}{27}\cdot(-1) = \frac{80}{27}$$

$f'(x)=\sin^2(\pi x)$ and $f(1/4)=2$. Estimate $f(1/5)$.

$$f(1/5)\approx f(1/4)+f'(1/4)\cdot\left(\frac{1}{5}-\frac{1}{4}\right)$$

x_0+h $\quad x_0 \quad h$

$$\approx 2+\sin^2\left(\frac{\pi}{4}\right)\cdot\left(-\frac{1}{20}\right)$$

$$\approx 2+\frac{1}{2}\left(-\frac{1}{20}\right) = 2-\frac{1}{40} = \frac{79}{40}.$$

Newton's Method: Formula - Use

$\hookrightarrow$ Approx roots to $f(x)=0$.

$\rightarrow$ Guess x_0.

1st Newton iterate $\longrightarrow$ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Use one iteration of Newton's method to approximate $\sqrt{48}$ from a guess of $\underset{x_0}{\overset{\shortparallel}{7}}$.

hm. what equation?

$$\underbrace{x^2-48}_{f(x)}=0. \qquad f'(x)=2x$$

$$x_1 = 7 - \frac{f(7)}{f'(7)} = 7 - \frac{1}{14} = \frac{97}{14}.$$

Use one iteration of Newton's method to approximate a root of $\underbrace{2x^3+4x^2-8x+3=0}_{f(x)=0}$ from a guess of $\boxed{x_0=1.}$

$$f(x) = 2x^3+4x^2-8x+3$$

$$f'(x) = 6x^2+8x-8$$

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{6} = \frac{5}{6}.$$

Mean Value Theorem: Formula - Use

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a value c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Verify the conclusion of the mean value theorem for $f(x) = x^3 - 4x^2 + x + 6$ on the interval $[-1,2]$. $f'(x) = 3x^2 - 8x + 1$

Find a value c between -1 and 2 so that

$$f'(c) = \frac{f(2) - f(-1)}{2 - -1}$$

$$3c^2 - 8c + 1 = \frac{0 - 0}{3}$$

$$3c^2 - 8c + 1 = 0 \iff c = \frac{8 \pm \sqrt{64 - 12}}{6}$$

$$\cancel{c = \underbrace{\frac{8 + \sqrt{52}}{6}}_{> 2}} \quad \text{or} \quad c = \frac{8 - \sqrt{52}}{6} \checkmark$$

$$= \frac{8 - \sqrt{4 \cdot 13}}{6}$$

$$= \frac{4 - \sqrt{13}}{3}$$

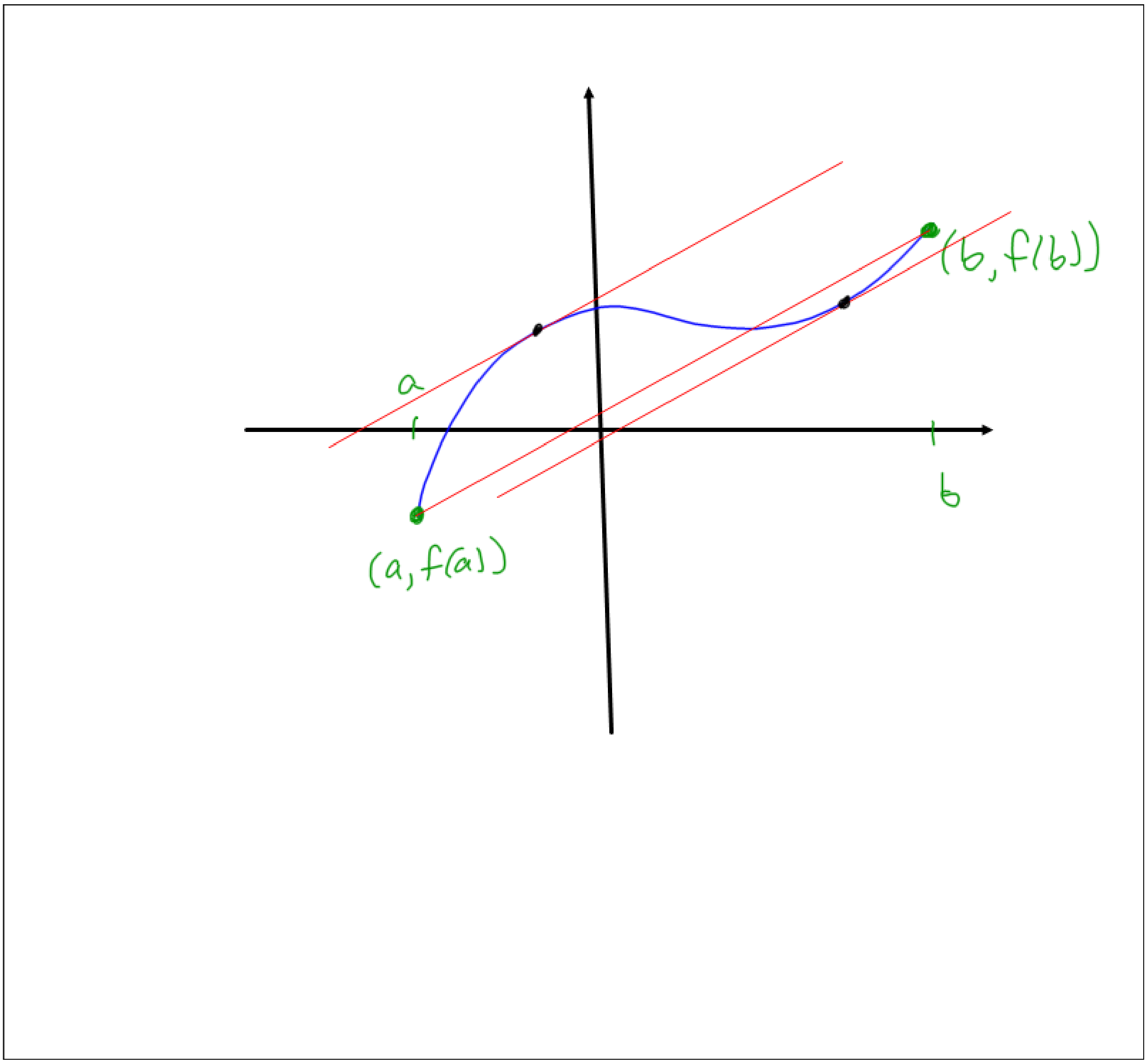
btwn -1 and 2.

Let f be a function such that $f(1) = 2.$ The value c that satisfies the mean value theorem on the interval ~~[1,5] is such~~ that $f'(c) = 2.$ What is $f(5)$?

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

$$2 = \frac{f(5) - 2}{4}$$

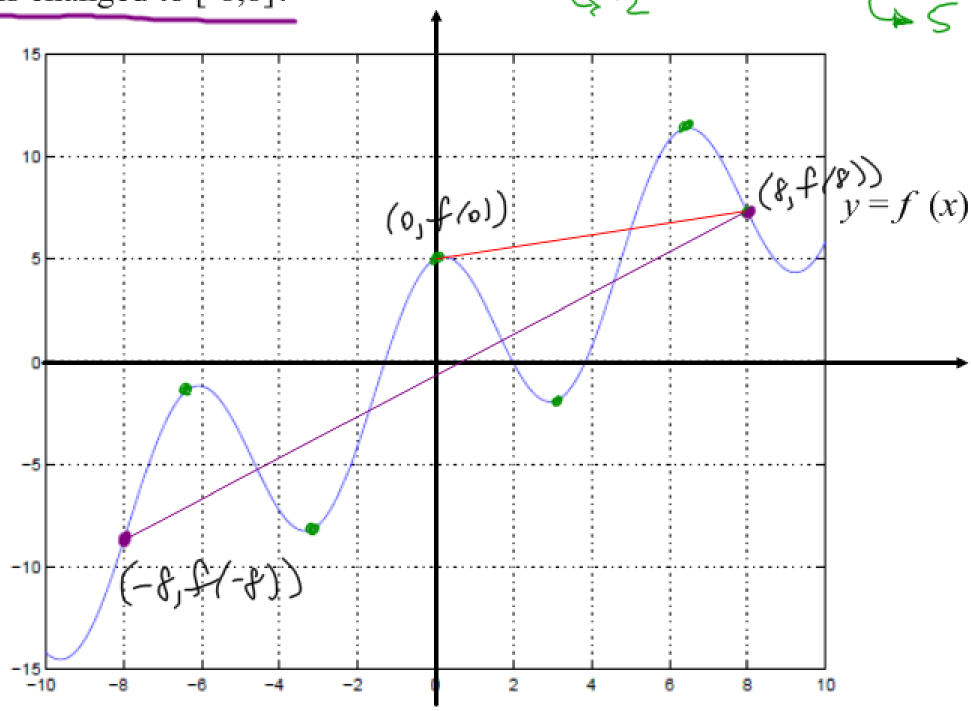
$$f(5) = 10.$$


(b, f(b))
a
b
(a, f(a))

The graph of $f(x)$ is shown below. How many values of c satisfy the conclusion of the mean value theorem on the interval [0,8]? What if the interval is changed to [-8,8]?

↳ 2

↳ 5



Increase/Decrease and Critical Numbers:

Critical numbers: values in the domain of f where either $f' = 0$ or f' dne.

- f is increasing on an interval I iff $f(x) < f(y)$ when $x < y$ in I.

  [If $f'(x) > 0$ except at finitely many values on I, then f is increasing on I.

- f is decreasing on an interval I iff $f(x) > f(y)$ when $x < y$ in I.

  [If $f'(x) < 0$ except at finitely many values on I, then f is decreasing on I.

Find the critical numbers, the interval(s) of increase, and the interval(s) of decrease for the function $f(x) = -x^3 + 6x^2 + 15x - 2$.

polynomial.

Domain: $(-\infty, \infty)$.

$f'(x) = -3x^2 + 12x + 15$

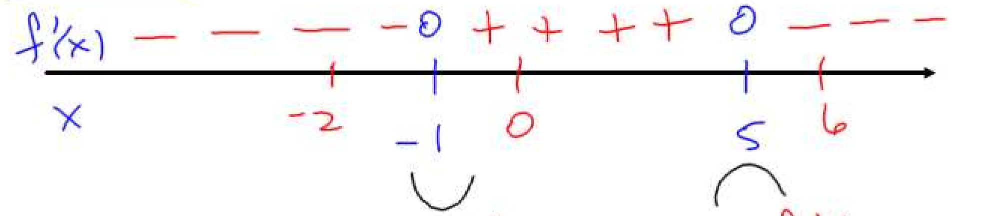
Set $f'(x) = 0$. $\quad -3x^2 + 12x + 15 = 0$

$-3(x^2 - 4x - 5) = 0$

$\boxed{-3(x-5)(x+1)} = 0$

$x = -1,\ x = 5$ ← C.n.

Slope chart:



$f'(-2) = -$ $\quad f'(0) = +$ $\quad f'(6) = -$

f is increasing on $[-1, 5]$.

f is decreasing on $(-\infty, -1]$ and $[5, \infty)$.

Classifying Critical Numbers:

First Derivative Test: ← slope chart

Second Derivative Test: Spse f'' exists on an open interval containing a c.n. $\tilde{x}$.

If $f''(\tilde{x}) > 0 \Rightarrow f$ has a loc. min. at $\tilde{x}$.

If $f''(\tilde{x}) < 0 \Rightarrow f$ has a loc. max. at $\tilde{x}$.

Classify the critical number $x = 0$ for the function $f(x) = x^2\cos(x)$.

Classify the critical numbers for the function $f(x) = -x^3 + 6x^2 + 15x - 2$ using

(a) the first derivative test.

(b) the second derivative test.

Classify the critical number $x = 0$ for the function $f(x) = x^2\cos(x)$.

$$f'(x) = x^2(-\sin(x)) + 2x\cos(x)$$

Note: $f'(0) = 0$.

Let's try the second derivative test.

$$f''(x) = x^2(-\cos(x)) + 2x(-\sin(x)) + 2x(-\sin(x)) + 2\cos(x).$$

$$\Rightarrow f''(0) = 0 + 0 + 0 + 2 > 0$$

$\therefore$ f has a local min at $x = 0$.

Classify the critical numbers for the function $f(x) = -x^3 + 6x^2 + 15x - 2$ using

(a) the first derivative test.

(b) the second derivative test.

a polynomial

$f'(x) = -3x^2 + 12x + 15$ exists for all x.

Set $f'(x) = 0$.

$$-3x^2 + 12x + 15 = 0$$
$$-3(x-5)(x+1) = 0$$
$x = -1$ and $x = 5$

c.n.

Slope chart:

$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$
x		-2		-1	0			5	6	

$\cup$ at -1, $\cap$ at 5

$f'(-2) = -$ $\quad$ $f'(0) = +$ $\quad$ $f'(6) = -$

$\therefore$ from the 1st derivative test,

f has a local min at $x = -1$

f has a local max at $x = 5$.

$$f''(x) = -6x + 12$$

$f''(-1) = 18 > 0 \Rightarrow f$ has a local min at $x = -1$.

$f''(5) = -18 < 0 \Rightarrow f$ has a local max at $x = 5$.

Absolute Maximums and Minimums for a Function on a Closed Bounded Interval:

If we want to find the extreme values of a continuous function on [a,b] then

3 Step Process:

1. Evaluate f(a) and f(b).
2. Evaluate f at each c.n. in [a,b].
3. Compare.

Find the absolute maximum and minimum values for the function
$f(x) = -x^3 + 6x^2 + 15x - 2$ on the interval [-2,1].

polynomial ⇒ continuous.

(1) $f(-2) = 8 + 24 - 30 - 2 = 0$

$f(1) = -1 + 6 + 15 - 2 = 18$

(2) $f'(x) = -3x^2 + 12x + 15$

$= -3(x-5)(x+1)$

c.n. $x = -1$, $\cancel{x = 5}$

$f(-1) = 1 + 6 - 15 - 2 = -10$

(3) Compare.

The abs. max value is 18, and it occurs at $x = 1$.

The abs. min value is -10, and it occurs at $x = -1$.

What if the function is not set on a closed bounded interval?

Find the absolute minimum value for the function $f(x) = x^4 - 8x^2 + 3$.

Slope chart

$$f'(x) = 4x^3 - 16x$$

$$f'(x) = 0 \iff 4x^3 - 16x = 0$$

$$\boxed{4x(x^2-4)} = 0$$

C.n. $\longrightarrow$ $x = 0,\ x = -2,\ x = 2$

slope chart

$f'(x)$	$----$	0	$++++$	0	$---$	0	$++++$
x	-3	-2	-1	0	1	2	3

$f'(-3) = -$ $f'(-1) = +$ $f'(1) = -$ $f'(3) = +$

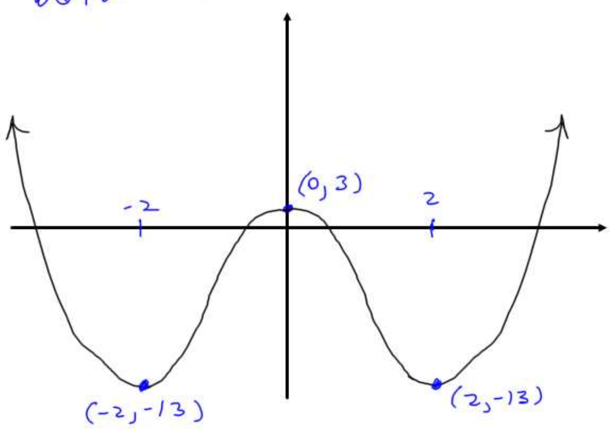
B/C f is decreasing on $(-\infty, -2]$ and increasing on $[2, \infty)$, the abs. min occurs at either

$f(x) = x^4 - 8x^2 + 3$ $x = -2$ or $x = 2$.

$f(-2) = 16 - 32 + 3$ $f(2) = -13$

$= -13$

$\therefore$ The abs. min value of f is -13. It occurs at both $x = -2$ and $x = 2$.



Concavity and Inflection:

Definition: f is concave up on an interval I iff f' is increasing on I.

If $f''(x) > 0$ at all but finitely many values on I then f is C.U. on I.

f is concave down on an interval I iff f' is decreasing on I.

If $f''(x) < 0$ at all but finitely many values on I then f is C.D. on I.

Inflection occurs at values in the domain of f where concavity changes.

Give the intervals of concave up and concave down for the function $f(x) = -x^3 + 6x^2 + 15x - 2$. Also give any inflection numbers.

$$f'(x) = -3x^2 + 12x + 15$$

$$f''(x) = -6x + 12.$$

$$f''(x) = 0 \quad \text{iff} \quad x = 2$$

Concavity chart

$f''(x)$	$+ + + +$	$+ \; 0$	$- - -$	$- -$
x	1	2	3	

$f''(1) = 6 > 0$ $\qquad$ $f''(3) = -6$

f is C.U. on $(-\infty, 2]$.

f is C.D. on $[2, \infty)$.

f has inflection at $x = 2$.

Give the intervals of concave up and concave down for the function $f(x) = x^4 - 8x^2 + 3$. Also give any inflection numbers.

$$f'(x) = 4x^3 - 16x$$

$$f''(x) = 12x^2 - 16$$

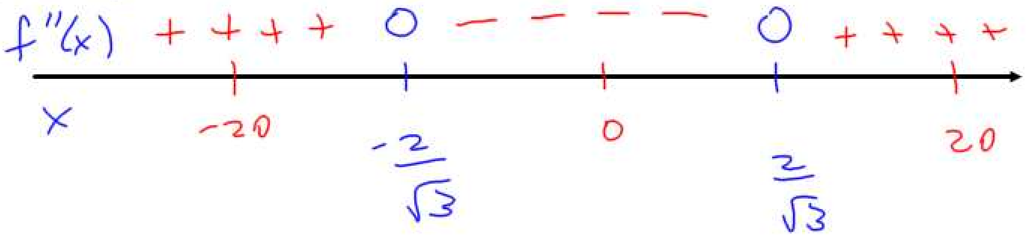
$$f''(x) = 0 \iff 12x^2 - 16 = 0$$

$$3x^2 - 4 = 0$$

$$x^2 = \frac{4}{3}$$

$$x = \pm \frac{2}{\sqrt{3}}$$

Concavity chart



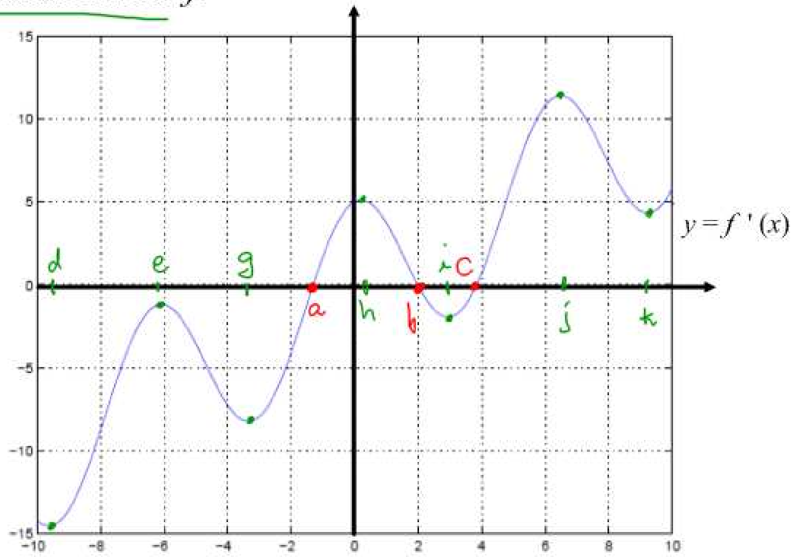
$f''(-20) = +$ $\quad$ $f''(0) = -$ $\quad$ $f''(20) = +$

f is c.u. on $\left(-\infty, -\frac{2}{\sqrt{3}}\right]$ and $\left[\frac{2}{\sqrt{3}}, \infty\right)$.

f is c.D. on $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$.

f has inflection at $-\frac{2}{\sqrt{3}}$ and $\frac{2}{\sqrt{3}}$.

The graph of $f\,'(x)$ is shown below on the interval [-10,10]. Give the intervals of increase, decrease, concave up, and concave down for f on this interval. Also find and classify any critical numbers for f and list the inflection numbers for f.



$f'(x)$ – – – – – – – – – – – – – 0 + + + + 0 – – 0 + + + + + +

x a   b   c

f is increasing on [a, b] and [c, 10].

f is decreasing on [-10, a] and [b, c].

Recall: f is C.U. when f' is increasing.

f is C.U. on [d, e], [g, h], [i, j] and [k, 10].

f is C.D. on [-10, d], [e, g], [h, i] and [j, k].

inflection occurs at d, e, g, h, i, j and k.

f has c.n. at x = a, b, c.

f has local mins at x = a and x = c.

f has a local max at x = b.

Graphing:

5 step method:

1. Domain.
2. Asymptotes and edge behavior.
3. first derivative info
4. second derivative info
5. Graph.

Let
$$f(x)=\frac{9(3-x^2)}{x^3}. \text{ Then } f'(x)=\frac{9x^2-81}{x^4} \text{ and } f''(x)=\frac{324-18x^2}{x^5}$$

1. Determine the domain of f, the x- and y-intercepts of the graph (if any), ~~the symmetry (if any)~~, and the horizontal and vertical asymptotes of the graph (if any).
2. Determine the critical numbers of f, the intervals on which f is increasing and the intervals on which f is decreasing.
3. Determine the intervals on which the graph of f is concave up, the intervals on which the graph is concave down, and find the points of inflection.
4. Use the information from parts (a) - (c) to sketch an accurate graph of f.

$$f(x)=\frac{9(3-x^2)}{x^3}. \text{ Then } f'(x)=\frac{9x^2-81}{x^4} \text{ and } f''(x)=\frac{324-18x^2}{x^5}$$

1. Domain: All x except $x=0$.

x-intercepts: Set $y=0$. (i.e. $f(x)=0$)

$9(3-x^2)=0$

$\boxed{x=-\sqrt{3},\ x=\sqrt{3}}$

y-intercepts: Set $x=0$.

$f(0)=$ OUCH!

No y-intercept. $x=0$ is NOT in the domain.

H.A.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\cancel{x^2}\left(\frac{27}{x^2}-9\right)}{x^{\cancel{3}}}$$

$$=\lim_{x\to-\infty}\frac{\overset{\to 0}{\frac{27}{x^2}}-9}{x}=0$$

$$\lim_{x\to\infty} f(x) = \text{same deal} = \cdots = 0$$

$y=0$.

V.A.:

$$f(x)=\frac{27-9x^2}{x^3}$$

$x=0$

2. $f'(x) = \dfrac{9x^2-81}{x^4}$

defined for all x except $x=0$, but $x=0$ is not in the domain of f.

$f'(x)=0$ iff

$9x^2-81=0$

$x^2-9=0$

$x=\pm 3.$

$\boxed{f'(x) = \dfrac{9x^2-81}{x^4}}$

Slope Chart:

x	-4	-3	-1	0	1	3	4
$f'(x)$	+ + +	0	− − −		− − −	0	+ + + +
f shape		∩				∪	

$f'(-4)=+$ $\quad f'(-1)=-$ $\quad f'(1)=-$ $\quad f'(4)=+$

f is increasing on $(-\infty,-3]$ and $[3,\infty)$.

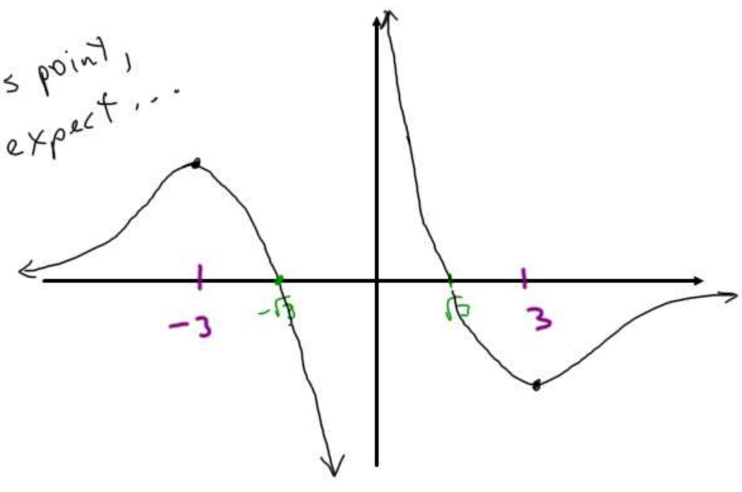
f is decreasing on $[-3,0)$ and $(0,3]$.

C.n.: $x=-3$, $x=3$

f has a local max at $x=-3$

f has a local min at $x=3$.

As this point, we expect...



3. $f''(x) = \dfrac{324 - 18x^2}{x^5}$

exists at all x except $x=0$, and $x=0$ is <u>not</u> in the domain of f.

$f''(x) = 0$ iff

$324 - 18x^2 = 0$

$x^2 = 18$

$x = \pm 3\sqrt{2}$

<u>Concavity chart</u>:

$f''(x)$	+ + + +	0	− − −		+ + +	0	− − − −
x	-32	$-3\sqrt{2}$	-1	0	1	$3\sqrt{2}$	100

$f''(-32) = +$ $\quad f''(-1) = -$ $\quad f''(1) = +$ $\quad f''(100) = -$

f is C.U. on $(-\infty, -3\sqrt{2}]$ and $(0, 3\sqrt{2}]$.

f is C.D. on $[-3\sqrt{2}, 0)$ and $[3\sqrt{2}, \infty)$.

f has inflection at $x = -3\sqrt{2}$ and $x = 3\sqrt{2}$

4. <u>Graph</u>. Plot $(-\sqrt{3}, 0)$, $(\sqrt{3}, 0)$

<u>V.A.</u> $x = 0$, <u>H.A.</u> $y = 0$,

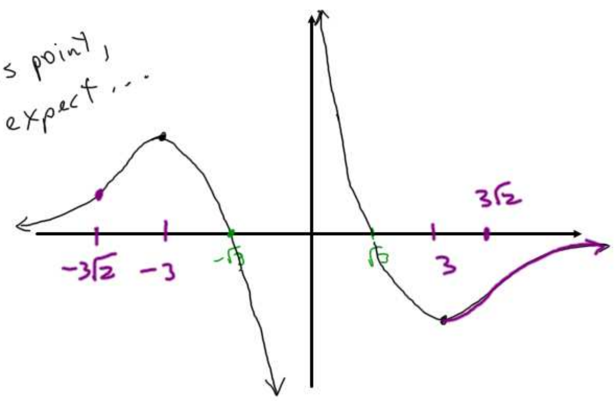
<u>C.P.</u> at $(-3, f(-3)) = (-3, 1)$

$(3, f(3)) = (3, -1)$

<u>infl. points</u>: $\left(-3\sqrt{2}, f(-3\sqrt{2})\right) = \left(-3\sqrt{2}, \frac{5}{2\sqrt{2}}\right)$

$\left(3\sqrt{2}, f(3\sqrt{2})\right) = \left(3\sqrt{2}, \frac{-5}{2\sqrt{2}}\right)$

As this point, we expect...



Graph the function $f(x) = \dfrac{x-1}{x^2 - 5x + 6}$

NO TIME

Max/Min Word Problems:

What are the dimensions of the base of the rectangular box of greatest volume that can be constructed from 100 square inches of cardboard if the base is to be twice as long as it is wide? Assume that the box has top.

You

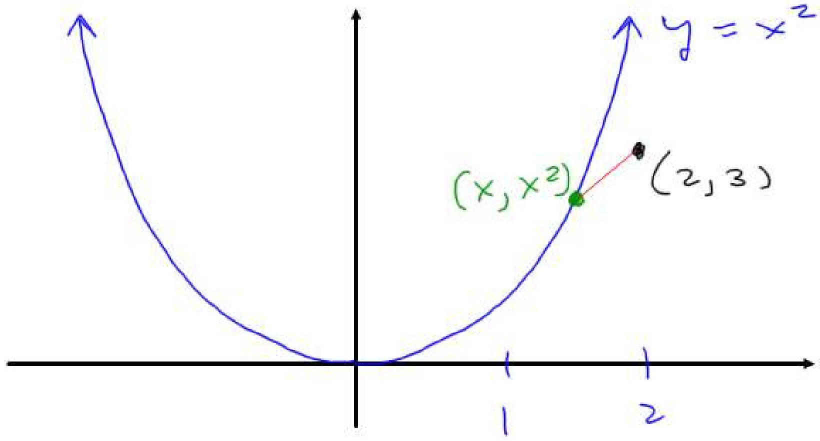
Find the point on the curve $y = x^2$ that is closest to the point (2,3).

US

Give the dimensions of the rectangle with greatest area that has its base on the x-axis and its upper vertices on the parabola $y = 4 - x^2$.

US

Find the point on the curve $y = x^2$ that is closest to the point (2,3).



distance $= \sqrt{(x-2)^2 + (x^2-3)^2}$

Minimize $f(x) = \sqrt{(x-2)^2 + (x^2-3)^2}$

exists for all x.

$$f'(x) = \frac{2(x-2) + 2(x^2-3)\cdot 2x}{2\sqrt{(x-2)^2 + (x^2-3)^2}}$$

$f'(x) = 0$ iff

$$2(x-2) + 2(x^2-3)\cdot 2x = 0$$

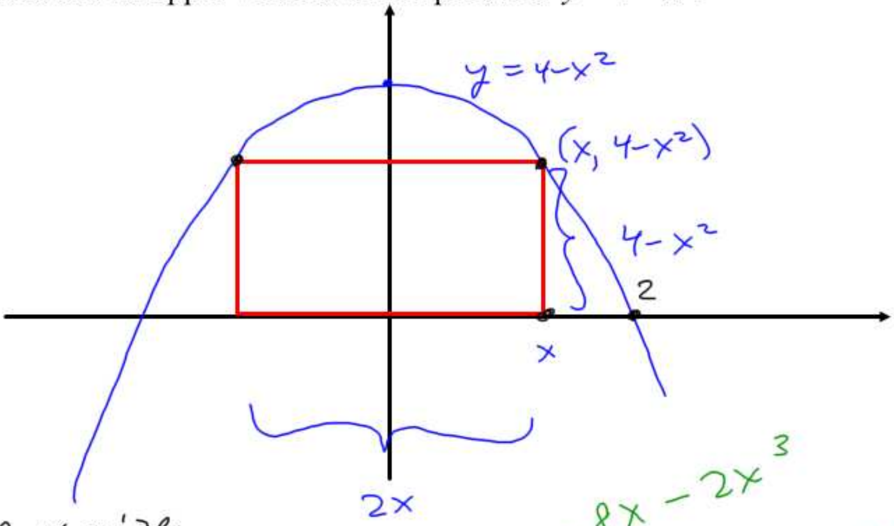
$$2x - 4 + 4x^3 - 12x = 0$$

$$\boxed{4x^3 - 10x - 4 = 0}$$

Solve! . . .

If this problem were properly rigged (as your exam problem will be), we'd be able to immediately determine the value(s) of x that make this zero. Then we would finish the problem.

Give the dimensions of the rectangle with greatest area that has its base on the x-axis and its upper vertices on the parabola $y = 4 - x^2$.



Maximize

$$A(x) = 2x(4-x^2) = 8x - 2x^3, \quad \boxed{0 \le x \le 2}$$

1. $A(0) = 0$, $A(2) = 0$

2. $A'(x) = 8 - 6x^2$

$8 - 6x^2 = 0 \qquad x = \pm\sqrt{\frac{4}{3}} = \cancel{\pm}\frac{2}{\sqrt{3}}$

btwn 0 and 2 ← $x = \frac{2}{\sqrt{3}}$ $\quad A\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}\left(4 - \frac{4}{3}\right)$

$$= \frac{32}{3\sqrt{3}}$$

3. Compare. The abs. max area is $\frac{32}{3\sqrt{3}}$, and it occurs when $x = \frac{2}{\sqrt{3}}$.

$$\boxed{\begin{aligned} \text{Base} &= 2x = \frac{4}{\sqrt{3}} \\ \text{Height} &= 4 - x^2 = 4 - \frac{4}{3} = \frac{8}{3}. \end{aligned}}$$