Notes:

- See the course calendar page for information concerning EMCFs, Homework and Quizzes.
- Test 2 is ***Oct 4-8*** in CASA. Scheduling is available on CourseWare starting September 22nd. A practice exam will be available on CourseWare soon. **Your grade on the practice exam will count as a quiz score.** I posted a review problem set.

12:01 am

New

Recall...

The Chain Rule

If u is differentiable at x and f is differentiable at $u(x)$, then the $\boxed{\text{composition}}$ $f \circ u$ is differentiable at x and $(f \circ u)'(x) = f'(u(x))\,u'(x)$.

i.e. $$\frac{d}{dx} f\big(u(x)\big) = f'\big(u(x)\big)\, \frac{d}{dx} u(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example:

product

Suppose $G(x) = u(x)\cos(v(x)\pi)$, $u(1) = 2$, $v(1) = 3$, $u'(1) = -2$, and $v'(1) = 3$. Find $G'(1)$.

1. $G'(x) = u(x)\left(-\sin(v(x)\pi)\cdot v'(x)\pi\right) + \cos(v(x)\pi)\,u'(x)$

2. $G'(1) = u(1)\left(-\sin(v(1)\pi)\cdot v'(1)\pi\right) + \cos(v(1)\pi)\cdot u'(1)$

$= 2\left(-\sin(3\pi)\cdot 3\pi\right) + \cos(3\pi)\cdot(-2)$

$= 2$

Example:

Suppose $G(x) = u(x)/v(x)$, $u(1) = 2$, $v(1) = 3$, $u'(1) = -3$, and $v'(1) = -1$. Find $G'(1)$.

$G(x) = \dfrac{u(x)}{v(x)}$

1. $G'(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}$

2. $G'(1) = \ldots$ all you.

Popper P06

Note: Decimal answers should be given to 4 decimal places of accuracy.

Example: -23.1572873 should be given as -23.1572.

1. $f(x) = x\sin(3x) - 2x^2$. $f'(1) =$

2. $g(x) = 3(x^2 - 1)^3$. What is the slope of the normal line to the graph of g at the point where $x = 2$?

$g'(x) = 9(x^2-1)^2 \cdot 2x$

$g'(2) = 9 \cdot 9 \cdot 4 = 324$

$-\frac{1}{g'(2)} = -1/324$

3.7

Implicit Differentiation

The Basic Idea…

Consider the equation

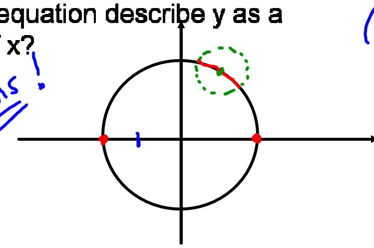
$$x^2 + y^2 = 1$$

circle radius 1 centered at $(0,0)$

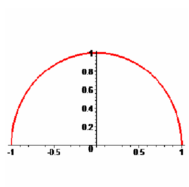
* Does this equation describe y as a function of x?

NO

Fails!

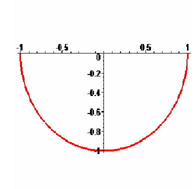


If we restrict the view, we see the graph of a function.



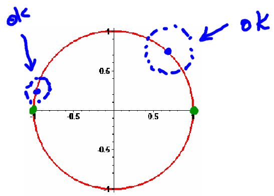
$$x^2 + y^2 = 1, \quad y \geq 0$$

The same is true with the restriction below.



$$x^2 + y^2 = 1, \quad y \leq 0$$

Remark: The points (-1,0) and (1,0) are the only places where we can't zoom in and get a $\boxed{\textit{local view}}$ that shows the graph of a function.



$$x^2 + y^2 = 1$$

$$y = \pm\sqrt{1-x^2}$$

Another Example

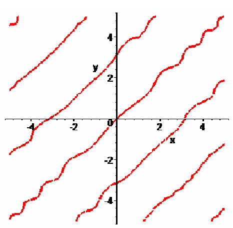
Consider the equation

$$\sin(y-x) + \frac{\sin(xy)}{5} = 0$$

Does this equation describe y as a function of x?

The Graph

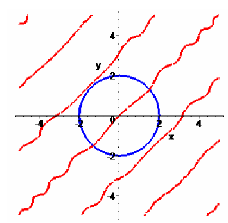
Not the graph of a function.



$$\sin(y-x) + \frac{\sin(xy)}{5} = 0$$

Let's restrict our view to the portion of the graph inside the blue circle below.

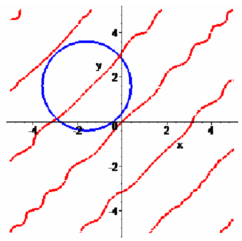
The portion in the blue circle looks like the graph of a function.



$$\sin(y-x) + \frac{\sin(xy)}{5} = 0$$

What about this view?

ditto



You can identify many others!!

The Point

The equations

$$x^2 + y^2 = 1$$

$$\sin(y - x) + \frac{\sin(xy)}{5} = 0$$

both have solution sets which *locally* look like the graph of a function.

i.e. They both (locally) describe *y* as a function of x.

The Basic Question

Can we differentiate *y* as a function of *x*?

(at places where it appears locally to be a function of x)

Simple

Consider $x^2 + y^2 = 1$

at points on the circle. $\leftrightarrow$ $\frac{dy}{dx} = ?$

1. Treat y like a diff function of x.

2. Diff wrt x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx} 1$$

$$2x + \underbrace{2y \frac{dy}{dx}}_{\text{chain rule}} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$ at points (x, y) on the circle.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$x \qquad y$

$\| \qquad \|$

Question : What is the derivative at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$?

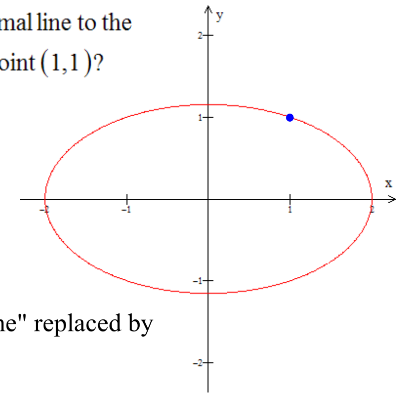
Hm... is this on $x^2 + y^2 = 1$?

Yes

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)} = -\frac{\sqrt{2}/2}{\sqrt{2}/2} = -1.$$

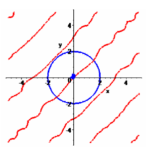
Popper P06

3. What is the slope of the normal line to the graph of $x^2 + 3y^2 = 4$ at the point $(1,1)$?



4. Repeat #3 with "normal line" replaced by "tangent line".

Consider $\sin(y-x) + \frac{\sin(xy)}{5} = 0$



(0,0) solves this equation, and the graph looks like a function of x near (0,0). Assuming y is a differentiable function of x near this point, find dy/dx at (0,0).

1. Treat y like a diff function of x.
2. Diff wrt x.

$$\frac{d}{dx}\left(\sin(y-x) + \frac{1}{5}\sin(xy)\right) = \frac{d}{dx}0$$

$$\cos(y-x)\cdot\left[\frac{dy}{dx} - 1\right] + \frac{1}{5}\cos(xy)\cdot\left[x\frac{dy}{dx} + y\right] = 0$$

Subst $x=0$, $y=0$.

$$\frac{dy}{dx} - 1 + \frac{1}{5}\left[0 + 0\right] = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,0)} = 1.$$

Typically...

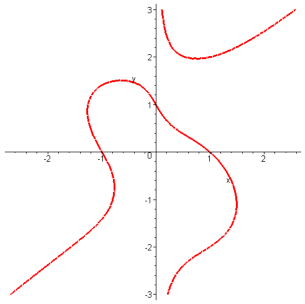
Suppose $f(x,y) = C$ gives y locally as a function of x. If you treat y like a differentiable function of x and you can solve for dy/dx after differentiating the equation

$$f(x,y) = C,$$

then dy/dx exists.

This is the essence of a result called the Implicit Function Theorem.

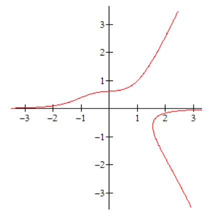
Example: Use the implicit function theorem to show that $x^4 + 3xy - xy^3 + y = 1$ yields y as a function of x (at least implicitly) for (x, y) near $(0,1)$, and find dy/dx at $(0,1)$.



Also give the equations for the tangent and normal lines at (0,1).

See the video.

Example: The equation $xy^2 + 2x^3y - 3y^3 = 2y - 2$ has the graph given below. Give a formula for dy/dx at points on the graph where y is locally a function of x. Also, note that (1,1) is on the graph of this function. Give the y-intercept of the tangent line to the graph at (1,1).



1. Treat y like a diff function of x.
2. Diff wrt x.

$$\frac{d}{dx}\left(xy^2 + 2x^3y - 3y^3\right) = \frac{d}{dx}(2y-2)$$

$$x \cdot 2y\frac{dy}{dx} + y^2 + 2x^3\frac{dy}{dx} + y \cdot 6x^2 - 9y^2\frac{dy}{dx} = 2\frac{dy}{dx}$$

See the video.