Notes:

- See the course calendar page for information concerning EMCFs, Homework and Quizzes.
- Test 2 is ***Oct 4-8*** in CASA. Scheduling is available on CourseWare starting September 22nd. A practice exam will be available on CourseWare soon. **Your grade on the practice exam will count as a quiz score.** I will post a review problem set soon.

at 12:01 am

Recall... The Chain Rule

If u is differentiable at x and f is differentiable at $u(x)$, then the composition $f \circ u$ is differentiable at x and $(f \circ u)'(x) = f'(u(x))u'(x)$.

$$\text{i.e.} \quad \frac{d}{dx} f\left(u(x)\right) = f'\left(u(x)\right) \frac{d}{dx} u(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example:

Suppose $G(x) = \underbrace{u(x)\cos(v(x)\pi)}_{\text{product}}$, $u(1) = 2$, $v(1) = 3$, $u'(1) = -2$, and $v'(1) = 3$. Find $G'(1)$.

1. $G'(x) = u(x)\left(-\sin(v(x)\pi) \cdot \pi v'(x)\right) + \cos(v(x)\pi)\, u'(x)$

2. $G'(1) = u(1)\left(-\sin(v(1)\pi)\, \pi v'(1)\right) + \cos(v(1)\pi)\, u'(1)$

$= 2\left(-\sin(3\pi)\, 3\pi\right) + \cos(3\pi)(-2)$

$= 2$

Example:

Suppose $G(x) = u(x)/v(x)$, $u(1) = 2$, $v(1) = 3$, $u'(1) = -3$, and $v'(1) = -1$. Find $G'(1)$.

$G(x) = \dfrac{u(x)}{v(x)}$

1. $G'(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}$

2. $G'(1) = \dfrac{v(1)u'(1) - u(1)v'(1)}{v(1)^2}$

$= \dfrac{3\cdot(-3) - 2\cdot(-1)}{9}$

$= \dfrac{-9+2}{9} = \dfrac{-7}{9}$

Implicit Differentiation

The Basic Idea…

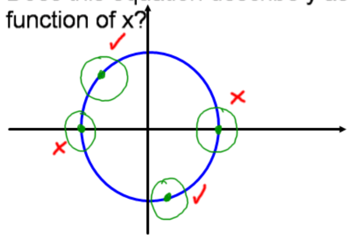
Consider the equation

$$x^2 + y^2 = 1$$

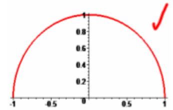
circle of radius 1 centered at (0,0)

Does this equation describe y as a function of x?

NO

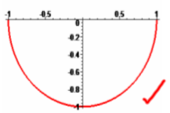


If we restrict the view, we see the graph of a function.



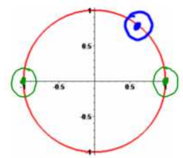
$$x^2 + y^2 = 1, \quad y \geq 0$$

The same is true with the restriction below.



$$x^2 + y^2 = 1, \quad y \leq 0$$

Remark: The points (-1,0) and (1,0) are the only places where we can't zoom in and get a *local view* that shows the graph of a function.



$$x^2 + y^2 = 1$$

Another Example

Consider the equation

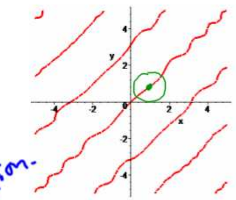
$$\sin(y-x)+\frac{\sin(xy)}{5}=0$$

more complicated

Does this equation describe y as a function of x?

The Graph

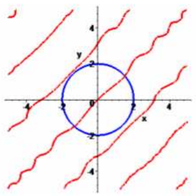
This is not the graph of a function.



$$\sin(y-x)+\frac{\sin(xy)}{5}=0$$

Let's restrict our view to the portion of the graph inside the blue circle below.

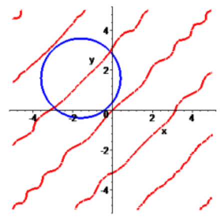
The portion inside the circle is the graph of a function.



$$\sin(y-x)+\frac{\sin(xy)}{5}=0$$

What about this view?

ditto



You can identify many others!!

"The Point"

The equations

$$x^2 + y^2 = 1$$

$$\sin(y - x) + \frac{\sin(xy)}{5} = 0$$

both have solution sets which *locally* look like the graph of a function.

i.e. They both (locally) describe *y* as a function of *x*.

The Basic Question

Can we differentiate *y* as a function of *x*?

(at places where it appears locally to be a function of x)

Consider $x^2 + y^2 = 1$

$$\frac{dy}{dx} = ?$$

Treat y like a diff. function of x.
Then diff. wrt x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}1$$

$$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

only holds for (x,y) on $x^2+y^2=1$ $\left\{ \frac{dy}{dx} = -\frac{x}{y} \right.$

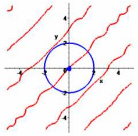
Question : What is the derivative at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$? ($x = \frac{\sqrt{2}}{2}$, $y = \frac{\sqrt{2}}{2}$)

Note: $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$ ✓

i.e. This point is on $x^2+y^2=1$.

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)} = -1$$

Consider $\sin(y-x)+\dfrac{\sin(xy)}{5}=0$



(0,0) solves this equation, and the graph looks like a function of x near (0,0). Assuming y is a differentiable function of x near this point, find dy/dx at (0,0).

Treat y like a diff. function of x.
Diff. wrt x.

$$\frac{d}{dx}\left(\sin(y-x)+\frac{1}{5}\sin(xy)\right)=\frac{d}{dx}0$$

$$\cos(y-x)\left[\frac{dy}{dx}-1\right]+\frac{1}{5}\cos(xy)\cdot\left[x\frac{dy}{dx}+y\right]=0$$

Subst. $x=0,\ y=0$

$$\frac{dy}{dx}-1+\frac{1}{5}\left[0+0\right]=0$$

$$\Rightarrow \frac{dy}{dx}=1 \quad \text{at } (0,0)$$

Typically...

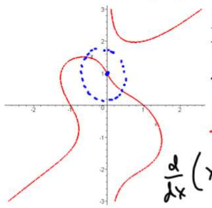
Suppose $f(x,y)=C$ gives y locally as a function of x. If you treat y like a differentiable function of x and you can solve for dy/dx after differentiating the equation

$$f(x,y)=C,$$

then dy/dx exists.

This is the essence of a result called the Implicit Function Theorem.

Example: Use the implicit function theorem to show that $x^4+3xy-xy^3+y=1$ yields y as a function of x (at least implicitly) for (x,y) near $(0,1)$, and find dy/dx at $(0,1)$.



Also give the equations for the tangent and normal lines at (0,1).

Treat y like a diff function of x, and diff wrt x.

$$\frac{d}{dx}\left(x^4+3xy-xy^3+y\right)=\frac{d}{dx}1$$

$$4x^3+3x\frac{dy}{dx}+y\cdot 3-\left(x\cdot 3y^2\frac{dy}{dx}+y^3\right)+\frac{dy}{dx}=0$$

To find $\frac{dy}{dx}$ at $(0,1)$, subst $x=0,\ y=1$.

$$0+0+3-(0+1)+\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-2$$

Also give the equations for the tangent and normal lines at (0,1).

$$\left.\frac{dy}{dx}\right|_{(0,1)}=-2$$

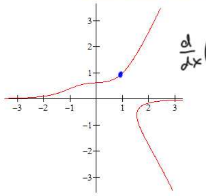
Tangent Line: Point $=(0,1)$
slope $=-2$

Equation: $y-1=-2x$
or $y=-2x+1$

Normal Line: Point $=(0,1)$
slope $=\frac{1}{2}$

Equation: $y-1=\frac{1}{2}x$
or $y=\frac{1}{2}x+1$.

Example: The equation $xy^2 + 2x^3y - 3y^3 = 2y - 2$ has the graph given below. Give a formula for dy/dx at points on the graph where y is locally a function of x. Also, note that (1,1) is on the graph of this function. Give the y-intercept of the tangent line to the graph at (1,1).



$$\frac{d}{dx}\left(xy^2 + 2x^3y - 3y^3\right) = \frac{d}{dx}(2y-2)$$

$$x\cdot 2y\frac{dy}{dx} + y^2 + 2x^3\frac{dy}{dx} + y\cdot 6x^2 - 9y^2\frac{dy}{dx} = 2\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2xy + 2x^3 - 9y^2 - 2\right) = -y^2 - 6x^2y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2 - 6x^2y}{2xy + 2x^3 - 9y^2 - 2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2 - 6x^2y}{2xy + 2x^3 - 9y^2 - 2}$$

Tangent Line at (1,1):

Point = (1,1)

$$\text{slope} = \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-7}{-7} = 1$$

$$\therefore \quad y - 1 = x - 1 \Rightarrow y = x$$

So, the y-intercept is 0.