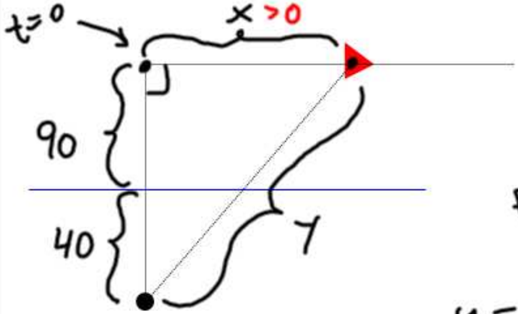
Related Rates

(Applied Chain Rule and Implicit Differentiation)

Setting: Two or more quantities are related by equation(s). All but one of the rates of change are known, and you are asked to find the other rate of change.

Example: A small boat is 90 feet offshore, and moving parallel to a straight beach. The boat is moving at a constant speed of 10 feet per second. At time $t = 0$, the boat is directly opposite a lifeguard station which is 40 feet from the water. How fast is the boat moving away from the lifeguard station when the distance between the boat and the lifeguard station is 150 feet?



$$\frac{dx}{dt} = 10 \text{ ft/sec}$$

Find $\dfrac{dy}{dt}$ when $y = 150$ feet.

$x^2 + 16900 = 22500$

$x^2 = 5600$

$x = \sqrt{5600}$

$= 20\sqrt{14}$

Note: $x^2 + (130)^2 = y^2$

Diff wrt t.

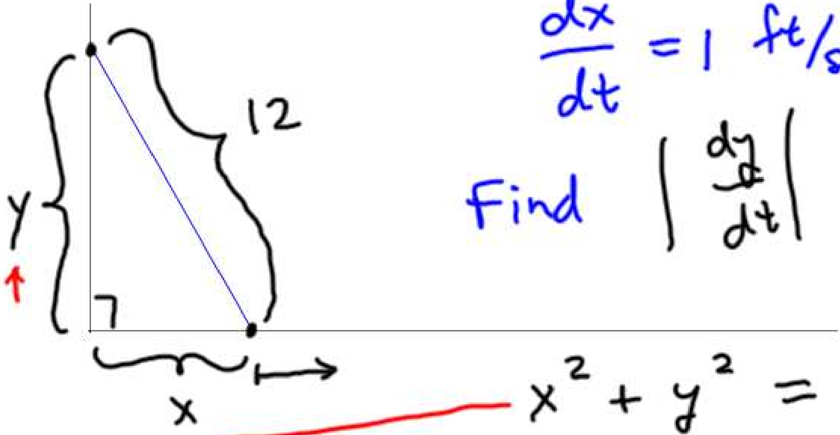
$$\cancel{2}x\frac{dx}{dt} + 0 = \cancel{2}y\frac{dy}{dt}$$

$20\sqrt{14} = ?$ $\frac{dx}{dt} = 10$ $y = 150$ $\frac{dy}{dt}$ ← find

$$200\sqrt{14} = 150\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{4}{3}\sqrt{14} \text{ ft/sec.} \approx 4.99 \text{ ft/sec}$$

Example: A 12 foot ladder is leaning against a wall. If the base of the ladder is moving away from the wall at the rate of 1 foot per second, at what rate will the top of the ladder be moving when the base of the ladder is 5 feet from the wall?



$$\frac{dx}{dt} = 1 \text{ ft/sec.}$$

Find $\left|\frac{dy}{dt}\right|$ when $x = 5$ ft.

$$x^2 + y^2 = 144$$

$25 + y^2 = 144$

$y^2 = 119$

$y = \sqrt{119}$

Diff wrt t.

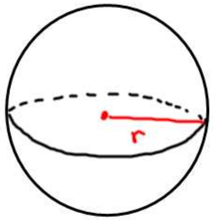
$$\cancel{2}\underset{5}{x}\underbrace{\frac{dx}{dt}}_{1} + \cancel{2}\underset{?=\sqrt{119}}{y}\underbrace{\frac{dy}{dt}}_{\text{Find}} = 0$$

$$5 + \sqrt{119}\,\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-5}{\sqrt{119}} \Rightarrow \left|\frac{dy}{dt}\right| = \frac{5}{\sqrt{119}} \text{ ft/sec}$$

Example: A large spherical balloon is inflated so that its volume is increasing at the rate of 3 cubic feet per minute.

- How fast is the radius of the balloon increasing at the instant when the diameter of the balloon is 1 foot?
- How fast is the surface area of the balloon increasing at the instant when the diameter is 1 foot?



V = volume of sphere

$$\frac{dV}{dt} = 3 \text{ ft}^3/\text{min.}$$

Find $\frac{dr}{dt}$ when $r = \frac{1}{2}$ ft.

We know $V = \frac{4}{3}\pi r^3$. Diff wrt t.

$$\underbrace{\frac{dV}{dt}}_{3} = 4\pi \underbrace{r^2}_{\frac{1}{2}} \underbrace{\frac{dr}{dt}}_{\text{Find}}$$

$$3 = \pi \frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{3}{\pi} \text{ ft/min.}}$$

$\boxed{\text{when } r = \tfrac{1}{2} \text{ ft.}}$

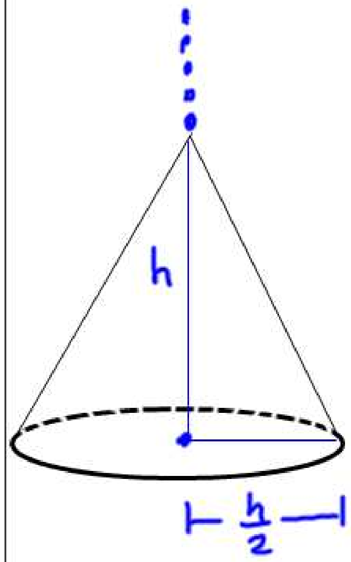
Now, find $\frac{dS}{dt}$ when $r = \frac{1}{2}$ ft.

<u>Note:</u> $S = 4\pi r^2$. Diff wrt t,

$$\underbrace{\frac{dS}{dt}}_{\text{Find}} = 8\pi \underbrace{r}_{\frac{1}{2}} \underbrace{\frac{dr}{dt}}_{\frac{3}{\pi}}$$

$$\Rightarrow \frac{dS}{dt} = 12 \text{ ft}^2/\text{min.}$$

Example: Sand is falling onto a conical pile so that the radius of the base of the pile is always equal to one half if its altitude. If the sand is falling at a rate of 6 cubic feet per minute, how fast is the altitude of the pile increasing when the pile is 9 feet deep?



$$\frac{dV}{dt} = 6 \text{ ft}^3/\text{min}$$

Find $\frac{dh}{dt}$ when $h = 9$ ft.

Note: $V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$$\Rightarrow V = \frac{1}{12}\pi h^3$$

Diff wrt t. $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ ← find

$\frac{dV}{dt} \leftarrow 6$, $h \leftarrow 9$

Subst. and solve for $\frac{dh}{dt}$.