Info

- Test 2 starts this week.
- EMCFs have been posted for the week.
- There is no homework due *next* Monday.
- There will be an EMCF due *next* Monday.
- There will be a quiz in lab on Friday.
- There is an online review tomorrow night.

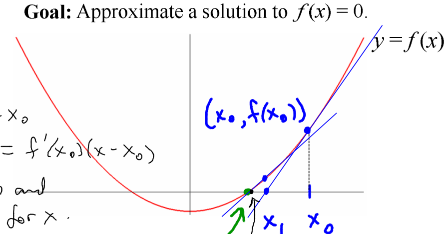
Differentials and Newton's Method

Section 3.9

(applications of tangent lines)

Newton's Method - Geometrically

* **Goal:** Approximate a solution to $f(x) = 0$.



T.L. at $x = x_0$

$y - f(x_0) = f'(x_0)(x - x_0)$

Set $y = 0$ and solve for x.

$-f(x_0) = f'(x_0)(x - x_0)$

$x = x_0 - \frac{f(x_0)}{f'(x_0)}$

x_1

True $= a$

x_2

wow!

Guess

1st Newton iterate or Newton update

Looks close.

Repeat starting at $x = x_1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ ... continue until you are satisfied

Newton's Method - Formula

Let f be a twice differentiable function and suppose a is a real number at which $f(a) = 0$. If $f'(a) \neq 0$ and x_0 is $\boxed{\text{sufficiently close}}$ to a, then the iteration

guess

True solution

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge (rapidly) to the root a.

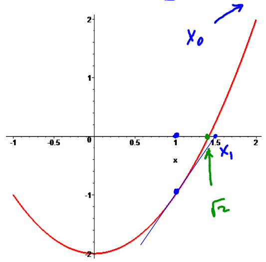
Very important!!

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\vdots$

Example: Use 2 iterations of Newton's method to approximate $\sqrt{2}$, starting from a guess of 1. Hint: Use the function $f(x) = x^2 - 2$.



$x^2 - 2 = 0.$

Note: $\sqrt{2}$ solves this.

$f(x) = x^2 - 2$

$f'(x) = 2x$

$x_0 = 1$

$f(1) = -1$

$f'(1) = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1.5$$

$f(\tfrac{3}{2}) = \tfrac{1}{4}$

$f'(\tfrac{3}{2}) = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{1/4}{3} = \frac{17}{12} = 1.41\overline{6}$$

close to $\sqrt{2}$

i	xi
0	1
1	1.5
2	1.416666667
3	1.414215686
4	1.414213562
5	1.414213562
6	1.414213562
7	1.414213562
8	1.414213562
9	1.414213562
10	1.414213562

This is the result of performing multiple Newton iterations in Excel. Notice that after the 4th Newton iterate, the values stay the same (in terms of the decimal places that are displayed).

Popper P10

1. Use one iteration of Newton's method from a guess of $x_0 = 2$ to approximate a positive solution to $x^2 - 2 = 0$.

2. What is the positive solution to the equation in problem 1?

3. Use one iteration of Newton's method from a guess of $x_0 = 2$ to approximate a solution to $x^3 - 2x - 5 = 0$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Example: Use several iterations of Newton's method from a guess of $x = 1$ to approximate a solution to $x^4 + x - 1 = 0$.

$f(x) = x^4 + x - 1$

$f'(x) = 4x^3 + 1$ $\quad x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{5} = .8$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} =$$

$x_3 =$

etc...

i	xi
0	1
1	0.8
2	0.731233596
3	0.72454848
4	0.724491963
5	0.724491959
6	0.724491959
7	0.724491959
8	0.724491959
9	0.724491959
10	0.724491959

Notice that the Newton iterates do not change after the 5th one. The convergence is very fast.

Example: Use Newton's Method to develop an algorithm for giving excellent approximations to square roots.

See Video

Definition: The differential of f at a with increment h is given by

$$* \quad df = f'(a)\,h$$

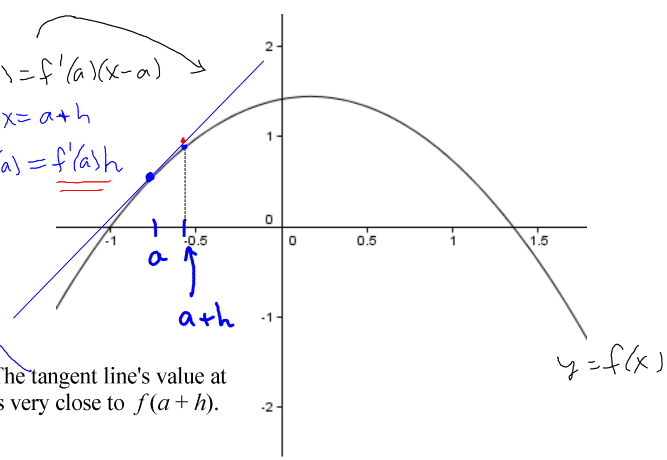
Approx of $f(a+h) - f(a)$

Geometric Interpretation

$y - f(a) = f'(a)(x-a)$

at $x = a+h$

$y - f(a) = f'(a)h$



Note: The tangent line's value at $a+h$ is very close to $f(a+h)$.

Differentials Can Be Used To Approximate Function Values

The differential of f at a with increment h is given by $df = f'(a)h$

Using the approximation $df \approx f(a+h) - f(a)$, the equation above becomes

$$f(a+h) \approx f(a) + f'(a)h$$

(this is a tangent line approximation)

Example: Give the differential of $f(x)=\sqrt{x}$ at $x=25$ with increment 0.1.

$f'(x)=\frac{1}{2\sqrt{x}}$

$df = f'(25)\cdot(0.1)$

$= \frac{1}{10}\cdot\frac{1}{10} = \frac{1}{100}$

Example: Use differentials to approximate $\sqrt{25.1}$.

$f(a+h)\approx f(a)+f'(a)h$

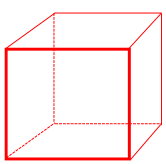
See the Video

Popper P10

4. Give the differential of $f(x)=x^2+2x-3$ at $x=1$ with increment 0.05.

$df = f'(a)h$

Example: A box is to be constructed in the form of a cube to hold 1000 cubic feet. Use a differential to estimate how accurately the edge must be made so that the volume will be correct to within 3 cubic feet.



Next Time...