Info

- **EMCFs are due every MWF morning.**
- **There is a quiz in lab Friday.**
- **There is no homework due on Monday.**
- **There will be an EMCF due on Monday.**
- **There is an online quiz due Monday.**
- **Practice Test 2 is posted.**
- **The slides and video are posted from last night's review.**
- **You should be registered for Test 2.**

more

Differentials and Newton's Method

Section 3.9

(tangent line approximation)

recall

Newton's Method - Formula

Goal: Approx a solution to $f(x) = 0$.

Let f be a twice differentiable function and $\boxed{\text{suppose } a \text{ is a real number}}$ at which $f(a) = 0$.

True solution

If $f'(a) \neq 0$ and x_0 is sufficiently close to a, then

guess

the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge (rapidly) to the root a.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$\vdots$

Example: Do one iteration of Newton's method from a guess of $\boxed{x_0 = 2}$ to approximate a solution to $x^4 + 2x - 3 = 0$. Then compute further Newton iterates using a calculator or other computing device.

$f(x) = x^4 + 2x - 3$ $\qquad f(2) = 17$

$f'(x) = 4x^3 + 2$ $\qquad f'(2) = 34$

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{17}{34} = \frac{3}{2}.$$

i	xi	
0	2.0000000000000000	$= x_0$
1	1.5000000000000000	$= x_1$
2	1.1733870967741900	$= x_2$
3	1.0265638836561600	$= x_3$
4	1.0006930733720100	$= x_4$
5	1.0000004801287200	$= x_5$
6	1.0000000000002300	$= x_6$
7	1.0000000000000000	$= x_7$
8	1.0000000000000000	$\vdots$
9	1.0000000000000000	
10	1.0000000000000000	

Example: Newton's method can go horribly wrong IF the initial guess is not sufficiently close to the actual solution. We can see this by exploring the equation $\frac{10x}{x^2+1} = 0$

See the video. An initial guess of 0.6 will cause things to go very bad!!

$$f(x) = \frac{10x}{x^2+1}$$

$$f'(x) = \frac{(x^2+1)\cdot 10 - 10\cdot 2x}{(x^2+1)^2} = \frac{-10x^2+10}{(x^2+1)^2}$$

Guess x_0 ← causes cycling.

$$x_1 = x_0 - \frac{\frac{10x_0}{x_0^2+1}}{\frac{-10x_0^2+10}{(x_0^2+1)^2}} = x_0 - \frac{\cancel{10}x_0(x_0^2+1)^{\cancel{2}}}{\cancel{(x_0^2+1)}(-\cancel{10}x_0^2+\cancel{10})}$$

$$x_1 = x_0 - \frac{x_0(x_0^2+1)}{-x_0^2+1}$$

Look for x_0 so that $x_1 = -x_0$

$$-\cancel{x_0} = \cancel{x_0} - \frac{\cancel{x_0}(x_0^2+1)}{-x_0^2+1}$$

we are not interested in $x_0 = 0$.

$$-1 = 1 - \frac{x_0^2+1}{-x_0^2+1}$$

$$-1 = 1 - \frac{x_0^2+1}{-x_0^2+1}$$

$$\frac{x_0^2+1}{-x_0^2+1} = 2 \qquad x_0^2+1 = -2x_0^2+2$$

$$3x_0^2 = 1$$

$$x_0^2 = 1/3$$

$$x_0 = \pm\sqrt{1/3}$$

Starting from either of these values will cause the iteration to oscillate back and forth. See the video.

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recall

The differential of f at a with increment h is given by $df = f'(a)h$

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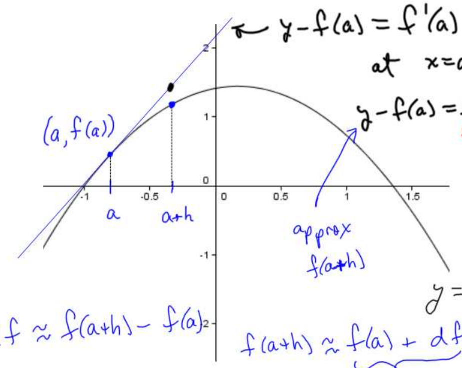
recall

Geometric Interpretation:



← $y - f(a) = f'(a)(x-a)$ at $x = a+h$

$y - f(a) = f'(a)h$

df

$$df \approx f(a+h) - f(a)$$

$$f(a+h) \approx \underbrace{f(a) + df}_{\text{T.L. approx}}$$

Differentials Can Be Used To Approximate Function Values

The differential of f at a with increment h is given by $df = f'(a)h$

Using the approximation $df \approx f(a+h) - f(a)$, the equation above becomes

$$f(a+h) \approx f(a) + f'(a)h$$

(this is a tangent line approximation)

Example: Use differentials to approximate $\sqrt{25.1}$.

$f(x) = \sqrt{x}$ $\qquad f'(x) = \frac{1}{2\sqrt{x}}$

Note: we know $f(25) = \sqrt{25} = 5$
AND this value should be close to $\sqrt{25.1}$.

Let's use differential approx to improve it.

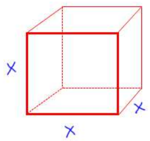
$f(\underset{a}{25} + \underset{h}{.1}) \approx f(\underset{a}{25}) + f'(\underset{a}{25}) \cdot (\underset{h}{.1})$

$\approx 5 + \frac{1}{10} \cdot \frac{1}{10} = 5.01$

wow!

$\boxed{\sqrt{25.1} = 5.0099900199501400\ldots}$

Example: A box is to be constructed in the form of a cube to hold 1000 cubic feet. Use a differential to estimate how accurately the edge must be made so that the volume will be correct to within 3 cubic feet.



$V = x^3$

For $V = 1000$, we need $x = 10$ ft

$dV = 3x^2 \cdot h$

$= 300 \cdot h$

estimate of volume error

$3 = 300h \qquad h = 0.01$

Keep side lengths "roughly" btwn

$10 - 0.01$ and $10 + 0.01$ feet