Info

- EMCFs and Homework will be posted for next week.
- Homework is posted and due on Monday.
- A Quiz will be given in lab today.
- An Online Quiz will be due on Monday at 11:59pm.
- Test 3 registration has already started.

The Second Derivative Test
for Classifying Critical Numbers

Suppose $f'(c) = 0$ and $f''(x)$ exists in an open neighborhood containing $x = c$.

If $f''(c) > 0$ then f has a local minimum at $x = c$.

If $f''(c) < 0$ then f has a local maximum at $x = c$.

If $f''(c) = 0$ then anything is possible, and the test fails to give information. *

Example: Use the second derivative test to classify the critical numbers of $f(x) = -2x^3 + 6x^2 + 18x + 2$.

← polynomial

$f'(x) = -6x^2 + 12x + 18$ ← exists everywhere

set $f'(x) = 0 \iff -6x^2 + 12x + 18 = 0$

$-6(x^2 - 2x - 3) = 0$

$(x-3)(x+1) = 0$

c.n. $x = -1$, $x = 3$

$f''(x) = -12x + 12$

$f''(-1) = 24 > 0 \Rightarrow f$ has a local min at $x = -1$

$f''(3) = -24 < 0 \Rightarrow f$ has a local max at $x = 3$

Example: $x = 1$ is a critical number of $f(x) = 2x(x-1)^3 + 3(x-1)^4$. Explain why the second derivative test cannot be used to classify this critical number. Then use the first derivative test to classify the critical number $x = 1$.

$$f'(x) = 2x \cdot 3(x-1)^2 + (x-1)^3 \cdot 2 + 12(x-1)^3$$

$$f'(x) = 6x(x-1)^2 + 14(x-1)^3$$

$$f''(x) = 6x \cdot 2(x-1) + (x-1)^2 \cdot 6 + 42(x-1)^2$$

$$= 12x(x-1) + 48(x-1)^2$$

$$f''(1) = 0 \Rightarrow \text{the test fails}$$

Let's try the first the deriv. test

$$f'(x) = 6x(x-1)^2 + 14(x-1)^3$$

$$= (x-1)^2 [6x + 14(x-1)]$$

$$= (x-1)^2 [20x - 14]$$

exists for all x

$$f'(x) = 0 \iff x = 1 \text{ and } x = \frac{14}{20} = \frac{7}{10}$$

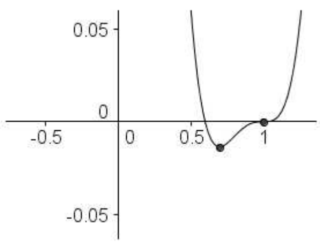
$f'(x)$		0	+ + + +		0	+ + + +	
x		$\frac{7}{10}$		$\frac{9}{10}$	1		2

$$f'\left(\tfrac{9}{10}\right) = + \qquad f'(2) = +$$

f

shape

$\therefore$ f has neither a local max nor a local min at $x = 1$.



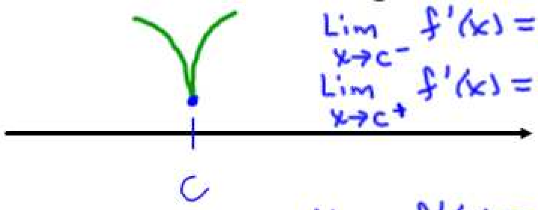
Moving towards graphing...

Identifying Cusps and Vertical Tangents:

Cusp

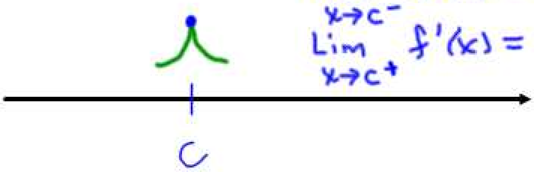
$f'(c)$ dne

$f(x) = x^{2/3}$



$\lim_{x\to c^-} f'(x) = -\infty$

$\lim_{x\to c^+} f'(x) = +\infty$



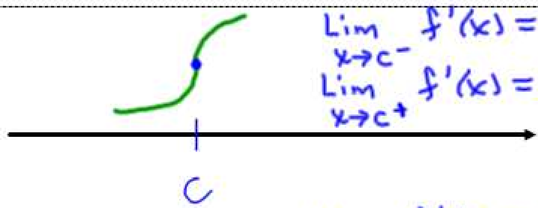
$\lim_{x\to c^-} f'(x) = +\infty$

$\lim_{x\to c^+} f'(x) = -\infty$

Vertical Tangents

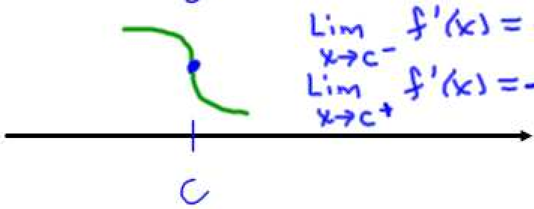
$f'(c)$ dne

$f(x) = x^{1/3}$



$\lim_{x\to c^-} f'(x) = +\infty$

$\lim_{x\to c^+} f'(x) = +\infty$



$\lim_{x\to c^-} f'(x) = -\infty$

$\lim_{x\to c^+} f'(x) = -\infty$

Determine whether or not the graph of f has a vertical tangent or a vertical cusp at c.

21. $f(x) = (x+3)^{4/3}$; $c = -3$.

$$f'(x) = \frac{4}{3}(x+3)^{1/3}$$

<u>check:</u> $f'(-3) = 0$ $\therefore$ neither occurs.

28. $f(x) = 4 - (2-x)^{3/7}$; $c = 2$.

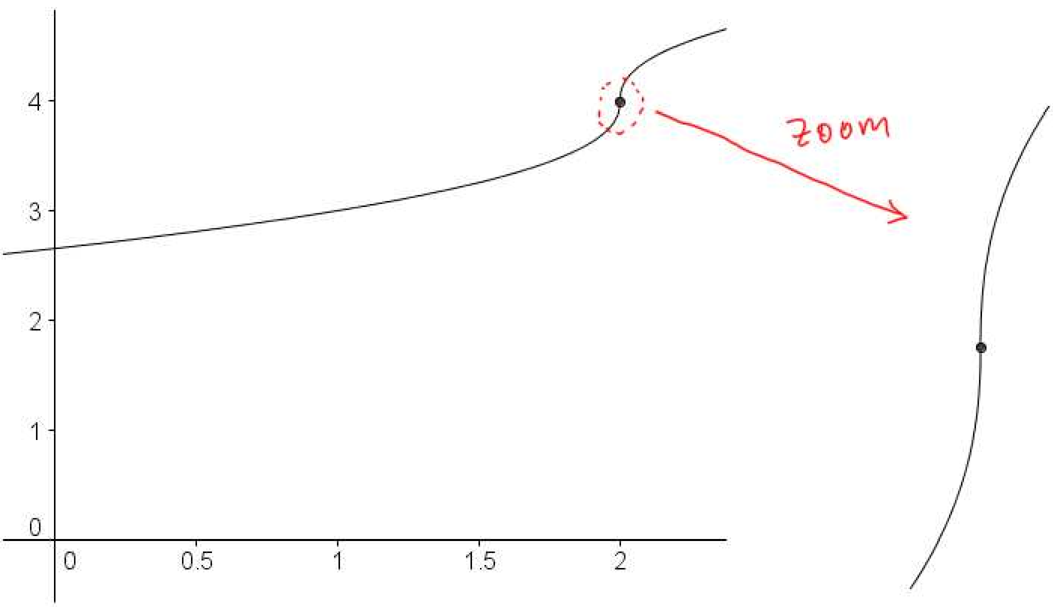
$$f'(x) = -\frac{3}{7}(2-x)^{-4/7}(-1) = \frac{3}{7} \cdot \frac{1}{(2-x)^{4/7}}$$

<u>check:</u> $f'(2)$ dne ✓

$\lim\limits_{x \to 2^-} f'(x) = +\infty$

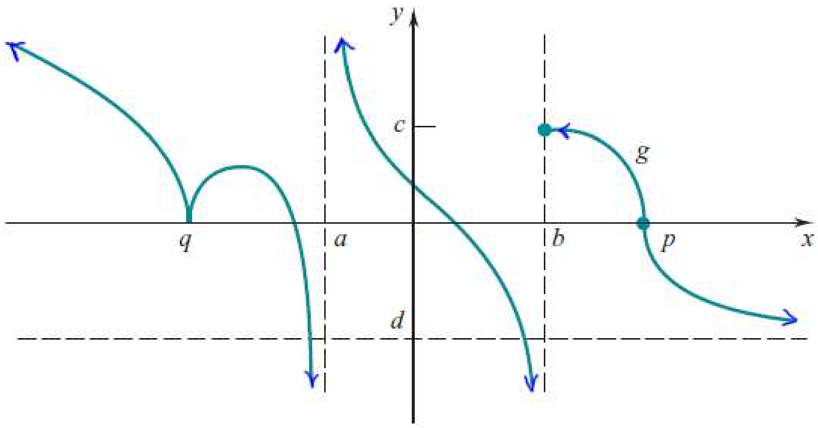
$\lim\limits_{x \to 2^+} f'(x) = +\infty$

f has a vertical tangent at $\underline{\underline{x=2}}$



Book Exercise:

2. The graph of a function g is given in the figure.



(a) As $x \to \infty, g(x) \to ?$ d (b) As $x \to b^+, g(x) \to ?$ c

(c) Give the equations of the vertical asymptotes, if any. $x = a$, $x = b$

(d) Give the equations of the horizontal asymptotes, if any. $y = d$

(e) Give the numbers c, if any, at which the graph of g has a vertical tangent line. $x = p$

(f) Give the numbers c, if any, at which the graph of g has a vertical cusp. $x = q$

Moving towards graphing...

Identifying Horizontal and Vertical Asymptotes:

??

A horizontal line that the graph "behaves like" at $x \to -\infty$ or $x \to +\infty$.

i.e. $y = c$ is a horizontal asymptote for f iff

either $\lim_{x \to -\infty} f(x) = c$

or $\lim_{x \to \infty} f(x) = c$

Example: Find the horizontal and vertical asymptotes for

$$f(x) = \frac{8-2x^2}{x^2-2x-8}$$

$$= \frac{2(4-x^2)}{(x-4)(x+2)} \qquad x \neq 4, \; x \neq -2$$

$$= \frac{2(2-x)\cancel{(2+x)}}{(x-4)\cancel{(x+2)}}$$

vertical asymptote at $x=4$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{8-2x^2}{x^2-2x-8}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^2}\left(\frac{8}{x^2}-2\right)}{\cancel{x^2}\left(1-\frac{2}{x}-\frac{8}{x^2}\right)}$$

$$= \lim_{x\to-\infty} \frac{\frac{8}{x^2}-2}{1-\frac{2}{x}-\frac{8}{x^2}} = -2$$

$\frac{8}{x^2} \to 0$, $\frac{2}{x} \to 0$, $\frac{8}{x^2} \to 0$

$\therefore$ $y=-2$ is a horizontal asymptote for f.

Similarly, $\lim_{x\to\infty} f(x) = \cdots = 2$