Info

- Homework and EMCFs are posted.
- The quiz in lab/workshop on Friday will cover area and u-substitution.
- Visit the Alpha Lambda Delta Bake Sale in the PGH Breezeway starting at 10am.

Review

Examples: $\int_{-1}^{2} \frac{x}{\sqrt{3x^2+2}}\,dx = \frac{1}{6}\int_{-1}^{2} (\underbrace{3x^2+2}_{u})^{-\frac{1}{2}} \underbrace{6x\,dx}_{du}$

$u = 3x^2+2$

$du = 6x\,dx$

$x = 2 \Rightarrow u = 14$

$x = -1 \Rightarrow u = 5$

$= \frac{1}{6}\int_{5}^{14} u^{-\frac{1}{2}}\,du$

$= \frac{1}{6}\, 2u^{1/2} \Big|_{5}^{14}$

$= \frac{1}{3}\left(\sqrt{14} - \sqrt{5}\right).$

Example: $\int_0^{\pi/4} \underbrace{\sin(2x)}_{\text{nearly } du}\big(\underbrace{\cos(2x)+2}_{u}\big)^3\,dx =$

$= -\frac{1}{2}\int_0^{\pi/4} (\underbrace{\cos(2x)+2}_{u})^3 \underbrace{(-2)\sin(2x)\,dx}_{du}$

$u = \cos(2x)+2$

$du = -2\sin(2x)\,dx$

$x = \pi/4 \Rightarrow u = 2$

$x = 0 \Rightarrow u = 3$

$= -\frac{1}{2}\int_{3}^{2} u^3\,du$

$= -\frac{1}{2}\cdot\frac{1}{4}u^4 \Big|_{3}^{2}$

$= -\frac{1}{8}(16 - 81) = \frac{65}{8}.$

Example: Suppose f is a continuous function and $\int_{-1}^{3} f(x)dx = 5$ and $\int_{-1}^{2} f(x)dx = 6$. Give the value for $\int_{2}^{3} f(x)dx$.

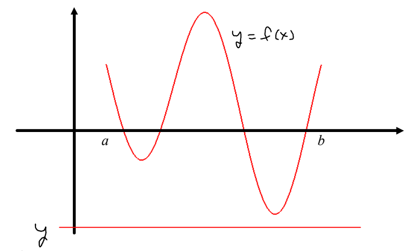
$\int_{-1}^{3} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$

$5 = 6 + \int_{2}^{3} f(x)\,dx$

$\Rightarrow \int_{2}^{3} f(x)\,dx = -1.$

Average Value of a Function

Question: The graph of *f* is shown below. Is there a value of *y* so that $y(b - a)$ is smaller than the net area bounded by the graph of *f* over the interval $[a,b]$?



$$\int_a^b C\,dx = Cx\Big|_a^b = C(b-a)$$

Need a value y so that

$$y(b-a) < \int_a^b f(x)\,dx$$

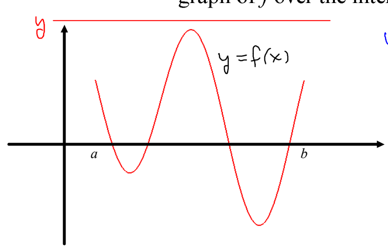
Recall: If $g(x) < f(x)$ (constant)

then

$$\int_a^b g(x)\,dx < \int_a^b f(x)\,dx$$

Yes

Question: The graph of *f* is shown below. Is there a value of *y* so that $y(b - a)$ is larger than the net area bounded by the graph of *f* over the interval $[a,b]$?



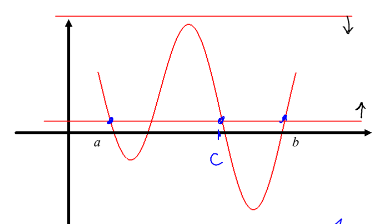
$$\int_a^b C\,dx = Cx\Big|_a^b = C(b-a)$$

we need a value y so that

$$y(b-a) > \int_a^b f(x)\,dx$$

Yes

Question: The graph of *f* is shown below. Is there a value of *y* so that $y(b - a)$ is equal to the net area bounded by the graph of *f* over the interval $[a,b]$?



Yes. In fact, if f is continuous on $[a,b]$ then there is at least one value c so that $a < c < b$ and

$$\boxed{f(c)(b-a) = \int_a^b f(x)\,dx}$$

Theorem: (The mean value theorem for integrals.) Suppose f is a continuous function on the interval $[a,b]$. Then there is a value c so that $a < c < b$, and

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note: The value $f(c)$ is called the average value (or mean value) of the function f over the interval $[a,b]$.

Pf. Define $F(x) = \int_a^x f(t)\,dt$. The MVThm (for derivatives) implies there is a value c so that $a<c<b$ and

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$

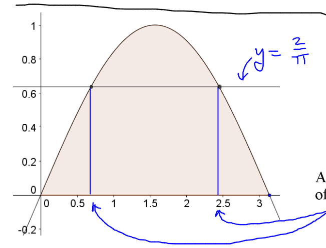
Note: Fund. Thm Calc $\Rightarrow F'(x) = f(x)$

$$\Rightarrow f(c) = \frac{1}{b-a}\left(\int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0}\right)$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx \quad \#$$

Example: Give the average value of $f(x) = \sin(x)$ on the interval $[0,\pi]$.

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\left(-\cos(x)\right)\Big|_0^{\pi}$$



$$= \frac{1}{\pi}\left(-(-1-1)\right)$$

$$= \frac{2}{\pi} \leftarrow \text{average value of } f(x) \text{ on } [0,\pi].$$

Although we were not asked, we can see that there are 2 values of c where the function $f(x)$ takes its average value.

Example: Give the average value of $f(x) = 3x^2 - x$ on the interval $[-1,3]$, and determine the value c where f has this value on the interval $(-1,3)$.

$$\boxed{\text{Average value of } f(x) \text{ on } [-1,3]} = \frac{1}{3-(-1)}\int_{-1}^{3} f(x)\,dx$$

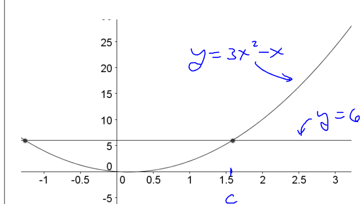
$$= \frac{1}{4}\int_{-1}^{3}(3x^2 - x)\,dx$$

$$= \frac{1}{4}\left(x^3 - \frac{1}{2}x^2\right)\Big|_{-1}^{3}$$

$$= \frac{1}{4}\left(\left(27 - \frac{9}{2}\right) - \left(-1 - \frac{1}{2}\right)\right)$$

$$= \frac{1}{4}\left(28 - \frac{8}{2}\right) = 6$$

$$\boxed{\text{The average value of } f(x) = 3x^2 - x \text{ on } [-1,3]} = 6$$



Now find c so that $\boxed{-1 < c < 3}$ and $f(c) = 6$.

Solve

$$3c^2 - c = 6$$

$$3c^2 - c - 6 = 0$$

$$c = \frac{1 \pm \sqrt{1+72}}{6}$$

i.e. $\cancel{c = \frac{1-\sqrt{72}}{6}}$ or $\boxed{c = \frac{1+\sqrt{72}}{6}}$

Note: $-1 < \frac{1+\sqrt{72}}{6} < 3$.