Test 4: (5.1 - 6.3)

We will review today and Monday.

Topics

1. Anti-derivatives.
2. Riemann sums (upper sums, lower sums).
3. Properties of the integral.
4. The fundamental theorem of calculus.
5. Average value and the mean value theorem for integrals.
6. u-substitution.
7. Area.
8. x and y integrations.
9. Volumes of revolution (disks, washers and shells).

Next Tuesday: Review online

8:30–10:30 pm

Examples:

$$\int \sin(2x)dx = -\frac{1}{2}\cos(2x) + C$$

$$\int \cos(3x)dx = \frac{1}{3}\sin(3x) + C$$

$$\int \sec^2(2x)dx = \frac{1}{2}\tan(2x) + C$$

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$u = 2x+1$

$du = 2dx$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$$\frac{1}{6}\int \frac{6x}{\sqrt{3x^2+2}}dx = \frac{1}{6}\int u^{-1/2}\,du$$

$u = 3x^2+2$

$du = 6x\,dx$

$$= \frac{1}{6}\cdot 2u^{1/2} + C$$

$$= \frac{1}{3}\sqrt{3x^2+2} + C$$

Examples: $\int_{-1}^{2} \frac{3x}{(x^2+1)^2}\,dx = \frac{3}{2}\int_{\boxed{2}}^{\boxed{5}} u^{-2}\,du = \frac{3}{2}(-1)u^{-1}\Big|_2^5$

$u = x^2+1$ $\quad x=2 \Rightarrow u=5$

$du = 2x\,dx$ $\quad x=-1 \Rightarrow u=2$

$= -\frac{3}{2}\left[\frac{1}{5} - \frac{1}{2}\right]$

P33

1. $9/20$

Give the average value of $f(x) = 2x^2 - 1$ on the interval $[-1,2]$, and verify the conclusion of the mean value theorem for integrals for this function on this interval.

Av. Value $= \frac{1}{2-1}\int_{-1}^{2}(2x^2-1)\,dx = \frac{1}{3}\left[\frac{2}{3}x^3 - x\right]\Big|_{-1}^{2}$

$= \frac{1}{3}\left[\left(\frac{16}{3} - 2\right) - \left(\frac{-2}{3} + 1\right)\right] = \frac{1}{3}(6-3) = 1$

Find c so that $-1 < c < 2$ and

$f(c) = 1$

$2c^2 - 1 = 1$

$2c^2 = 2$

$c^2 = 1$

$\cancel{c = -1}$ $\quad c = 1$

Note: $-1 < 1 < 2$.

$\frac{d}{dx}\int_{1}^{2x+3} \cos\left(\sqrt{t}+3\right)dt =$

$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$

$= \cos\left(\sqrt{2x+3} + 3\right)\cdot 2$

$= 2\cos\left(\sqrt{2x+3} + 3\right)$.

Examples:

$$\frac{d}{dx}\int_{x^2}^{2x+3}\cos\left(\sqrt{t}+3\right)dt = \frac{d}{dx}\int_{x^2}^{1}\cos(\sqrt{t}+3)\,dt + \frac{d}{dx}\int_{1}^{2x+3}\cos(\sqrt{t}+3)\,dt$$

$$= -\frac{d}{dx}\int_{1}^{x^2}\cos(\sqrt{t}+3)\,dt + 2\cos(\sqrt{2x+3}+3)$$

$$= -2x\cos(\sqrt{x^2}+3) + 2\cos(\sqrt{2x+3}+3)$$

$\sqrt{x^2} \stackrel{?}{=} x$

$\sqrt{(-3)^2} = 3$

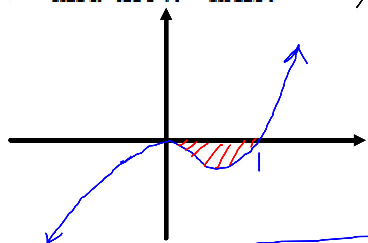
$\sqrt{x^2} = |x|$

Give the area bounded between the graph of $y = x^3 - x^2$ and the x-axis.

$x^3 - x^2 = 0$ $\quad x^2(x-1) = 0, \quad x=0, x=1$

P33

2.



$$\text{Area} = -\int_0^1 (x^3 - x^2)\,dx$$

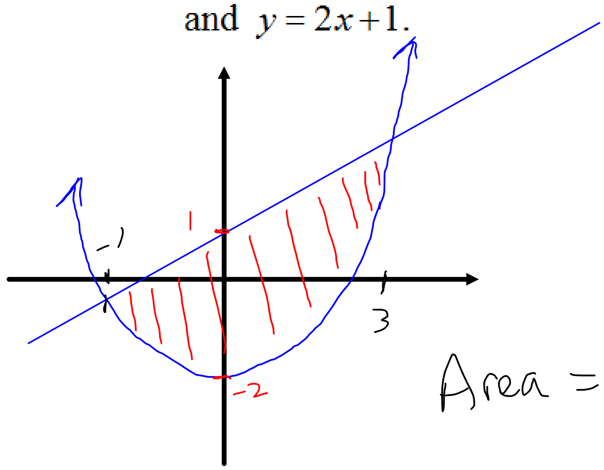
Give the area bounded between the graphs of $y = x^2 - 2$ and $y = 2x+1$.

$x^2 - 2 = 2x + 1$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, \quad x = -1.$



$$\text{Area} = \int_{-1}^{3} (\text{Top} - \text{Bottom})\,dx$$

$$= \int_{-1}^{3} \left(2x+1 - (x^2-2)\right)dx$$

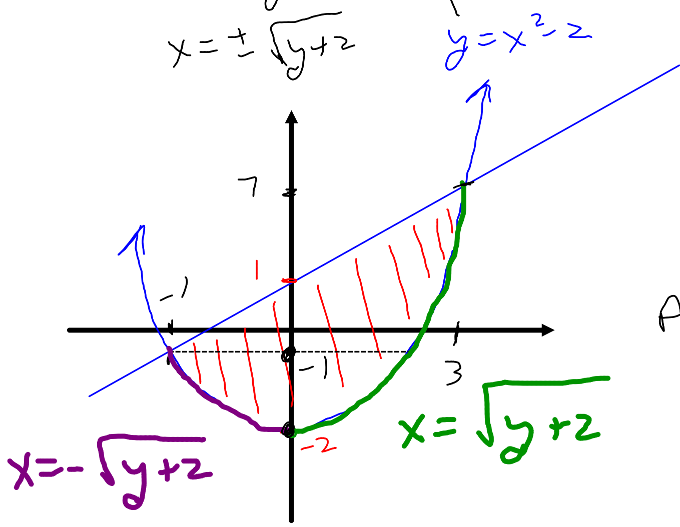
$$= \int_{-1}^{3} (2x + 3 - x^2)\,dx$$

$=$ you ...

Example: Give a formula involving integrals with respect to y, for the area bounded between the graphs of $y = x^2 - 2$ and $y = 2x + 1$.

$x^2 = y + 2$

$x = \pm\sqrt{y+2}$



$y = 2x + 1$

$y - 1 = 2x$

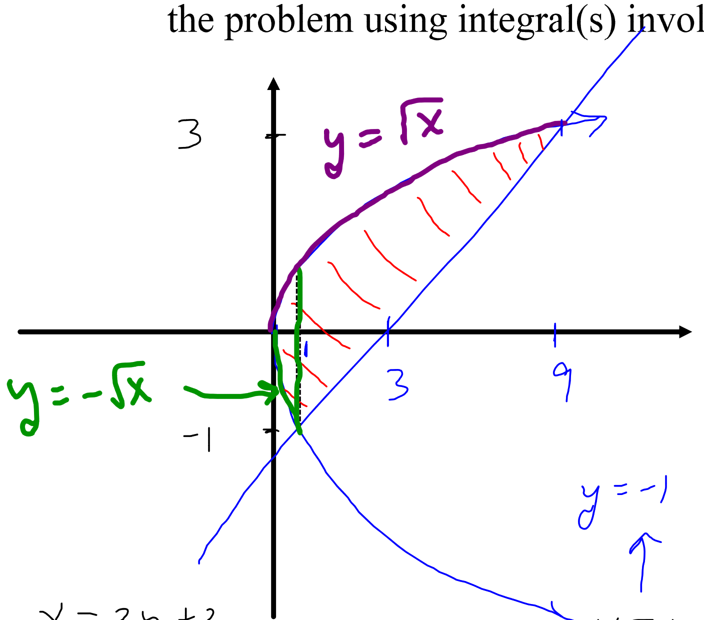
$x = \frac{1}{2}(y-1)$

$x = -1 \Rightarrow y = -1$

$x = 3 \Rightarrow y = 7$

$$\text{Area} = \int_{-2}^{-1} \left(\sqrt{y+2} - -\sqrt{y+2}\right) dy + \int_{-1}^{7} \left(\sqrt{y+2} - \tfrac{1}{2}(y-1)\right) dy$$

Example: Give a formula for the area of the region bounded by the curves $x = y^2$ and $x = 2y + 3$ using integral(s) involving y. Then repeat the problem using integral(s) involving x.



$y^2 = 2y + 3$

$y^2 - 2y - 3 = 0$

$\vdots$

$y = -1, \; y = 3$

$y = -1 \Rightarrow x = 1 \qquad y = 3 \Rightarrow x = 9$

$x = y^2 \qquad y = \pm\sqrt{x}$

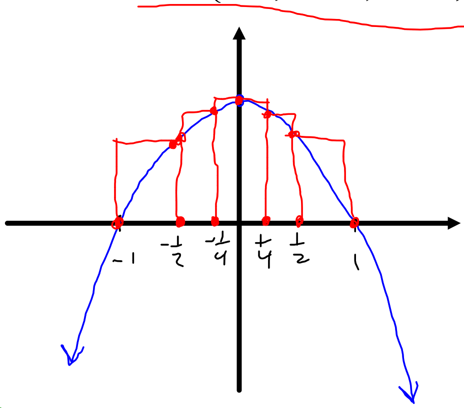
$x = 2y + 3$

$y = \frac{1}{2}(x - 3)$

y integral $\longrightarrow$ $\text{Area} = \int_{-1}^{3} \left(2y + 3 - y^2\right) dy$

$$\text{Area} = \int_0^1 \left(\sqrt{x} - -\sqrt{x}\right) dx + \int_1^9 \left(\sqrt{x} - \frac{1}{2}(x-3)\right) dx$$

Example: Give both the upper and lower Riemann sums for the function $f(x) = 1 - x^2$ on the interval [-1,1], with respect to the partition $P = \{ -1, -1/2, -1/4, 1/4, 1/2, 1 \}$.



Upper RS

$f(-\tfrac{1}{2})\cdot\tfrac{1}{2} + f(-\tfrac{1}{4})\cdot\tfrac{1}{4} + f(0)\cdot\tfrac{1}{2}$

$+ f(\tfrac{1}{4})\cdot\tfrac{1}{4} + f(\tfrac{1}{2})\cdot\tfrac{1}{2}$

= you do it.

P34

1. Give the lower Riemann sum.

$f(-1)\cdot\tfrac{1}{2} + f(-\tfrac{1}{2})\cdot\tfrac{1}{4} + f(-\tfrac{1}{4})\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{4} + f(1)\cdot\tfrac{1}{2}$

Examples: Give an anti-derivative $F(x)$ for the function $f(x) = \sin(\pi x) + 1$, satisfying $F(1) = 3$.

$$F(x) = -\frac{1}{\pi}\cos(\pi x) + x + C$$

$$3 = -\frac{1}{\pi}\cos(\pi) + 1 + C$$

$$\Rightarrow C = 2 - \frac{1}{\pi} \quad \Rightarrow \boxed{F(x) = -\frac{1}{\pi}\cos(\pi x) + x + 2 - \frac{1}{\pi}}$$

Suppose $G''(x) = \sin(\pi x) + 1$, $G(0) = 2$, and $G'(0) = 3$. Give $G(x)$.

$$G'(x) = -\frac{1}{\pi}\cos(\pi x) + x + C_1$$

$$3 = -\frac{1}{\pi}\cos(0) + 0 + C_1$$

$$\Rightarrow C_1 = 3 + \frac{1}{\pi}$$

$$\Rightarrow G'(x) = -\frac{1}{\pi}\cos(\pi x) + x + 3 + \frac{1}{\pi}$$

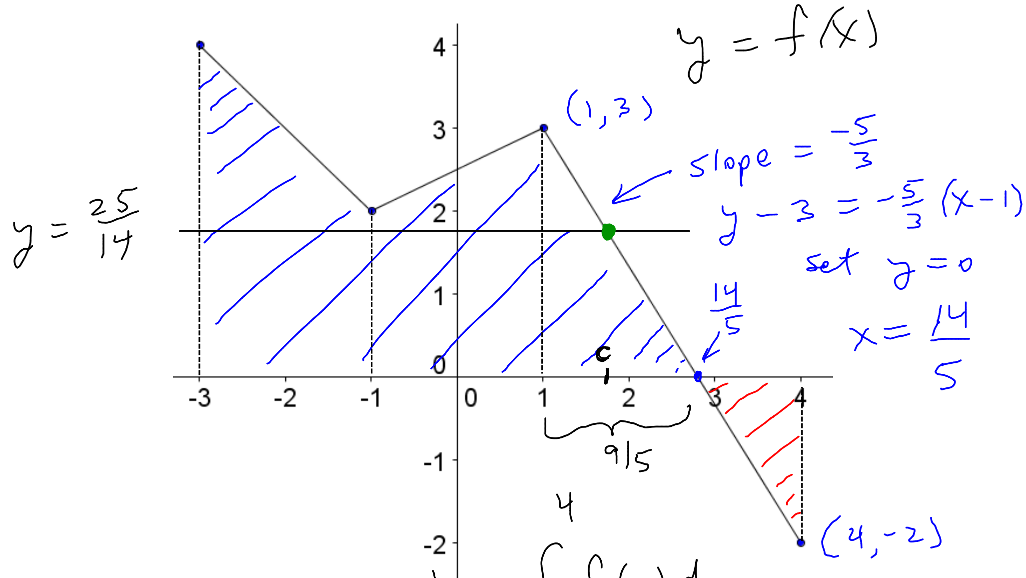
$$G(x) = \frac{-1}{\pi^2}\sin(\pi x) + \frac{x^2}{2} + \left(3 + \frac{1}{\pi}\right)x + C_2$$

$$2 = 0 + 0 + 0 + C_2$$

$$\Rightarrow C_2 = 2$$

$$\therefore \quad G(x) = \frac{-1}{\pi^2}\sin(\pi x) + \frac{x^2}{2} + \left(3 + \frac{1}{\pi}\right)x + 2$$

Example: Give the average value of the function shown below on the interval $[-3,4]$, and determine the number of values that satisfy the mean value theorem for integrals on this interval.



$$\text{Average Value} = \frac{1}{4--3}\int_{-3}^{4} f(x)\,dx$$

$$= \frac{1}{7}\left(\text{Area}(\text{blue}) - \text{Area}(\text{red})\right)$$

$$= \frac{1}{7}\left(\frac{1}{2}(4+2)\cdot 2 + \frac{1}{2}(2+3)\cdot 2 + \frac{1}{2}\cdot 3\cdot\frac{9}{5} - \frac{1}{2}\cdot 2\cdot\frac{6}{5}\right)$$

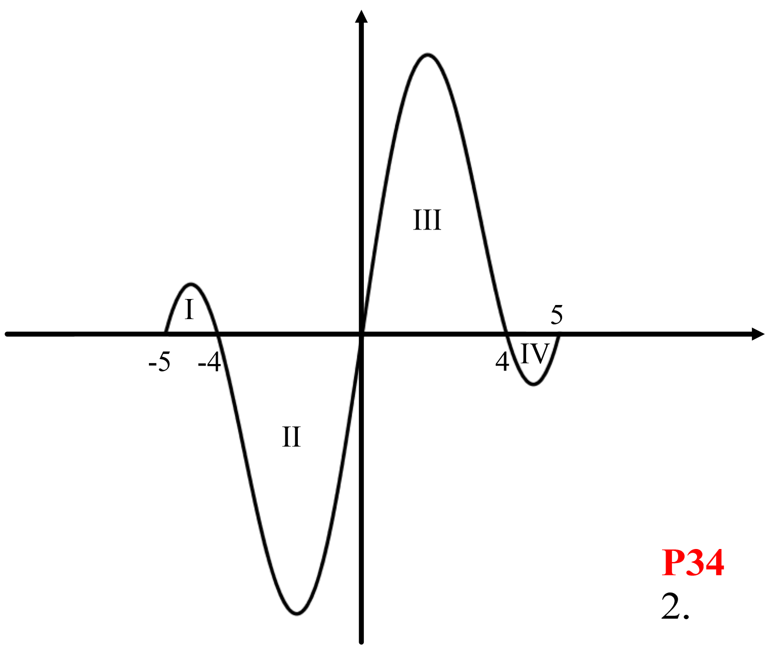
$$= \frac{1}{14}\left(12 + 10 + 3\right) = \frac{25}{14}$$

Average value

Note: $y = \frac{25}{14}$ intersects the graph 1 time on $[-3,4]$.

$\therefore$ One value satisfies the MVThm for integrals.

Example: The function f is graphed below. The area of region I is 1, the area of region II is 4, the area of region III is 4, and the area of region IV is 1.



$$\int_{-5}^{-4} f(x)dx = \text{Area}(\mathrm{I}) = 1$$

$$\int_{-5}^{0} f(x)dx = \text{Area}(\mathrm{I}) - \text{Area}(\mathrm{II})$$

$$= 1 - 4 = \underline{\underline{-3}}$$

$$\int_{-5}^{4} f(x)dx = \text{Area}(\mathrm{I}) - \text{Area}(\mathrm{II}) + \text{Area}(\mathrm{III})$$

$$= 1 - 4 + 4 = 1$$

P34

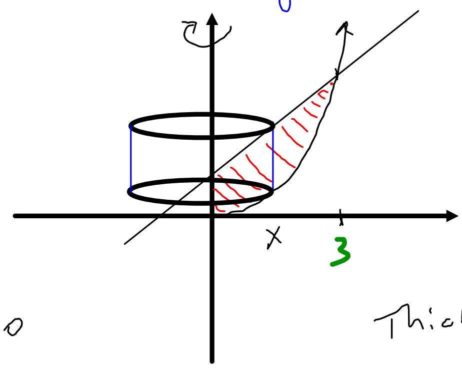
2. $\int_{0}^{5} f(x)dx =$

Example: A region in the first quadrant of the xy plane is bounded by the curves $y = x^2$ and $y = 2x + 3$. Rotate this region about the y-axis.

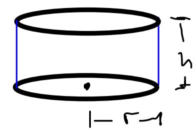
a) Give a formula involving integral(s) in x for the resulting volume. dx vertical line segments

b) Give a formula involving integral(s) in y for the resulting volume. dy horizontal line segments

a)



shells



$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\cancel{x=-1}$, $x = 3$

Thickness $= dx$

$x \quad 2x+3-x^2$

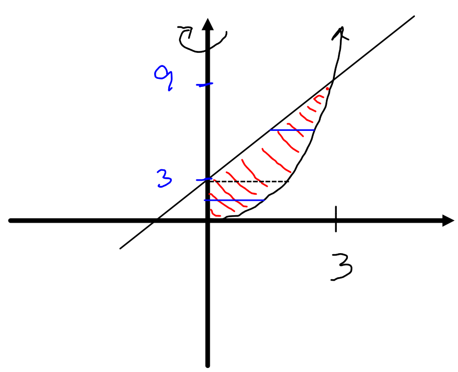
Surface Area (shell) $= 2\pi r h$

$= 2\pi x(2x+3-x^2)$

Inf Volume (shell) $= 2\pi x(2x+3-x^2)\,dx$

Full Volume $= \int_0^3 2\pi x(2x+3-x^2)\,dx$

b)



$0 \le y \le 3$

$3 \le y \le 9$

See the video