Test 4: (5.1 - 6.3)

We will review for the next 2 days.

There will also be an online live review with time and date posted on the course homepage.

Topics

1. Anti-derivatives.
2. Riemann sums (upper sums, lower sums).
3. Properties of the integral.
4. The fundamental theorem of calculus.
5. Average value and the mean value theorem for integrals.
6. u-substitution.
7. Area.
8. x and y integrations.
9. Volumes of revolution (disks, washers and shells).

Examples: $\int \sin(2x)dx = -\frac{1}{2}\cos(2x) + C$

$$\int \cos(3x)dx = \frac{1}{3}\sin(3x) + C$$

$$\int \sec^2(2x)dx = \frac{1}{2}\tan(2x) + C$$

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{2x+1}\;2dx = \frac{1}{2}\int \sqrt{u}\,du$$

$u = 2x+1$

$du = 2dx$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

$$\frac{1}{6}\int \frac{6x}{\sqrt{3x^2+2}}dx = \frac{1}{6}\int u^{-\frac{1}{2}}\,du$$

$u = 3x^2+2$

$du = 6x\,dx$

$$= \frac{1}{6}\cdot 2u^{1/2} + C$$

$$= \frac{1}{3}\sqrt{3x^2+2} + C$$

Examples: $\displaystyle\int_{-1}^{2}\frac{3x}{(x^2+1)^2}dx = 3\cdot\frac{1}{2}\int_{-1}^{2}\frac{2x}{(x^2+1)^2}dx = \frac{3}{2}\int_{\boxed{2}}^{\boxed{5}}u^{-2}du$

$u = x^2+1$ $\quad x=2 \Rightarrow u=5$

$du = 2x\,dx$ $\quad x=-1 \Rightarrow u=2$

$= \frac{3}{2}(-1)\frac{1}{u}\Big|_2^5 = -\frac{3}{2}\left[\frac{1}{5}-\frac{1}{2}\right]$

$=$ you.

Give the average value of $f(x)=2x^2-1$ on the interval $[-1,2]$, and verify the conclusion of the mean value theorem for integrals for this function on this interval.

$\displaystyle\frac{1}{2-^{-}1}\int_{-1}^{2}(2x^2-1)dx = \frac{1}{3}\left(\frac{2}{3}x^3-x\right)\Big|_{-1}^{2} = \frac{1}{3}\left[\left(\frac{16}{3}-2\right)-\left(-\frac{2}{3}+1\right)\right]$

$= \frac{1}{3}\left[6-3\right] = 1.$ $\therefore$ The average value is 1.

Find a value c so that $\boxed{-1<c<2}$ and $f(c)=1$.

$2c^2-1=1 \Rightarrow c^2=1$

$\cancel{c=-1}$ or $c=1.$

Note: $-1<1<2.$

$\displaystyle\frac{d}{dx}\int_{1}^{2x+3}\cos\left(\sqrt{t}+3\right)dt =$

$\displaystyle\frac{d}{dx}\int_a^{u(x)}f(t)dt = f(u(x))\,u'(x)$

$= \cos\left(\sqrt{2x+3}+3\right)\cdot 2$

$= 2\cos\left(\sqrt{2x+3}+3\right)$

Examples: $\dfrac{d}{dx}\displaystyle\int_{x^2}^{2x+3}\cos\left(\sqrt{t}+3\right)dt = \frac{d}{dx}\int_{x^2}^{1}\cos(\sqrt{t}+3)\,dt + \frac{d}{dx}\int_{1}^{2x+3}\cos(\sqrt{t}+3)\,dt$

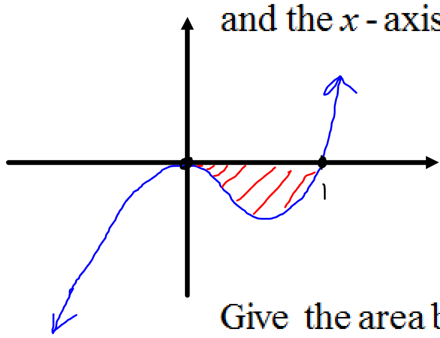
$$= -\frac{d}{dx}\int_{1}^{x^2}\cos(\sqrt{t}+3)\,dt + 2\cos\left(\sqrt{2x+3}+3\right)$$

$$= -2x\cos\left(\sqrt{x^2}+3\right) + 2\cos\left(\sqrt{2x+3}+3\right)$$

Note: $\sqrt{x^2} = |x|$

Give the area bounded between the graph of $y = x^3 - x^2$ and the x-axis.

$x^3 - x^2 = 0 \qquad x^2(x-1) = 0 \qquad x = 0,\ x = 1$



$$\text{Area} = -\int_0^1 (x^3 - x^2)\,dx = -\left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1$$

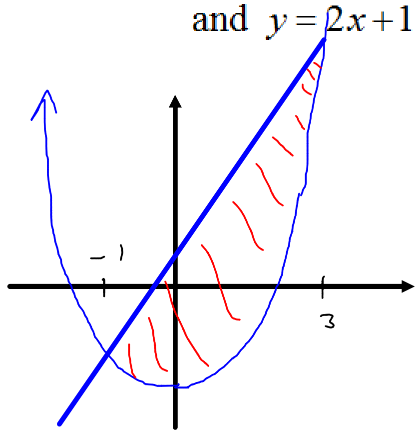
$$= -\left[\left(\frac{1}{4} - \frac{1}{3}\right) - 0\right] = \frac{1}{12}$$

Give the area bounded between the graphs of $y = x^2 - 2$ and $y = 2x + 1$.

$x^2 - 2 = 2x + 1$

$x^2 - 2x - 3 = 0 \qquad (x-3)(x+1) = 0$

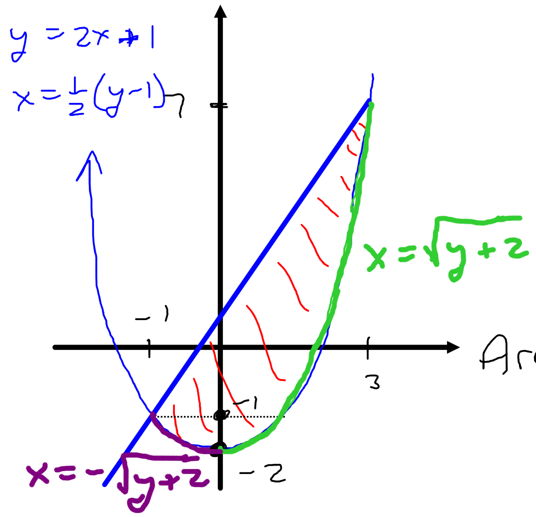
$x = -1,\ x = 3$



$$\text{Area} = \int_{-1}^{3}\left[\underbrace{(2x+1)}_{\text{Top}} - \underbrace{(x^2-2)}_{\text{Bottom}}\right]dx$$

$= \text{you.}$

Example: Give a formula involving integrals with respect to y, for the area bounded between the graphs of $y = x^2 - 2$ and $y = 2x + 1$.



$y = x^2 - 2$

$x^2 = y + 2$

$x = \pm\sqrt{y+2}$

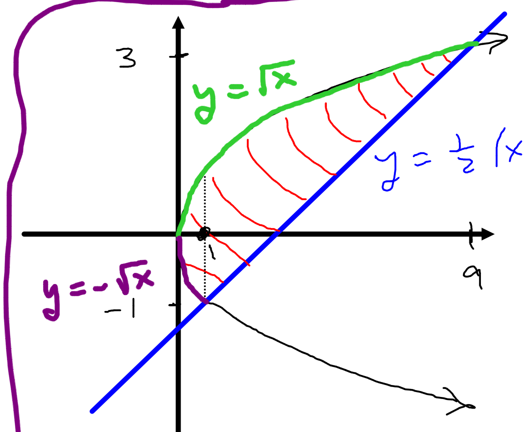
$x = -1 \Rightarrow y = -1$

$x = 3 \Rightarrow y = 7$

$$\text{Area} = \int_{-2}^{-1} \left(\sqrt{y+2} - -\sqrt{y+2}\right) dy + \int_{-1}^{7} \left(\sqrt{y+2} - \tfrac{1}{2}(y-1)\right) dy$$

$=$ you.

Example: Give a formula for the area of the region bounded by the curves $x = y^2$ and $x = 2y + 3$ using integral(s) involving y. Then repeat the problem using integral(s) involving x.



$y^2 = 2y + 3$

$y^2 - 2y - 3 = 0$

$(y+1)(y-3) = 0$

$y = -1, \; y = 3$

$$\text{Area} = \int_{-1}^{3} \left(2y + 3 - y^2\right) dy$$

$x = y^2 \Rightarrow y = \pm\sqrt{x}$

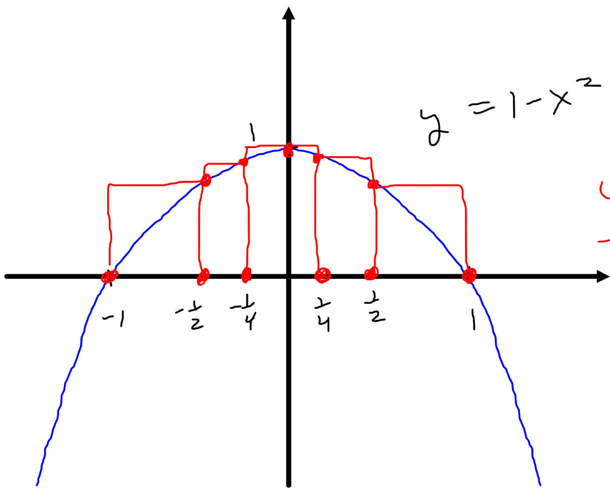
$x = 2y + 3 \Rightarrow y = \frac{1}{2}(x-3)$

$y = -1 \Rightarrow x = 1$

$y = 3 \Rightarrow x = 9$

$$\text{Area} = \int_0^1 \left(\sqrt{x} - -\sqrt{x}\right) dx + \int_1^9 \left(\sqrt{x} - \frac{1}{2}(x-3)\right) dx$$

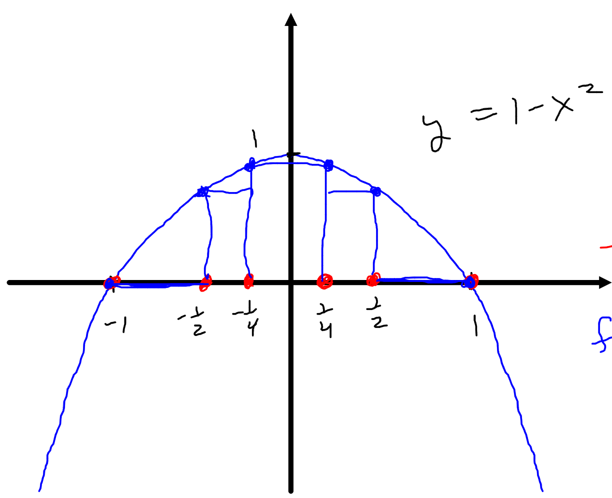
Example: Give both the upper and lower Riemann sums for the function $f(x) = 1 - x^2$ on the interval $[-1,1]$, with respect to the partition $P = \{ -1, -1/2, -1/4, 1/4, 1/2, 1 \}$.



Upper R.S.

$$f(-\tfrac{1}{2})\tfrac{1}{2} + f(-\tfrac{1}{4})\cdot\tfrac{1}{4} + f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{4})\cdot\tfrac{1}{4} + f(\tfrac{1}{2})\cdot\tfrac{1}{2}$$

$$= \frac{3}{4}\cdot\frac{1}{2} + \frac{15}{16}\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + \frac{15}{16}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{1}{2} = \cdots = \text{you}$$



Lower R.S.

$$f(-1)\cdot\tfrac{1}{2} + f(-\tfrac{1}{2})\cdot\tfrac{1}{4} + f(\tfrac{1}{4})\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{4} + f(1)\cdot\tfrac{1}{2}$$

$$= 0 + \frac{3}{4}\cdot\frac{1}{4} + \frac{15}{16}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{4} + 0 = \cdots = \text{you}.$$

Examples: Give an anti-derivative $F(x)$ for the function $f(x) = \sin(\pi x) + 1$, satisfying $F(1) = 3$.

$$F(x) = -\frac{1}{\pi}\cos(\pi x) + x + C$$

$$3 = -\frac{1}{\pi}\cos(\pi) + 1 + C$$

$$2 - \frac{1}{\pi} = C$$

$$\Rightarrow F(x) = -\frac{1}{\pi}\cos(\pi x) + x + 2 - \frac{1}{\pi}$$

Suppose $G''(x) = \sin(\pi x) + 1$, $G(0) = 2$, and $G'(0) = 3$. Give $G(x)$.

$$G'(x) = -\frac{1}{\pi}\cos(\pi x) + x + C_1$$

$$3 = -\frac{1}{\pi}\cos(0) + 0 + C_1$$

$$3 + \frac{1}{\pi} = C_1 \quad \Rightarrow G'(x) = -\frac{1}{\pi}\cos(\pi x) + x + 3 + \frac{1}{\pi}$$

$$G(x) = -\frac{1}{\pi^2}\sin(\pi x) + \frac{1}{2}x^2 + \left(3 + \frac{1}{\pi}\right)x + C_2$$

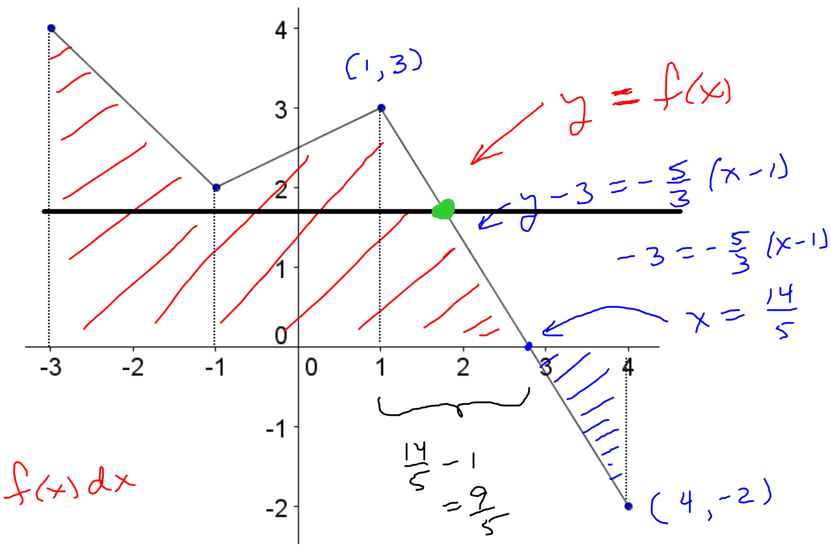
$$2 = -\frac{1}{\pi^2}\sin(0) + 0 + 0 + C_2$$

$$\Rightarrow C_2 = 2$$

$$\boxed{G(x) = -\frac{1}{\pi^2}\sin(\pi x) + \frac{1}{2}x^2 + \left(3 + \frac{1}{\pi}\right)x + 2}$$

Example: Give the average value of the function shown below on the interval [–3,4], and determine the number of values that satisfy the mean value theorem for integrals on this interval.

$y = \frac{25}{14}$



$$= \frac{1}{4 - -3}\int_{-3}^{4} f(x)\,dx$$

$$= \frac{1}{7}\left[\text{Area}(\quad) - \text{Area}(\quad)\right]$$

$$= \frac{1}{7}\left[\left(\frac{1}{2}(4+2)\cdot 2 + \frac{1}{2}(2+3)\cdot 2 + \frac{1}{2}\cdot 3\cdot\frac{9}{5}\right) - \frac{1}{2}\cdot 2\cdot\frac{6}{5}\right]$$

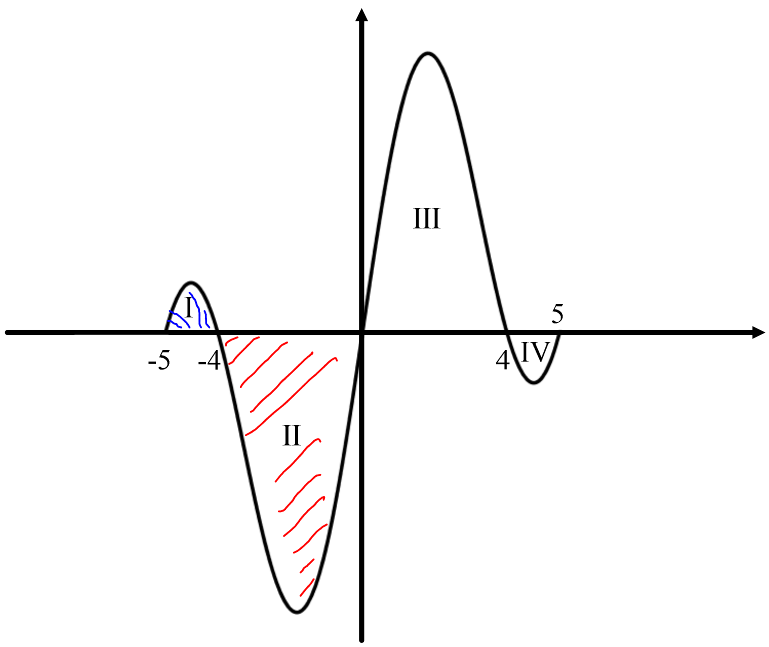
$$= \frac{1}{14}\left[12 + 10 + \frac{27}{5} - \frac{12}{5}\right] = \frac{1}{14}\left[22 + 3\right] = \boxed{\frac{25}{14}}$$

Average value.

Note: The graph of f(x) intersects $y = \frac{25}{14}$ at exactly one place on [−3, 4].

∴ The number of values that satisfy the MVThm for integrals is 1.

Example: The function f is graphed below. The area of region I is 1, the area of region II is 4, the area of region III is 4, and the area of region IV is 1.



$$\int_{-5}^{-4} f(x)dx = 1$$

$$\int_{-5}^{0} f(x)dx = \text{Area}(\text{I}) - \text{Area}(\text{II})$$

$$= 1 - 4 = -3$$

$$\int_{-5}^{4} f(x)dx = \int_{-5}^{0} f(x)dx + \int_{0}^{4} f(x)dx$$

$$= -3 + 4 = 1$$

$$\int_{0}^{5} f(x)dx = \text{Area}(\text{III}) - \text{Area}(\text{IV})$$

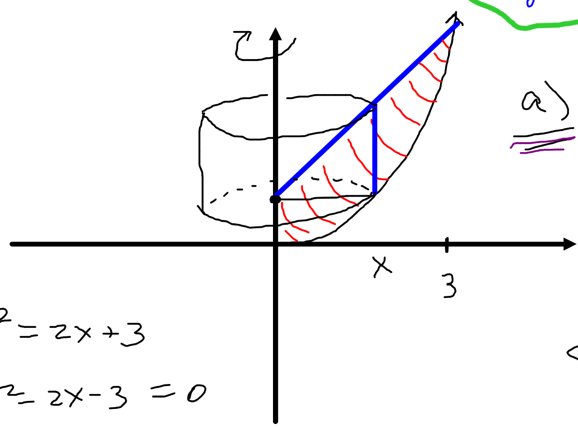
$$= 4 - 1 = 3$$

Example: A region in the first quadrant of the *xy* plane is bounded by the curves $y = x^2$ and $y = 2x + 3$. Rotate this region about the y-axis.

a) Give a formula involving integral(s) in x for the resulting volume. dx vertical line segments

b) Give a formula involving integral(s) in y for the resulting volume. dy horizontal line segments

cylindrical shell

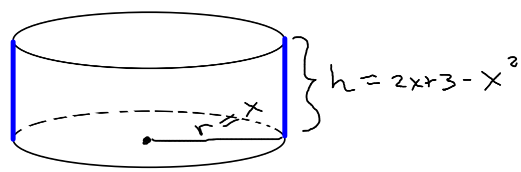


$x^2 = 2x + 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

a)



Thickness $= dx$

Surface Area $= 2\pi r h$

$= 2\pi x\,(2x+3-x^2)$

Inf. volume (shell) $= 2\pi x\,(2x+3-x^2)\,dx$

$$\text{Volume} = \int_0^3 2\pi x\,(2x+3-x^2)\,dx$$

Example: A region in the first quadrant of the *xy* plane is bounded by the curves $y = x^2$ and $y = 2x + 3$. Rotate this region about the x-axis.

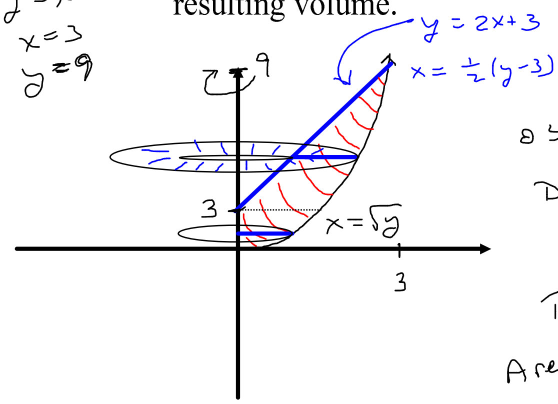
a) Give a formula involving integral(s) in x for the resulting volume.

b) Give a formula involving integral(s) in y for the resulting volume.

$y = x^2$

$x = 3$

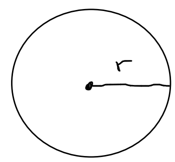
$y = 9$



b)

$0 \le y \le 3$

Disk



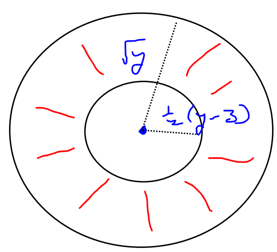
Thickness $= dy$

Area $= \pi r^2 = \pi(\sqrt{y})^2 = \pi y$

Inf. volume $= \pi y \, dy$

$3 \le y \le 9$

Washer



Thickness $= dy$

Area $= \pi(\sqrt{y})^2 - \pi\left(\frac{1}{2}(y-3)\right)^2$

Inf. Volume $= \left(\pi y - \frac{\pi}{4}(y-3)^2\right)dy$

Full Volume $= \int_0^3 \pi y \, dy + \int_3^9 \left(\pi y - \frac{\pi}{4}(y-3)^2\right)dy$