VECTORS

1. GEOMETRY Let $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2,\, y_2,\, z_2)$ be points in 3-space:

- **Distance Formula:** $d(P_1,\, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$

- **Midpoint Formula:** The midpoint of the line segment joining P_1 and P_2 is the point $P\left(\frac{x_1 + x_2}{2},\, \frac{y_1 + y_2}{2},\, \frac{z_1 + z_2}{2}\right)$

- **Equation for the sphere of radius r and center $\mathbf{P}\,(\mathbf{a}, \mathbf{b}, \mathbf{c})$:**
  $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

2. VECTORS A vector $\mathbf{a}$ in 3-space is represented by an ordered triple of real numbers, $\mathbf{a} = (a_1,\, a_2,\, a_3)$; a vector $\mathbf{a}$ in the plane is represented by an ordered pair of real numbers, $\mathbf{a} = (a_1,\, a_2)$. Let $\mathbf{a} = (a_1,\, a_2,\, a_3)$ and $\mathbf{b} = (b_1,\, b_2,\, b_3)$ be vectors in 3-space and let α be a real number (scalar). [Corresponding definitions hold for vectors in the plane.]

- **Vector Addition:** $\mathbf{a} + \mathbf{b} = (a_1 + b_1,\, a_2 + b_2,\, a_3 + b_3)$

- **Scalar Multiplication:** $\alpha\mathbf{a} = (\alpha a_1,\, \alpha a_2,\, \alpha a_3)$; $\alpha\mathbf{a}$ has the same direction as $\mathbf{a}$ if $\alpha > 0$ and opposite direction if $\alpha < 0$. If $\alpha = 0$, then $\alpha\mathbf{a} = (0, 0, 0) = \mathbf{0}$ is the zero vector. $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{a} = \alpha\mathbf{b}$ for some real number α.

- **Norm:** $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; $\mathbf{a}$ is a unit vector if $\|\mathbf{a}\| = 1$. If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}/\|\mathbf{a}\|$ is a unit vector in the direction $\mathbf{a}$.

  (1) $\|\mathbf{a}\| \geq 0$; $\|\mathbf{a}\| = 0$ if and only if $\mathbf{a} = \mathbf{0}$.

  (2) $\|\alpha\mathbf{a}\| = |\alpha|\, \|\mathbf{a}\|$

  (3) $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$

- **Unit Coordinate Vectors:** $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$

- **$\mathbf{i}, \mathbf{j}, \mathbf{k}$ - Representation:** $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

3. DOT PRODUCT $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. The dot product for 2-dimensional vectors is defined in the same way.

- **Properties:**

(1) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$	(2) $\mathbf{a} \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{a} = \mathbf{0}$
(3) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	(4) $(\alpha\mathbf{a}) \cdot (\beta\mathbf{b}) = \alpha\beta\, (\mathbf{a} \cdot \mathbf{b})$
(5) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$	

- **Geometric Interpretation:** $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$. $\mathbf{a}$ is perpendicular to $\mathbf{b}$ $(\mathbf{a} \perp \mathbf{b})$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- **Component & Projection of $\mathbf{a}$ on $\mathbf{b}$:** $\text{comp}_{\mathbf{b}}\, \mathbf{a} = \mathbf{a} \cdot \mathbf{u_b}$, $\mathbf{u_b} = \mathbf{b}/\|\mathbf{b}\|$; $\text{proj}_{\mathbf{b}}\, \mathbf{a} = (\mathbf{a} \cdot \mathbf{u_b})\mathbf{u_b}$
- **Direction Cosines:** $\mathbf{a} \cdot \mathbf{i} = \|\mathbf{a}\| \cos\alpha$, $\mathbf{a} \cdot \mathbf{j} = \|\mathbf{a}\| \cos\beta$, $\mathbf{a} \cdot \mathbf{k} = \|\mathbf{a}\| \cos\gamma$, α, β, and γ are called the direction angles of $\mathbf{a}$; $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines;

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

4. CROSS PRODUCT: Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in 3-space, $\mathbf{a} \neq \lambda\mathbf{b}$. $\mathbf{a} \times \mathbf{b}$ is the vector such that:

(1) $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{b}$.

(2) $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ (in this order) form a right-handed triple.

(3) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

- **Properties:**

(1) $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ (2) $(\alpha\mathbf{a}) \times (\beta\mathbf{b}) = \alpha\,\beta\,(\mathbf{a} \times \mathbf{b})$

(3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

- **Components of $\mathbf{a} \times \mathbf{b}$:**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

- **Triple Scalar Product:**

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is the volume of the parallelepiped having $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as sides.