

# UNSTABLE SOLUTIONS OF NON-AUTONOMOUS LINEAR DIFFERENTIAL EQUATIONS

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**Abstract.** The fact that the eigenvalues of the family of matrices  $A(t)$  do not determine the stability of non-autonomous differential equations  $x' = A(t)x$  is well known. This point is often illustrated using examples in which the matrices  $A(t)$  have constant eigenvalues with negative real part, but the solutions of the corresponding differential equation grow in time. Here we provide an intuitive, geometric explanation of the idea that underlies these examples. The discussion is accompanied by a number of animations and easily modifiable Mathematica programs. We conclude with a discussion of possible extensions of the ideas that may provide suitable topics for undergraduate research.

**1. Introduction.** A considerable part of an introductory course on differential equations is devoted to the study of linear, autonomous differential equations [2, 6, 13, 10, 16]. The relation between the behavior of the solutions of the equation  $x' = Ax$  and the eigenvalues of the matrix  $A$  forms one of the centerpieces of the subject: The fact that the solutions to this equation are stable whenever the spectrum of  $A$  is contained in the left half plane of  $\mathbb{C}$  is repeated throughout the course. It therefore comes as a surprise that for linear, non-autonomous equations

$$x' = A(t)x \tag{1.1}$$

the eigenvalues of the matrix  $A(t)$  are in general of no use in determining the stability of solutions [20, 21, 12, 7, 18, 11].

The most striking examples are provided by matrices  $A(t)$  with constant negative eigenvalues for which system (1.1) has exponentially growing solutions. The following example introduced by Vinograd [18, 21, 20] is typical

$$A(t) = \begin{bmatrix} -1 - 9 \cos^2(6t) + 12 \sin(6t) \cos(6t) & 12 \cos^2(6t) + 9 \sin(6t) \cos(6t) \\ -12 \sin^2(6t) + 9 \sin(6t) \cos(6t) & -1 - 9 \sin^2(6t) - 12 \sin(6t) \cos(6t) \end{bmatrix}. \tag{1.2}$$

The eigenvalues of  $A(t)$  for any  $t$  are  $-1$  and  $-10$ , yet it can be checked by direct substitution that the system has the following exponentially growing solution

$$x(t) = \begin{bmatrix} e^{2t}(\cos(6t) + 2 \sin(6t)) + 2e^{-13t}(2 \cos(6t) - \sin(6t)) \\ e^{2t}(2 \cos(6t) - \sin(6t)) - 2e^{-13t}(\cos(6t) + 2 \sin(6t)) \end{bmatrix}.$$

It appears that Poincaré and Lyapunov were aware of such examples. Many similar systems can be found in the literature: The example of Markus and Yamabe of an unstable system of the form (1.1) in which  $A(t)$  has complex eigenvalues with negative real parts is frequently cited [12, 7]. Other examples include that of Hinrichsen where  $A(t)$  has a single negative eigenvalue of geometric multiplicity 1, yet the system is unstable [9], and Wu where the eigenvalues of  $A(t)$  have opposite sign, yet all solutions are stable [20]. Moreover, these observations have motivated much work on the utility of using asymptotic quantities, like eigenvalues and their generalizations, to quantify the behavior of solutions over finite time intervals [5, 8, 11, 15].

There always appears to be something miraculous about these examples when they are first encountered. In most cases it is not explained how the form of the matrix  $A(t)$  was divined, or how the unstable solution was constructed. We have been unable to locate an explanation of the intuitive, geometrical reasons that make these examples “tick.”

Our goal here is to provide such an explanation. In the process we will show that all examples found in the literature belong to the same family. Along the way, we will introduce a number of ideas that are

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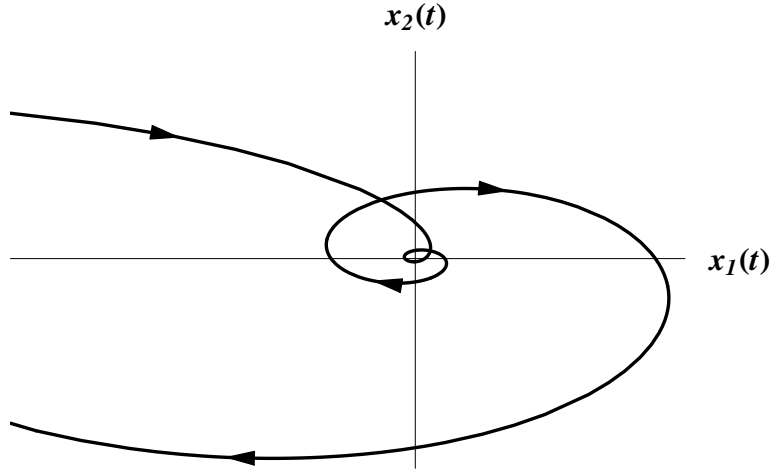


FIG. 1.1. An unstable solution to Vinograd's example.

accessible to students who have taken a semester of linear algebra and differential equations, but which lie outside the typical curricula for such classes. These include Lie brackets, Floquet exponents and different geometric ways of looking at linear transformations. For the ease of exposition we only discuss planar systems, although the ideas are easily extended.

Wherever possible, we present the motivations and proofs in an intuitive, geometric way. As an alternative, and in certain cases to complete the argument, we also frequently present analytic proofs of the claims. We have created a number of animations to accompany the explanations presented here, and we will refer to them at several points in the following exposition. The animations and the Mathematica notebooks used to create them can be found at

<http://www.math.uh.edu/~josic/nonautonomous>

**2. Instability of the Frozen Coefficient Equations.** Our goal is to explain how it is possible that the matrix  $A(t)$  in (1.1) has negative eigenvalues, and the solutions are unstable. For this to happen the norm of a solution,  $\|x(t)\| = \sqrt{x(t) \cdot x(t)}$ , must increase over time so that

$$\frac{d}{dt} \|x(t)\|^2 = 2x'(t) \cdot x(t) = 2[A(t)x(t)] \cdot x(t) > 0 \quad (2.1)$$

for at least some values of  $t$ . Fix a  $t_0 > 0$  and consider the *autonomous, frozen coefficient* system

$$x' = A(t_0)x, \quad (2.2)$$

obtained from (1.1) by “freezing” the matrix  $A(t)$  at time  $t_0$ . Condition (2.1) implies that there must be a  $t_0$  such that the frozen coefficient system (2.2) has solutions whose distance from the origin increases during some interval of time. This is illustrated in Figure 2.1 in the case of Vinograd's example.

Our first goal is therefore to characterize the class of matrices

$$\mathcal{B} = \{B \text{ is a } 2 \times 2 \text{ matrix whose eigenvalues have negative real part} \mid x \cdot Bx > 0 \text{ for some } x \in \mathbb{R}^2\}.$$

By (2.1), solutions to (1.1) can only be unstable if  $A(t) \in \mathcal{B}$  for some  $t$ . Note that  $x \cdot A(0)x = 1 > 0$  where  $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and  $A(t)$  is as in (1.2).

We also note that similar ideas are used in [15, 5] to argue that asymptotic quantities like Lyapunov exponents may not be useful in quantifying the increase in uncertainty over finite time.

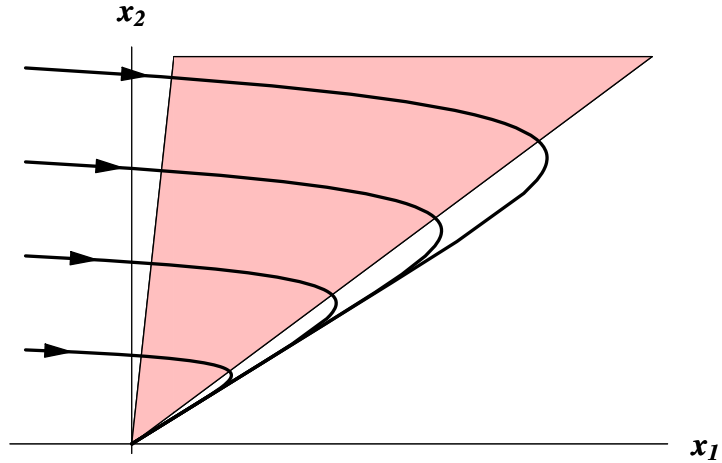


FIG. 2.1. Stable solutions to the frozen coefficient equation  $x' = A(0)x$  where  $A(t)$  is given by (1.2). The distance of solutions from the origin,  $|x(t)|$ , increases in the shaded region.

**Remark.** The conclusion that there must exist an  $x$  and a  $t$  such that  $x \cdot A(t)x > 0$  for (1.1) to have unstable solutions can be reached using Lyapunov functions: If  $x \cdot x' = x \cdot A(t)x \leq 0$  for any  $t$  and  $x$ , then let  $V(x) = x \cdot x$  to conclude that  $\|x(t)\|$  cannot increase. This follows from the same computation as above:

$$V'(x) = 2(x' \cdot x) = 2(A(t)x \cdot x) \leq 0,$$

and no solutions can cross the level curves of  $V(x)$ .

**2.1. Real Eigenvalues.** The case of real and complex eigenvalues are geometrically somewhat distinct and we deal with them separately. We first consider  $2 \times 2$  matrices  $B$  with distinct, real eigenvalues  $\lambda_1 < \lambda_2 \leq 0$ . Since we are only interested in finding  $x \in \mathbb{R}^2$  for which  $x \cdot Bx > 0$ , we can rotate the coordinates and assume that the eigenvector corresponding to eigenvalue  $\lambda_1$  lies on the horizontal axis. Let  $\delta$  be the angle between the two eigenvectors, and assume that  $0 < \delta < \pi$ . Then  $B$  has the form

$$B = \begin{bmatrix} \lambda_1 & (\lambda_2 - \lambda_1) \cot(\delta) \\ 0 & \lambda_2 \end{bmatrix}. \quad (2.3)$$

Let

$$r(x) = x \cdot Bx = \lambda_1 x_1^2 + \lambda_2 x_2^2 + (\lambda_2 - \lambda_1) x_1 x_2 \cot(\delta).$$

To check whether  $r(x)$  can be positive it is sufficient to locate its maxima on the unit circle. Setting  $x(\theta) = (\cos \theta, \sin \theta)$ , we find that

$$\frac{dr(x(\theta))}{d\theta} = (\lambda_2 - \lambda_1) \cos(\delta - 2\theta) \csc(\delta),$$

and that  $r(x(\theta))$  attains its maximum at  $\theta_{\max} = \delta/2 + \pi/4$ . Thus, the maximum of  $r(x) = x \cdot Bx$  on the unit circle is

$$r(x(\theta_{\max})) = \frac{\lambda_1 + \lambda_2 + (\lambda_2 - \lambda_1) \csc(\delta)}{2}. \quad (2.4)$$

Figure 2.2 shows a polar plot of  $r(x(\theta))$  where  $B = A(0)$  for  $A(t)$  as in Vinograd's example (see (1.2)).

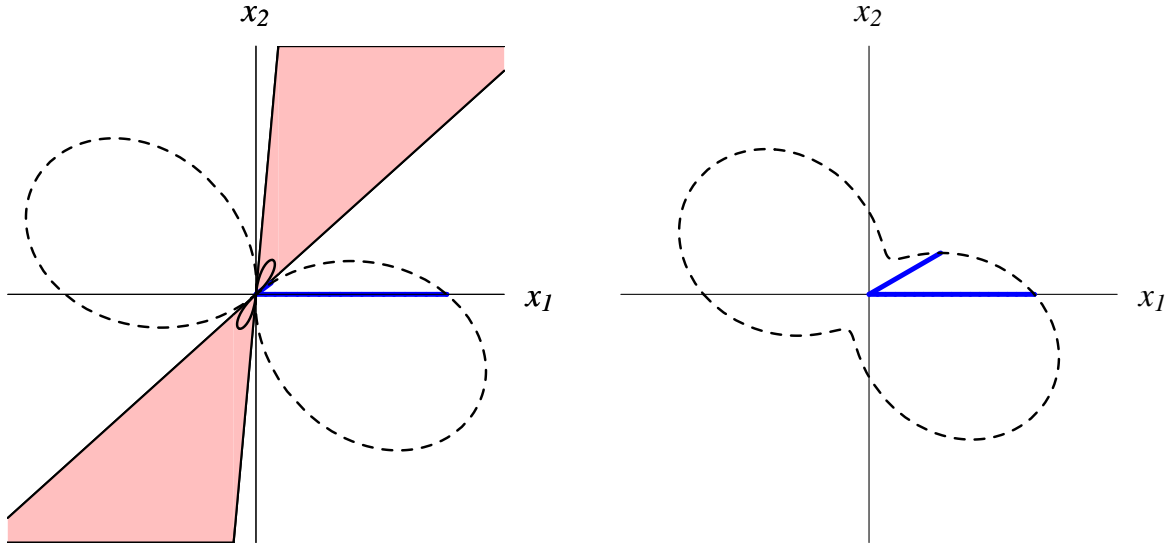


FIG. 2.2. Polar plots of  $r(x(\theta))$ . The part of the graph corresponding to positive values of  $r$  is plotted using a solid line, while a dashed line denotes negative values. The thick lines correspond to eigenvectors of  $B$  (with magnitude equal to their corresponding eigenvalues). In the shaded regions,  $x \cdot Bx > 0$ . On the left  $B = A(0)$  for  $A(t)$  defined in (1.2) so that  $\lambda_1 = -10$ ,  $\lambda_2 = -1$ , and  $\delta = \cot^{-1}(4/3)$  and the maximum of  $r$  on the unit circle is 2. On the right,  $\lambda_1 = -4$ ,  $\lambda_2 = -2$ , and  $\delta = \frac{\pi}{6}$  so that the maximum of  $r$  on the unit circle is  $-1$ .

Setting  $p = \lambda_2/\lambda_1$ , and observing that  $0 \leq p < 1$  we conclude from (2.4) that  $x \cdot Bx > 0$  for some values of  $x$  if and only if

$$\sin(\delta) < \frac{1-p}{1+p}. \quad (2.5)$$

This simple relation between the geometry of the action of  $B$  and the existence of a region in which  $x \cdot Bx > 0$  is summarized in Figure 2.3.

Following [17], we refer to  $B$  as *normal* when  $\sin(\delta) = 1$  and *far from normal* when  $\sin(\delta) \ll 1$ . It is well-known that, for  $A$  far from normal, the behavior of solutions to  $x' = Ax$  over a finite time interval is difficult to determine by the sign of the eigenvalues alone [17]. Note that matrices in  $\mathcal{B}$  are far from normal ( $\sin(\delta) \ll 1$ ) or have unequal eigenvalues ( $p \ll 1$ ). Animation 1 on the accompanying website shows  $r(x(\delta))$  changing as  $\delta$  increases (i.e., as the eigenvectors of  $B$  grow apart). Note that the size of the cone in which  $r(x(\delta))$  is positive increases as  $\delta$ , and therefore  $\sin(\delta)$ , approaches 0.

If  $B$  has only one eigenvalue, the computations are similar. A rotation can be applied to put the matrix in the form

$$B = \begin{bmatrix} \lambda & c \\ 0 & \lambda \end{bmatrix}$$

where  $c = 0$  if the geometric multiplicity of  $\lambda$  is 2.

Maximizing  $x \cdot Bx$  on the unit circle, we see that  $B \in \mathcal{B}$  if

$$\lambda + \left| \frac{c}{2} \right| > 0.$$

**2.2. Complex Eigenvalues.** The case of complex eigenvalues is treated similarly. Any  $2 \times 2$  matrix with a complex conjugate pair of eigenvalues  $\lambda_{1,2} = k \pm \sigma i$  and eigenvectors  $\begin{bmatrix} 1 \\ a \pm bi \end{bmatrix}$  can be written in

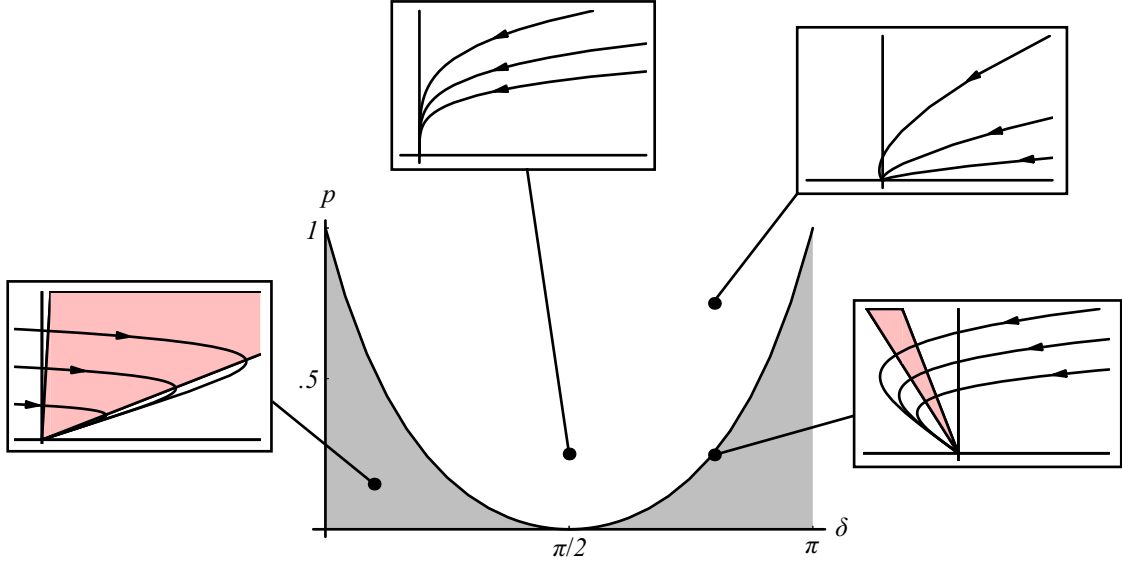


FIG. 2.3. Matrices corresponding to values of  $\delta$  and  $p$  that lie in the shaded region have solutions that tend away from the origin. Insets show solutions to  $x' = Bx$  for different values of  $B$ . As in Figure 2.1, solutions increase in norm inside the shaded regions of the insets.

the form

$$B = \begin{bmatrix} 1 & 1 \\ a+bi & a-bi \end{bmatrix} \begin{bmatrix} k+\sigma i & 0 \\ 0 & k-\sigma i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ a+bi & a-bi \end{bmatrix}^{-1}$$

where  $b > 0$ . We again assume that the eigenvalues of  $B$  have negative real part, so that  $k < 0$ . Changing  $a$  corresponds to a rotation of the coordinate system. As in the previous section, such rotations have no impact on whether a matrix lies in  $\mathcal{B}$ . Therefore, we assume that  $a = 0$ , so that

$$B = \begin{bmatrix} 1 & 1 \\ bi & -bi \end{bmatrix} \begin{bmatrix} k+\sigma i & 0 \\ 0 & k-\sigma i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ bi & -bi \end{bmatrix}^{-1} = \begin{bmatrix} k & \frac{\sigma}{b} \\ -\sigma b & k \end{bmatrix}. \quad (2.6)$$

The solution to  $x' = Bx$  with initial condition  $[x_0, y_0]$  is

$$\begin{bmatrix} e^{kt}(x_0 b \cos(\sigma t) + y_0 \sin(\sigma t)) \\ e^{kt}(y_0 \cos(\sigma t) - x_0 b \sin(\sigma t)) \end{bmatrix}.$$

Here,  $k$  represents the decay rate,  $\sigma$  the frequency, and  $b$  the eccentricity of the spiral solutions. In particular, if  $k = 0$ , solutions are ellipses with a horizontal major axis if  $b < 1$  and a vertical major axis if  $b > 1$ . As  $b$  tends away from 1, these solutions become more eccentric.

As in the case of real eigenvalues, we set  $x(\theta) = (\cos(\theta), \sin(\theta))$  and find that the maximum of  $r(x(\theta))$  is

$$k + \left| \frac{(b^2 - 1)\sigma}{2b} \right| \quad (2.7)$$

Let  $q = \frac{\sigma}{k}$  so that  $B \in \mathcal{B}$  if and only if

$$|q| > \left| \frac{2b}{b^2 - 1} \right|. \quad (2.8)$$

The points in the shaded region illustrated in Figure 2.4 satisfy the inequality (2.8), so that the corresponding matrices lie in  $\mathcal{B}$ . Geometrically, matrices in  $\mathcal{B}$  correspond to autonomous systems with eccentric solutions (small  $b$ ) or solutions that oscillate fast compared to the decay rate.

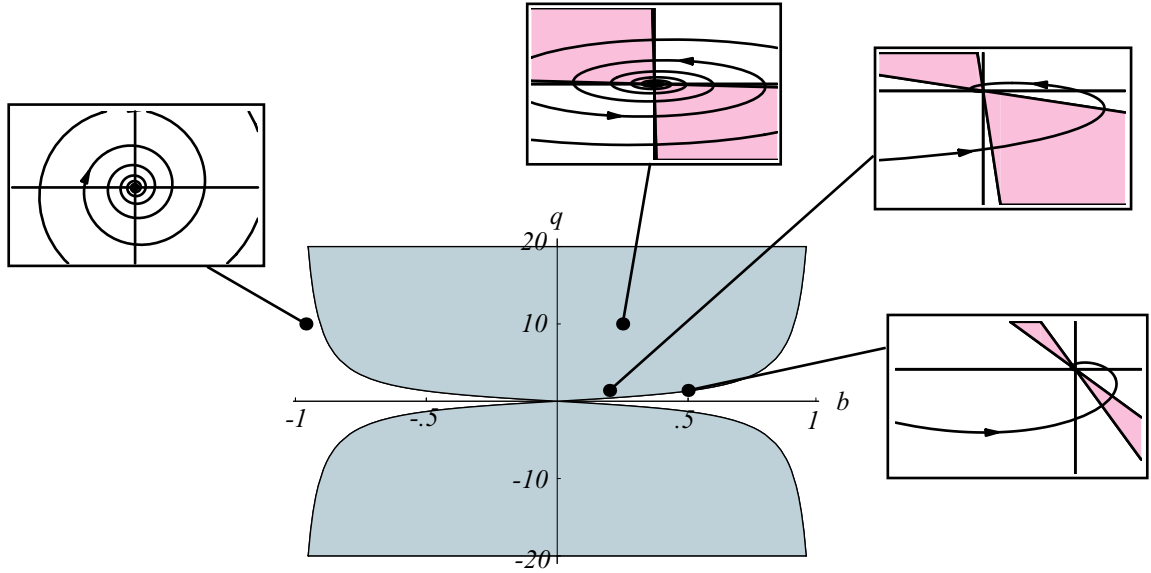


FIG. 2.4. Solutions with values of  $q$  and  $b$  in the shaded region belong to  $\mathcal{B}$ . The shadings and insets are as in Figure 2.3.

**3. Stability in Linear non-Autonomous Equations.** The discussion in this section assumes a basic understanding of matrix exponentials. We remind the reader that the exponential of a matrix  $A$  is defined as

$$e^A = \sum_{n=1}^{\infty} \frac{A^n}{n!}.$$

For a discussion of the basic properties and applications of matrix exponentials, we refer the reader to [10].

We now return to the original problem of constructing specific non-autonomous equations with unstable solutions. This can be done in a very simple way: take one of the matrices from the class  $\mathcal{B}$  described in the previous section, and rotate the corresponding vector field at a constant angular velocity.

More precisely, let  $B \in \mathcal{B}$ , and let  $G(\omega) = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ , so that

$$R(t, \omega) = e^{tG(\omega)} = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix} \quad (3.1)$$

rotates the plane by an angle  $\omega t$ , or, equivalently, at angular velocity  $\omega$ . Let  $A(t) = R(t, \omega)B[R(t, \omega)]^{-1}$ , so that the vector field  $A(t)x$  is obtained from  $Bx$  by a rotation through an angle  $\omega t$ . The equations we consider henceforth will be of the form

$$x' = A(t)x = (R(t, \omega)B[R(t, \omega)]^{-1})x. \quad (3.2)$$

Animation 2 on the accompanying webpage illustrates how such a vector field evolves in time.

To solve the equation  $x' = A(t)x$  we move to rotating coordinates and define  $y = [R(t, \omega)]^{-1}x$ . There are two contributions to  $\frac{dy}{dt}$ : The first,  $A(0)y$ , comes from the original vector field which is autonomous in the rotating frame. The second contribution, due to the rotation of the plane, equals  $-G(\omega)y$  and points in the direction opposite to the one in which the plane is rotated (see Figure 3.1). We therefore obtain

$$y' = [A(0) - G(\omega)]y = [B - G(\omega)]y \quad (3.3)$$

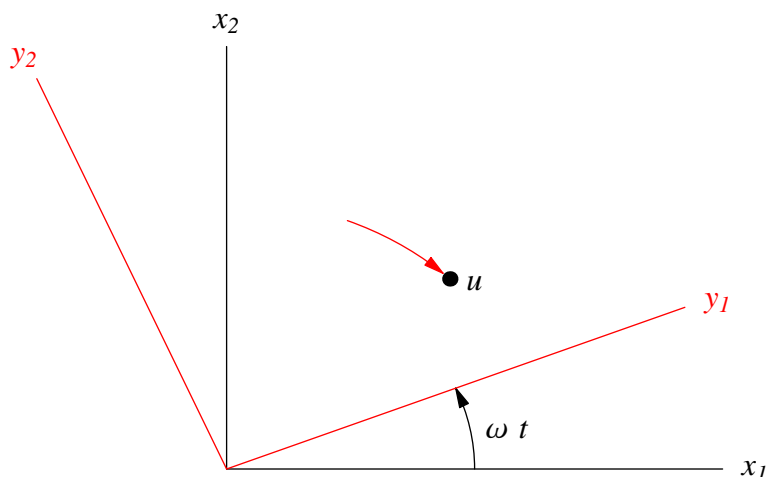


FIG. 3.1. The coordinate system  $y$  is rotating counterclockwise with respect to the coordinate system  $x$ . Therefore, a point  $u$  that is fixed in the original coordinate system, appears to rotate clockwise with angular velocity  $-\omega$  in the rotating coordinates  $y$ .

so that

$$y(t) = e^{[B-G(\omega)]t}y(0).$$

Inverting the change of coordinates gives  $x(t) = R(t, \omega)y(t)$ , from which we obtain the solution to (3.2)

$$x(t) = R(t, \omega)e^{[B-G(\omega)]t}x(0). \quad (3.4)$$

A similar approach to solving equations of this form can be found in [11].

**3.1. Behavior of solutions.** The behavior of the solutions in (3.4) is fairly easy to understand: take the solution  $e^{[B-G(\omega)]t}x(0)$  to the autonomous system (3.3) and, as it evolves in time, spin it around the origin with angular velocity  $\omega$ . Therefore, only if the linear, autonomous system (3.3) has unstable solutions can the solution to the non-autonomous equation be unstable.

As an example, consider  $A(t)$  defined in (1.2). In this case  $A(t)$  has the form given in (3.2) with  $\omega = -6$  and  $B = \begin{bmatrix} -10 & 12 \\ 0 & -1 \end{bmatrix}$ . It can be easily checked that  $B - G(-6)$  has eigenvalues 2 and -13, and hence solutions of the corresponding system are unstable. In the well-known example by Markus and Yamabe [12],  $B = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & -1 \end{bmatrix}$  has eigenvalues  $\frac{1}{4}(-1 \pm \sqrt{7})$ ,  $\omega = -1$ , and  $B - G(-1)$  has eigenvalues -1 and  $\frac{1}{2}$ .

The behavior of the solutions is best understood by viewing Animations 3 and 4 on the accompanying webpage. In Animation 3 we show a solution of Vinograd's example (1.2) in the fixed coordinate frame. The two blue lines represent the eigendirections corresponding to the positive and negative eigenvalues of the matrix  $B - G(-6)$ , rotating at angular velocity  $\omega = -6$ . We have chosen a solution which starts close to the stable eigendirection, and therefore comes close to the origin before diverging along the unstable direction.

Animation 4 shows the same solution in the *rotating* ( $y$ ) coordinate system. In this case, the two eigendirections, again shown in blue, are fixed. The actual solution to the non-autonomous system is obtained by tracing the solution of the autonomous system  $y' = [B - G(-6)]y$ , while at the same time rotating the paper on which the solution is being drawn in the counterclockwise direction at angular velocity  $\omega = 6$ .

**3.2. Condition for existence of unstable solutions.** Our original problem of constructing examples of families of matrices  $A(t)$  with constant eigenvalues with negative real part resulting in unstable differential

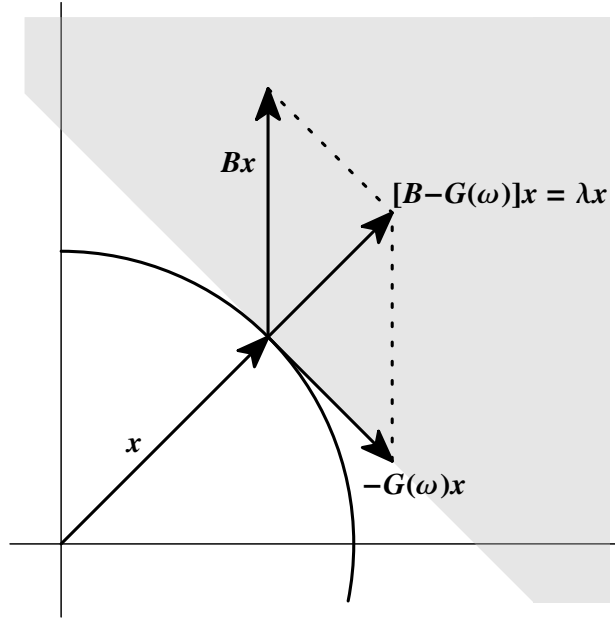


FIG. 3.2. Only if there exists a vector  $Bx$  which points outside the unit circle can we find an  $\omega$  such that  $(B - G(\omega))x = \lambda x$  for a positive  $\lambda$ .

equations (1.1) has now been reduced to the following: Given a matrix  $B$ , find a matrix  $G(\omega)$  such that  $B - G(\omega)$  has positive eigenvalues. To do this it is sufficient to find  $x$  and  $\omega$  so that

$$[B - G(\omega)]x = \lambda x \text{ for } \lambda > 0. \quad (3.5)$$

We claim that this is possible if and only if  $B \in \mathcal{B}$ . An immediate, geometrical proof is illustrated in Figure 3.2: the vector  $-G(\omega)x$  has length  $\omega$  and is tangent to the unit circle for any value of  $\omega$ . Therefore, the vector  $Bx - G(\omega)x$  for  $|x| = 1$  will point to the same side of the tangent line to the unit circle as  $Bx$ . In particular, only if  $Bx$  points to the outside of the tangent line to the unit circle (the lightly shaded half-plane in Figure 3.2), that is  $B \in \mathcal{B}$ , will it be possible to find  $x$  and  $\omega$  satisfying (3.5). On the other hand, using the parallelogram rule for vector addition, and the fact that we are free to choose the length of  $\|G(\omega)x\| = \omega$ , we can see that  $Bx - G(\omega)x$  can point in any direction inside the shaded region in Figure 3.2. In particular, for any  $x$  such that  $x \cdot Bx > 0$  there exists an  $\omega$  such that  $[B - G(\omega)]x = \lambda x$  for some  $\lambda > 0$ .

The following Theorem provides a quantitative statement of this result.

**THEOREM 3.1.** *Let  $B$  be a real-valued diagonalizable  $2 \times 2$  matrix. Let  $v_1$  and  $v_2$  be eigenvectors associated with eigenvalues  $\lambda_1$  and  $\lambda_2$  respectively. If  $B \in \mathcal{B}$  then (3.2) has unstable solutions if and only if  $\omega$  is in the non-empty interval*

$$I = \left( D - \sqrt{D^2 - \lambda_1 \lambda_2}, D + \sqrt{D^2 - \lambda_1 \lambda_2} \right)$$

where

$$D = \frac{v_1 \cdot v_2}{2 \det \begin{bmatrix} v_1 & v_2 \end{bmatrix}} (\lambda_1 - \lambda_2)$$

and  $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$  denotes the  $2 \times 2$  matrix with columns  $v_1$  and  $v_2$ .



*Proof.* First note that even if  $B$  has complex eigenvalues  $D$  is real since  $v_1 \cdot v_2$  is real and both  $\det(\begin{bmatrix} v_1 & v_2 \end{bmatrix})$  and  $(\lambda_1 - \lambda_2)$  are imaginary.

The eigenvalues of  $B - G(\omega)$  are

$$\eta_1, \eta_2 = \frac{\lambda_1 + \lambda_2 \pm \sqrt{(\lambda_1 - \lambda_2)^2 + 8D\omega - 4\omega^2}}{2}$$

Of these  $\eta_1$  has maximal real part, and an unstable solution to (1.1) exists whenever  $\eta_1 > 0$ . It can be checked algebraically that  $\eta_1 = 0$  when  $\omega = D \pm \sqrt{D^2 - \lambda_1\lambda_2}$  and that  $\eta_1 > 0$  whenever  $\omega \in I$ .

It remains to be shown that  $I$  is non-empty assuming that  $B \in \mathcal{B}$ . By the preceding above, we must show that  $D^2 - \lambda_1\lambda_2 > 0$ . We break the proof into 2 cases.

**Case 1:  $B$  has real eigenvalues.** We can assume  $B$  is of the form (2.3). Then

$$D = \frac{\cot(\delta)(\lambda_1 - \lambda_2)}{2}$$

and

$$D^2 - \lambda_1\lambda_2 = \frac{1}{4} \cot^2(\delta)(\lambda_1 - \lambda_2)^2 - \lambda_1\lambda_2$$

Therefore  $D^2 - \lambda_1\lambda_2 > 0$  whenever

$$|\csc(\delta)(\lambda_1 - \lambda_2)| > |\lambda_1 + \lambda_2| \quad (3.6)$$

Since  $B \in \mathcal{B}$ , we have that  $\lambda_1 < \lambda_2 < 0$  and  $0 < \delta < \pi$ . Thus (3.6) is equivalent to

$$\lambda_1 + \lambda_2 + (\lambda_2 - \lambda_1) \csc(\delta) > 0$$

Therefore, it follows from equation (2.4) that  $D^2 - \lambda_1\lambda_2 > 0$ .

**Case 2:  $B$  has complex eigenvalues.** We can assume that  $B$  is of the form (2.6). Then

$$D = -\frac{(b^2 + 1)\sigma}{2b}$$

and

$$D^2 - \lambda_1\lambda_2 = \left(\frac{b^2 + 1}{2b}\right)^2 \sigma^2 - \sigma^2 - k^2 \quad (3.7)$$

Since  $k < 0$ , (3.7) is positive when

$$k + \left|\frac{(b^2 - 1)\sigma}{2b}\right| > 0.$$

Therefore, equation (2.7) implies that  $D^2 - \lambda_1\lambda_2 > 0$ .  $\square$

The length of the interval of  $\omega$  that yields unstable solutions to  $x' = A(t)x$  is  $\sqrt{D^2 - \lambda_1\lambda_2}$ . In Figure (3.3) we show the length of this interval as a function of the ratio  $p = \frac{\lambda_2}{\lambda_1}$  and the angle  $\delta$  between the eigenvectors (compare with Figure 2.3).

**4. Additional Topics.** We next present an outline of additional topics that tie in naturally to the ideas we have presented. Some of these can serve as starting points for further study, while others can lead to research projects for undergraduates.

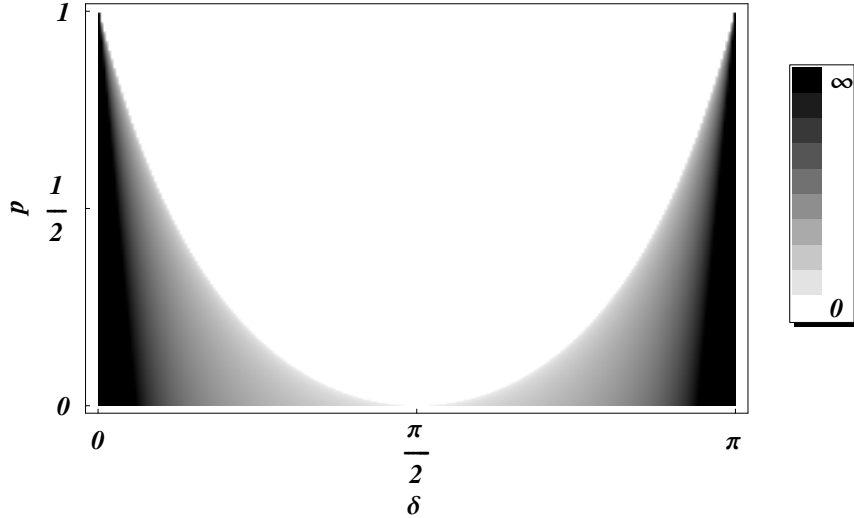


FIG. 3.3. The length of the interval  $I$  described in Theorem 3.1. The white region corresponds to matrices outside of  $\mathcal{B}$  for which  $I$  is empty.

**4.1. Floquet Theory.** Since the eigenvalues of  $A(t)$  do not determine the stability of the solution to equation (3.2), it is natural to ask whether there are generalizations that are useful in this regard. In the case that  $A(t)$  is periodic, the answer is provided by Floquet exponents.

Suppose that  $A(t)$  is a  $T$ -periodic continuous  $n \times n$  matrix function. Floquet's Theorem [7] states that the fundamental solution to  $x' = A(t)x$  can be written in the form

$$x(t) = P(t)e^{Mt} \quad (4.1)$$

where  $P(t)$  has period  $T$  and  $M$  is constant. The eigenvalues of  $e^{MT}$  are called characteristic multipliers of  $A(t)$ , and a Floquet exponent of  $A(t)$  is a complex number  $\mu$  such that  $e^{\mu T}$  is a characteristic multiplier of  $A(t)$ . In particular, the eigenvalues of  $M$  are Floquet exponents of  $A(t)$ . The system  $x' = A(t)x$  is asymptotically stable if the real parts of the Floquet exponents are negative [7].

For systems of the form (3.2),  $A(t)$  has period  $T = \frac{2\pi}{\omega}$  and the discussion in Section 3 implies that  $P(t) = e^{G(\omega)t}$  and  $M = B - G(\omega)$ . Thus the eigenvalues of  $B - G(\omega)$  are the Floquet exponents the system (3.2) and the signs of their real parts determine the stability of the system. This is the same conclusion we reached in Section 3.

If the matrix  $A(t)$  is not periodic the situation is a bit more complicated. In this case Lyapunov exponents [3], or the dichotomy spectrum [14] can be used as generalization of the Floquet exponents.

**4.2. Stable Systems with Constant Eigenvalues of Opposite Sign.** We have discussed the existence of unstable systems of the form  $x' = A(t)x$  where the eigenvalues of  $A(t)$  are constant with negative real part. In this section, we introduce an example of a stable system of the form (1.1) where the eigenvalues of  $A(t)$  are real and constant, but of opposite signs. In other words, while the origin is a saddle for the frozen coefficient system  $x' = A(t_0)x$ , it is asymptotically stable for the non-autonomous system.

In this example, due to Wu [20],

$$A(t) = \begin{bmatrix} -\frac{11}{2} + \frac{15}{2} \sin(12t) & \frac{15}{2} \cos(12t) \\ \frac{15}{2} \cos(12t) & -\frac{11}{2} - \frac{15}{2} \sin(12t) \end{bmatrix}$$

and has constant eigenvalues -13 and 2. This family of matrices can again be written as  $A(t) = R(t, \omega)B[R(t, \omega)]^{-1}$

where  $B = \frac{1}{2} \begin{bmatrix} -11 & 15 \\ 15 & -11 \end{bmatrix}$ ,  $\omega = -6$ , and  $R(t, \omega)$  is defined in (3.1). The eigenvalues of  $B - G(-6)$  are -10 and -1, and hence the non-autonomous system  $x' = A(t)x$  is stable.

It is not possible to use these ideas directly to construct examples in which  $A(t)$  has two constant eigenvalues with positive real part, but the corresponding non-autonomous system is stable.

**4.3. Discrete Systems.** Given  $x_0 \in \mathbb{R}^m$  and an  $m \times m$  matrix  $D$ , consider the discrete dynamical system  $z_{n+1} = Dz_n = D^n z_0$ . Solutions of this system are stable if the eigenvalues of  $D$  lie inside the unit circle, and diverge to infinity if at least one of the eigenvalues lies outside of it [4]. The discrete counterpart to linear, non-autonomous systems has the form

$$z_{n+1} = D(n)z_n \quad (4.2)$$

where  $\{D(n)\}$  is a sequence of matrices depending on the discrete “time” variable  $n$ .

The analog of the examples considered heretofore in the case of continuous time, consist of matrices  $D(n)$  with eigenvalues of modulus less than one and a  $z_0$  such that the sequence  $\{z_n\}$  defined by (4.2) is unbounded. The analog of the set of matrices  $\mathcal{B}$  defined in (2) is the set  $\mathcal{C}$  consisting of matrices with eigenvalues of modulus less than one, which satisfy

$$|Ax| > |x| \quad (4.3)$$

for some values of  $x$ . The set of matrices  $\mathcal{C}$  can be characterized geometrically following the ideas in Section 2. In particular, as in the continuous case, for every matrix  $A \in \mathcal{C}$  there exists a wedge  $W(A)$  such that any  $x \in W(A)$  satisfies (4.3).

It is now easy to construct a sequence  $\{D(n)\} \subset \mathcal{C}$  defining a sequence (4.2) which is unbounded. Choose an initial value  $z_0$  and a matrix  $D(0)$  in  $\mathcal{C}$  such that  $z_0 \in W(D(0))$ . The matrices  $D(n)$  can now be chosen recursively from  $\mathcal{C}$  so that each iterate  $z_n$  lies in  $W(D(n))$  in such a way that the sequence  $\{z_n\}$  diverges.

Although this construction gives the desired example, it seems inelegant and artificial. On the other hand, the examples discussed in Section 3 can be used to obtain a discrete counterpart more directly. Fix  $\epsilon > 0$  and let  $\Phi^n(t)$  be the fundamental solution matrix of (1.1) such that  $\Phi^n(n\epsilon) = Id$ . Define the sequence of matrices  $D(n) = \Phi^n((n+1)\epsilon)$ . Then system (4.2) is the stroboscopic map associated to the continuous system (1.1). Therefore, if  $A(t)$  is chosen as in the Vinograd example, there will be solutions that diverge to infinity. For  $\epsilon$  sufficiently small the  $D(n)$  will have eigenvalues inside the unit circle.

**4.4. Other Extensions.** Following Wu [19, 21] we can derive solutions to a more general class of non-autonomous differential equations which includes equations of the type in (3.2). Let  $A(t)$  be differentiable, and suppose there is a constant matrix  $A_1$  such that

$$A_1 A(t) - A(t) A_1 - A'(t) = [A(t), A_1] = 0,$$

where  $[A(t), A_1]$  is the Lie bracket [1]. We claim that  $e^{-A_1 t} A(t) e^{A_1 t}$  is constant, and so equals  $A(0)$ . This is a consequence of the fact that  $A_1$  is an infinitesimal generator of  $e^{A_1 t}$ . More directly

$$\begin{aligned} \frac{d}{dt}(e^{-A_1 t} A e^{A_1 t}) &= -A_1 e^{-A_1 t} A e^{A_1 t} + e^{-A_1 t} A' e^{A_1 t} + e^{-A_1 t} A e^{A_1 t} A_1 \\ &= -e^{-A_1 t} A_1 A e^{A_1 t} + e^{-A_1 t} (A_1 A - A A_1) e^{A_1 t} + e^{-A_1 t} A A_1 e^{A_1 t} = 0 \end{aligned}$$

Let  $A_2 = A(0) - A_1$ , then  $e^{A_1 t} e^{A_2 t} x_0$  is the general solution to  $x'(t) = A(t)x$ . The proof again follows by defining coordinates in the moving frame  $y = e^{-A_1 t} x$ , so that  $x' = A_1 e^{A_1 t} y + e^{A_1 t} y'$ , and

$$\begin{aligned} y' &= e^{-A_1 t} x' - e^{-A_1 t} A_1 e^{A_1 t} y = e^{-A_1 t} x' - A_1 y \\ &= e^{-A_1 t} A x - A_1 y = (e^{-A_1 t} A e^{A_1 t} - A_1) y = (A(0) - A_1) y = A_2 y \end{aligned}$$

Therefore  $y = e^{A_2 t} y(0)$  and  $x = e^{A_1 t} e^{A_2 t} x(0)$ . The solution of equation (3.2) is obtained by setting  $A(0) = B$  and  $A_1 = G(\omega)$ , so that  $A_2 = B - G(\omega)$ , and  $e^{A_1 t} = R(t, \omega)$ . However, more general matrices  $A_1$  can be chosen as we show in Section 4.4.

As a particular example, we can replace  $G(\omega)$  by

$$H(\omega, b) = \begin{bmatrix} 0 & \frac{-\omega}{b} \\ \omega b & 0 \end{bmatrix}$$

so that  $e^{H(\omega, b)t}$  is an elliptical rotation. The resulting system can be analyzed using the ideas presented in Section 3. In particular, since  $A(t) = e^{H(\omega, b)t} B e^{-H(\omega, b)t}$  is periodic, the stability of the system depends on its Floquet exponents which are the eigenvalues of  $B - H(\omega, b)$  (see Section 4.1). For instance, if  $B \in \mathcal{B}$  is of the form (2.3), solutions to  $x' = A(t)x$  are unstable whenever  $b\omega(\lambda_1 - \lambda_2) > (\lambda_1\lambda_2 + \omega^2) \tan(\delta)$ . Animation 5 on the accompanying webpage shows an unstable solution to a system of this form.

As another example, we can replace  $G(\omega)$  by

$$F(\mu_1, \mu_2) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

In this case,  $A(t) = e^{F(\mu_1, \mu_2)t} B e^{-F(\mu_1, \mu_2)t}$  is not periodic and therefore does not lend itself to the same analysis as the previous examples. The fundamental solution to  $x' = A(t)x$  is  $e^{F(\mu_1, \mu_2)t} e^{(B - F(\mu_1, \mu_2))t}$ . If  $B \in \mathcal{B}$  is of the form (2.3) then

$$A(t) = \begin{bmatrix} \lambda_1 & e^{(\mu_1 - \mu_2)t}(\lambda_2 - \lambda_1) \cot(\delta) \\ 0 & \lambda_2 \end{bmatrix}$$

and solutions to (1.1) are stable whenever  $\mu_1 - \mu_2 > -\lambda_2$ . Animations 6 and 7 on the accompanying webpage show solutions to systems of this form.

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