Network architecture and spatio-temporally symmetric dynamics

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Abstract

We examine the relation between the structure of a network and the spatio-temporally symmetric periodic dynamics it can support. If we are looking for solutions in which no cell is stationary, then we show that only networks in which all cells interact with each other, or which contain a single group of interacting cells which drive the remainder of the network can exhibit such dynamics robustly. These characteristics of network architecture are not captured by the typical statistical quantities used to describe network structure. We illustrate the existence of spatio-temporally periodic solutions through a direct construction using ideas from coupled cell theory and the theory of weakly coupled oscillators, and show that these solutions can be stable in a very large region of parameter space. While we consider only a special type of network behavior, these ideas extend to more general architectures and dynamics.

Key words: Spatio-temporal patterns, coupled cell network
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1 Introduction

The components of a dynamically evolving network are frequently observed to exhibit coherent behavior ranging from synchrony and phase locking [31,27], to reliably recurring complex

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patterns of activity [21,20,22]. Network architecture plays a significant role in determining the types of coherent behavior a network can exhibit [24,25,36,37]. Moreover, numerical and experimental studies have shown that there is frequently a relationship between network architecture, and its function [32,34,18,8].

It is therefore important to understand the relation between network structure and dynamics. While methods of statistical physics are frequently useful in the analysis of large networks with sufficiently coherent (or incoherent) behavior [27,1] they are not as effective in the case of networks of small to intermediate size. Such networks are of particular interest as they appear frequently in isolation [17,11,28], and as functional subunits of a larger network [26,28,38,29].

The main question we address in this article is how network architecture can determine what types of spatiotemporally structured periodic dynamics the network can support. Such solutions have been studied extensively in the limit of weak interaction between the cells [19,25,2], and have been particularly useful in modeling rhythmogenesis in neural circuits controlling motor behavior [26,28,33]. Our goal is to investigate the relations between network structure and the existence and stability of these solutions. We are interested in stability to both perturbations in phase space (*dynamical* stability), and perturbations of the vector field (*structural* stability). Moreover, we also consider the stability of solutions under the more restricted class of perturbations that respect network architecture. In the interest of simplifying the statement and proof of the results we consider only the case in which all cells in the network oscillate, although the results can be extended to the more general case.

Although we consider the stability of a particular type of network dynamics, the main ideas are applicable in much more general situations, and can be used to demonstrate that only networks with a particular architecture can support stable attractors with no stationary cells. We note that these features are not captured by the statistical measures typically used to characterize network structure such as the degree distribution (See also [3]).

The existence of stable spatiotemporally symmetric solutions is shown by direct construction using equations of the type that can be obtained by averaging in networks of weakly interacting elements [19]. Since these solutions are shown to be stable in a very large region of parameter space, it is likely that they are easy to observe. The particular solutions we construct can be viewed as generalizations of ones studied previously [10,2].

Coupled cell theory provides a natural mathematical framework to study these questions [14]. In this theory a *cell* refers to a system of differential equations and a *coupled cell system* is a collection of cells that are coupled together. The *network architecture* is a directed graph that indicates which cells are identical, which cells are coupled to which, and which couplings are identical. Therefore a coupled cell system is simply a class of differential equations that model a network with a given architecture. A set of equations that are compatible with a given network architecture will be called *admissible*.

In the next two sections we motivate the discussion and provide a heuristic statement of the main results, as well as a precise definitions of spatiotemporally symmetric solutions. We review the results for the existence of spatiotemporally symmetric solutions of equivariant equations in Section 4. Coupled cell theory is reviewed in Section 5, and is used to state our main results precisely in Section 6. The results and the main idea of the construction are illustrated in several examples in Section 7. The remainder of the paper is devoted to the proof of the main results using an explicit construction. All steps in the construction are illustrated using a simple example that accompanies the proof.

2 A heuristic statement of the results

To illustrate the types of solutions that we will consider and the relation between structural and dynamical stability and network architecture, consider the three-cell networks in Figure 1.

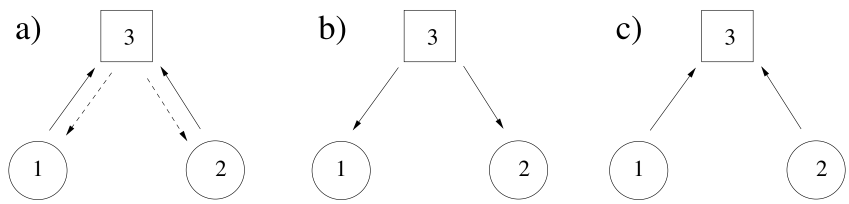

Fig. 1. These networks support different types of nontrivial, structurally stable periodic orbits.

Each of these networks is symmetric under reflection in the vertical axis going through cell 3. As we will see in Section 3, certain periodic solutions to the model equations inherit the symmetries of the network. In particular, if the individual cells in these networks are assumed to have more than two degrees of freedom, then all three networks can support both periodic solutions in which cells 1 and 2 are one half period out of phase, while cell 3 oscillates at twice the frequency (see Figure 2), and solutions in which cells 1 and 2 are synchronous, while cell 3 oscillates with an independent frequency .

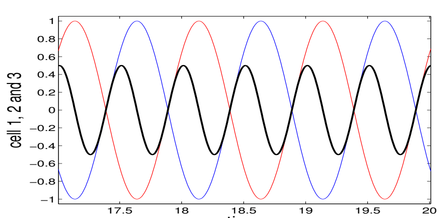

Fig. 2. The networks shown in Figure 1 can support periodic solutions in which cells 1 and 2 (large amplitudes) are one half period out of phase, while cell 3 (small amplitude) oscillates at twice their frequency. For networks a) and b) this solution can be hyperbolic and attracting [12].

One of the main questions we ask in this paper is when can such solutions be stable in the dynamical and structural sense. The situation is relatively simple in the case of networks a) and b) in Figure 1: In network a) cells interact with one another directly or through intermediary cells (the network is *strongly connected*). While in network b) not all cells interact, there is still a single "root" cell that drives the remainder of the network. In fact, both networks a) and b) can support spatiotemporally symmetric solutions which are hyperbolic, and dynamically and structurally stable.

Network c) is quite different: There are no cells driving both cells 1 and 2. Moreover, perturbations applied to cell 1 cannot be felt by cell 2, and similarly perturbations to cell 2 cannot be felt by cell 1. It turns out that the periodic solutions in this system *can be neither hyperbolic nor attracting*, unless either cell 1 or cell 2 are not oscillating. Indeed, the differential equations corresponding to network c) have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1) \\ \dot{x}_2 &= f(x_2) \\ \dot{x}_3 &= g(x_3, x_1, x_2) \end{aligned} \tag{1}$$

and thus the following Remark applies:

Remark 1 *If the differential equation $x' = F(x)$, $x = (x_1, x_2, x_3)$, given by*

$$\begin{aligned} \dot{x}_1 &= f(x_1) \\ \dot{x}_2 &= g(x_2) \\ \dot{x}_3 &= h(x_3, x_1, x_2) \end{aligned} \tag{2}$$

has a hyperbolic *periodic solution $x(t) = (x_1(t), x_2(t), x_3(t))$, then either $x_1(t)$ or $x_2(t)$ is constant. That is, not more than one root can oscillate in a hyperbolic periodic solution.*

Indeed, the linear equation determining the Floquet multipliers is

$$\dot{\xi}(t) = (d_{x(t)}F)\xi(t)$$

and the matrix $d_{x(t)}F$ is lower triangular. Therefore the evolution operator is also lower triangular, and has two eigenvalues equal to 1. These eigenvalues correspond to the two diagonal blocks describing the evolution in the spaces of the variables x_1 and x_2.

Note that one can also give a direct argument, which is simple for an attracting *periodic solution of (2): if both $x_1(t)$ or $x_2(t)$ were oscillating, then the solution $x^\varepsilon(t) = (x_1(t), x_2(t+\varepsilon), x_3^\varepsilon(t))$ with initial condition $(x_1(0), x_2(\varepsilon), x_3(0))$ will never converge to $x(t)$, despite starting as close to it as we desire. To show that there are no* hyperbolic *solutions in which both x_1 and x_2 oscillate, consider the vector from $(x_1(0), x_2(0), x_3(0))$ to $(x_1(0), x_2(\varepsilon), x_3(0))$ and notice that this vector stays of (approximatively) constant length under both forward and backward evolution. But for small ε this vector is not tangent to the solution $x(t)$, hence it is a second "direction" that is neutral for the evolution, proving that the solution is not hyperbolic.*

On the other hand, network c) admits solutions for which cells 1 and 2 *evolve in perfect synchrony* since the manifold defined by $x_1 = x_2$ is invariant. This observation motivates a more specific question: Can a periodic solution for which cells 1 and 2 oscillate and evolve synchronously be stable under perturbations that preserve the network architecture, that is perturbations that preserve the form of the equations given in (1)? We will show that the answer to this question is positive.

These observations can be used to motivate a heuristic statement of our findings.

"Theorem:" Consider a symmetric coupled cell system composed of cells whose internal dynamics is at least two-dimensional. There is an admissible vector field supporting periodic, spatio-temporally symmetric solutions determined by network symmetries. In these solutions all cells oscillate. Moreover

(1) If the network is strongly connected (all cells influence each other, as in network a), or there is a cell or a strongly connected group of cells that drives all the other cells (as in network b), then such solutions can be structurally and dynamically stable.
(2) Otherwise, solutions cannot be dynamically or structurally stable unless some of the cells are not oscillating. However, under appropriate conditions, the oscillating solutions persist under perturbations that respect network architecture.

While we concentrate on the stability of spatiotemporally symmetric solutions, the conditions for hyperbolicity that we discuss are applicable to more general attractors. In particular, as discussed in Remark 1, networks of the type illustrated in Figure 1 c) with multiple "roots" cannot support hyperbolic attractors in which all cells oscillate.

"Theorem:" Only networks that are strongly connected, or have a single strongly connected group of cells that drives all the rest, can support attracting or hyperbolic sets for which each cell is not stationary.

We also note that the conditions for stability we consider are not directly related to the statistical properties frequently used to describe network architecture, such as degree distribution and local connectivity. These results suggests that such statistical measures cannot fully characterize the dynamical behavior of a network, regardless of size (see also [3]).

3 Equivariant equations and spatiotemporally symmetric solutions

Our main goal in this section is to review the theory of equivariant differential equations in the context of networks, and show that spatiotemporally symmetric solutions of a certain type can be naturally expected in such systems.

Consider again the networks shown in Figure 1. These networks are symmetric under the permutation of cells 1 and 2 in the sense that if these two cells are physically interchanged along with their connections, the architecture of the networks is unaltered.

This symmetry must be reflected in the equations that model the network. To make this relation precise we note that the action of interchanging the two cells in the network can be viewed as an action of the group $\mathbb{Z}_2$ generated by the permutation $1 \to 2$ and $2 \to 1$ (using standard notation we denote this perturbation by $(1\ 2)$). More generally, we can define the action of a group of permutations G on Euclidean space in the following way: Suppose that $x \in \mathbb{R}^{mK}$ so that $x = (x_1, x_2, \ldots, x_K)$ where $x_i \in \mathbb{R}^m$. An element γ of the permutation group on K symbols acts on $\mathbb{R}^{mK}$ by $\gamma(x_1, x_2, \ldots, x_K) = (x_{\gamma^{-1}(1)}, x_{\gamma^{-1}(2)}, \ldots, x_{\gamma^{-1}(K)})$. The following statement defines equivariance under general group actions, although we consider only the described action by permutations.

Definition 2 *Let G act on $\mathbb{R}^{mK}$ and let $F : \mathbb{R}^{mK} \to \mathbb{R}^{mK}$. Then F is G-*equivariant *if $F(\gamma x) = \gamma F(x)$ for all $\gamma \in G, x \in \mathbb{R}^{mK}$.*

Indeed, equations (1) are $\mathbb{Z}_2$-equivariant, as can be checked directly. It is in this sense that the symmetry of the network is reflected in the equations that are used to model it. Note that while all three networks depicted in Figure 1 are $\mathbb{Z}_2$ symmetric, their structure is quite different. Indeed, symmetries capture only some aspects of network structure.

Symmetries of admissible differential equations are reflected directly in the structure of their solutions. If F is G-equivariant, and $x(t)$ is a solution of $x' = F(x)$, then $y(t) = \gamma x(t)$ is the solution of the same differential equation for any $\gamma \in G$.

The periodic solutions *fixed* by some elements of the symmetry group G form a special class and reflect some of the symmetries of the network itself.

Definition 3 *Let $x(t)$ be a periodic solution of a G-equivariant differential equation $x' = f(x)$. The subgroup $K \subset G$ of symmetries that fix $x(t)$* pointwise *is the group of spatial symmetries of $x(t)$. The subgroup $H \subset G$ that fixes $x(t)$* orbitwise *is the group of* spatio-temporal *symmetries. Therefore*

$$K := \{g \in G \mid g(x(t)) = x(t) \text{ for all } t\} \tag{3}$$
$$H := \{g \in G \mid g(\{x(t)\}_t) = \{x(t)\}_t\}, \tag{4}$$

This definition implies that if $\gamma \in K$ and $\gamma^{-1}(i) = j$, then a periodic solution $x(t)$ with spatial symmetries K must satisfy $x_i(t) = x_j(t)$ for all $t \in \mathbb{R}$. In other words, all the cells that lie on an orbit of K are synchronous. Note that, by uniqueness of solutions and the fact that G-symmetries take solutions into solutions, for each $\gamma \in H$ there is a unique $s \in \mathbb{R}/\mathbb{Z}$ such that $\gamma(x(t)) = x(t + sT)$ for all $t \in \mathbb{R}$, where T is the period of the solution $x(t)$. Therefore a solution $x(t)$ with spatio-temporal symmetries H satisfies $x_i(t + \theta_{i,j}) = x_j(t)$ whenever $\gamma^{-1}(i) = j$ with $\gamma \in H$. Therefore, while K determines which cells evolve synchronously, H determines which cells evolve identically up to a phase shift.

Example 4 *Consider the network depicted in Figure 3, and the associated network equations*

$$\begin{array}{lll} \dot{x}_1 = f(x_1, x_3, x_4) & \dot{x}_2 = f(x_2, x_1, x_5) & \dot{x}_3 = f(x_3, x_2, x_6) \\ \dot{x}_4 = f(x_4, x_6, x_1) & \dot{x}_5 = f(x_5, x_4, x_2) & \dot{x}_6 = f(x_6, x_5, x_3). \end{array} \tag{5}$$

This network is symmetric under the action of $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ *where the generator of* $\mathbb{Z}_2 \times \{\mathbf{1}\}$ *is* $(1\ 4)(2\ 5)(3\ 6)$ *and the generator of* $\{\mathbf{1}\} \times \mathbb{Z}_3$ *is* $(1\ 2\ 3)(4\ 5\ 6)$*. If we choose* K *to be the subgroup generated by* $(1\ 4)(2\ 5)(3\ 6)$*, and choose* $H = G$*, then the pairs of cells* $(1, 4), (2, 5)$ *and* $(3, 6)$ *evolve synchronously, while each cell in the triplets* $(1, 2, 3)$ *and* $(4, 5, 6)$ *is one third of a period out of phase with the preceding one (see Figure 3).*

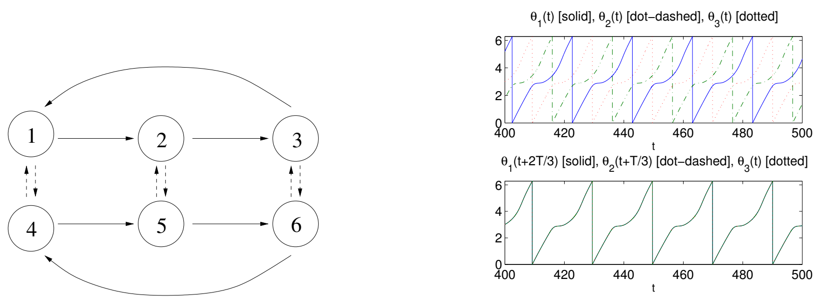

Fig. 3. This $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetric network supports several types of spatio-temporally symmetric solutions. The figure on the right, top shows the timeseries of the phases of cells 1, 2 and 3 in a network of reduced Hodgkin-Huxley equations [13]. Cells 4, 5 and 6 evolve synchronously with cells 1, 2, and 3, respectively. At the bottom, the shift of the timeseries by one third of the period shows that all cells evolve identically up to a phase shift.

4 Results for equivariant differential equations: The H/K Theorem

The question we want to address is the following: Given a network whose symmetries are described by a group G, what are the pairs $K \subset H$ in G that describe possible spatio-temporally symmetric solutions in the system? In other words, what spatio-temporally symmetric periodic solutions can the network support? Moreover, are there equations which are compatible with the structure of the network for which such periodic solutions are hyperbolic, or even attracting?

These questions have been completely answered in the case of G-equivariant systems by the following theorem

Theorem 5 ([6]) *Assume that a* G*-equivariant ordinary differential equation admits a periodic solution with spatial symmetry* K *and spatio-temporal symmetry* H*. Then:*

(a) H/K *is cyclic;*
(b) K *is an isotropy subgroup;*
(c) $\dim \mathrm{Fix}(K) \geq 2$*, and if* $\dim \mathrm{Fix}(K) = 2$ *then either* $K = H$ *or* H *is the normalizer of* K *in* G*;*
(d) H *fixes a component of* $\mathrm{Fix}(K) \setminus (\bigcup_{\gamma \notin K} \mathrm{Fix}(\gamma))$*.*

Conversely, if conditions (a)-(d) hold there exists a G*-equivariant system of ordinary differential equations that admit an attracting periodic solution with* (H, K) *as symmetries.*

However, the symmetries of a network do not fully describe its structure, and the results for equivariant systems do not carry over to the network setting as the following example shows.

Example 6 *Consider the two networks shown in Figure 4, and assume that each cell evolves in* $\mathbb{S}^1$, i.e. *both are phase oscillator networks. Note that for both networks* $G = \mathbb{Z}_2$*, since both are symmetric under reflection.*

In the case of the network on the left, it is easy to construct examples in which $H = G = \mathbb{Z}_2$ *and* K *consists of the identity so that the two cells oscillate one half period out of phase [2]. Indeed, such examples occur frequently in practice.*

On the other hand, a solution with $H = G = \mathbb{Z}_2$ *cannot exist if phase oscillators form the network shown on the right. In this case* H *is generated by* $(1\ 2)(3)(4)$ *which means that cells 1 and 2 oscillate one half period out of phase, while cell 4 (and cell 3) oscillates at one half period out of phase* with itself. *This implies that cell 4 oscillates at twice the frequency of cells 1 and 2. However, it is easy to check that the subspace defined by* $x_1 = x_4$ *is invariant. It can be shown that any solution with spatiotemporal symmetry* H *must cross this subspace [12], a contradiction.*

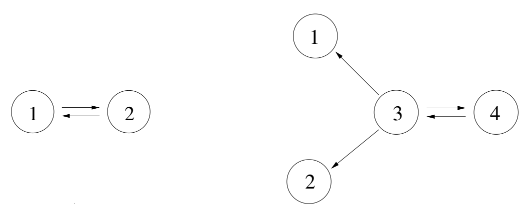

Fig. 4. Both of these networks are $\mathbb{Z}_2$-symmetric. However, if these are networks of phase oscillators, then only the left network can support a solution with $H = \mathbb{Z}_2$.

The goal of the construction developed in the remainder of this manuscript is to show that, if the internal phase spaces of the cells comprising the network is at least two dimensional, then any network with symmetries given by G can support spatio-temporally symmetric solutions with symmetries $K \subset H \subset G$ satisfying easily verifiable conditions. The case of one dimensional internal dynamics is more delicate and will be considered elsewhere. To state this result precisely we need to review parts of the theory of coupled cell systems [14,16].

5 Review of coupled cell theory

In this section we review a few definitions of coupled cell networks that permit multiple arrows and self-couplings following [16], and introduce notation that will be used subsequently.

5.1 Coupled cell networks and admissible vector fields.

The following is a precise definition of what is meant by a coupled cell network.

Definition 7 ([16]) A *coupled cell network* $\mathcal{G}$ consists of:

(a) A finite set $\mathcal{C} = \{1, \ldots, N\}$ of *nodes* or *cells*.
(b) An equivalence relation $\sim_C$ on cells in $\mathcal{C}$.
 The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.
(c) A finite set $\mathcal{E}$ of *edges* or *arrows*.
(d) An equivalence relation $\sim_E$ on edges in $\mathcal{E}$.
 The *type* or *coupling label* of edge e is the $\sim_E$-equivalence class $[e]_E$ of e.
(e) Two maps $\mathcal{H} : \mathcal{E} \to \mathcal{C}$ and $\mathcal{T} : \mathcal{E} \to \mathcal{C}$.
 For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e.

We also require a *consistency condition*:

(f) Equivalent arrows have equivalent tails and heads. That is, if $e_1, e_2 \in \mathcal{E}$ and $e_1 \sim_E e_2$, then

$$\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2) \qquad \mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$$

Note that self-coupling is permitted, since it is possible that $\mathcal{H}(e) = \mathcal{T}(e)$. Multiple arrows between two cells are also permitted, since it is possible to have $\mathcal{H}(e_1) = \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) = \mathcal{T}(e_2)$ for $e_1 \neq e_2$.

To every coupled cell network there corresponds a class of vector fields that are "compatible" with the labeled graph structure of the network. Such vector fields are said to be *admissible* for the given coupled cell network. A system of admissible differential equations will also be called a coupled cell system. For a precise definition of admissibility we refer the reader to [16].

We point out that if two cells are $\sim_{\mathcal{C}}$-equivalent, $c \sim_{\mathcal{C}} d$, then the phase space of their internal variables, x_c and x_d, are canonically identified.

5.2 Input equivalence

We say that two cells c and d are input equivalent, denoted $c \sim_{\mathcal{I}} d$, if there is an edge-type preserving bijection between the edges whose head is c and those whose head is d. If no edges end in c and d, we say that $c \sim_{\mathcal{I}} d$ iff $c \sim_{\mathcal{C}} d$. By property (f) of Definition 7, $c \sim_{\mathcal{I}} d$ implies that $c \sim_{\mathcal{C}} d$.

For an admissible vector field, components corresponding to input equivalent cells are described by the same function (but with possibly different inputs).

Example 8 *Consider the coupled cell system (network) in Figure 1c). The set* $\mathcal{C} = \{1, 2, 3\}$ *represents the three cells in the network, and* $1 \sim_C 2$*. The set of edges* $\mathcal{E} = \{e_1, e_2\}$ *consists of the two edges from cells 1 and 2 to cell 3 and* $e_1 \sim_E e_2$*, which indicates that these two edges represent couplings of equal type. The tail and head maps are* $\mathcal{T}(e_1) = 1$*,* $\mathcal{T}(e_2) = 2$*,* *and $\mathcal{H}(e_1) = \mathcal{H}(e_2) = 3$. Note that condition (f) of Definition 7 is satisfied. The admissible equations are those given by* (1). *Cells 1 and 2 are input equivalent for all networks in Figure 1.*

5.3 Quotient networks

We describe here the quotient network induced on the fixed-point space $\mathrm{Fix}(K)$ of a group K of symmetries of the network. For the general case, see [16].

Definition 9 *By a* symmetry of a coupled cell network $\mathcal{G}$ *we mean a cell-type preserving permutation κ of the cells which extends to an edge-type preserving permutation $\tilde{\kappa}$ of the edges, compatible with the head and tail maps (that is, $\tilde{\kappa}(e) \sim_{\mathcal{E}} e$ and $\mathcal{T}(\tilde{\kappa}(e)) = \kappa(\mathcal{T}(e))$, $\mathcal{H}(\tilde{\kappa}(e)) = \kappa(\mathcal{H}(e))$). In particular, if $\kappa(c) = d$ then $c \sim_{\mathcal{I}} d$, and one can check that κ is a symmetry of any $\mathcal{G}$-admissible vector field.*

By a group of symmetries of $\mathcal{G}$ *we mean a group K of permutations of the cells whose elements are symmetries of $\mathcal{G}$.*

The above permutations κ form a group G. This is exactly the group G of symmetries of the admissible equations for the coupled cell system.

Let K be a group of symmetries of $\mathcal{G}$. Then each orbit of K consist of input-equivalent cells. Define its fixed-point space by

$$\begin{aligned}\mathrm{Fix}(K) &:= \{x_c = x_d \mid \text{ there is } \kappa \in K \text{ such that } \kappa(c) = d\} \\ &= \{x \in \mathcal{P} \mid \kappa(x) = x \text{ for all } \kappa \in K\}\end{aligned}$$

where x_c is the internal variable of the cell c and $\mathcal{P}$ stands for the total space, of $(x_c)_{c \in \mathcal{C}}$.

The *quotient network* $\mathcal{G}/K$ is constructed as follows. The cells of $\mathcal{G}/K$ are the equivalence classes $\mathcal{C}/K$, that is, the cell-orbits of K. Pick a representative $\hat{c} \in K(c)$ from each orbit $K(c) \in \mathcal{C}/K$. The edges of $\mathcal{G}/K$ are given by the edges whose head is one of these cells $\hat{c}$, and they are "attached" as follows: if $\mathcal{H}(e) = \hat{c}$ and $\mathcal{T}(e) = d$, then in $\mathcal{G}/K$ this becomes an edge with head $K(c)$ and tail $K(d)$, and inherits the same edge-type as e. [Since K acts by network symmetries, a different representative from a K-orbit of cells would give the same edges in $\mathcal{G}/K$.]

It is easy to see that $\mathrm{Fix}(K) \cong \{(x_{\hat{c}})_{\hat{c}} \mid \hat{c} \text{ the representative of } K(c), c \in \mathcal{C}\}$, hence $\mathcal{G}/K$ is indeed realized on $\mathrm{Fix}(K)$.

A $\mathcal{G}$-admissible vector field on $\mathcal{P}$ restricts to a $\mathcal{G}/K$-admissible vector field on $\mathrm{Fix}(K)$. Conversely, any $\mathcal{G}/K$-admissible vector field on $\mathrm{Fix}(K)$ can be lifted to a $\mathcal{G}$-admissible vector field on $\mathcal{P}$. Instead of describing the general construction [16, Theorem 5.2], we will discuss in §8.11 only how to lift the vector fields we are interested in. For more details, see [16, §5].

Example 10 *Consider the coupled cell in Figure 3 discussed in example 4. If K is the group generated by* $(1\ 4)(2\ 5)(3\ 6)$*, then* $\mathrm{Fix}(K)$ *is defined by the equalities $x_1 = x_4, x_2 = x_5$, and $x_3 = x_6$. The quotient network is shown in Figure 5, and the equations restricted to* $\mathrm{Fix}(K)$ *are $\mathcal{G}/K$-admissible and take the form*

$$\dot{x}_1 = f(x_1, x_3, x_1) \qquad \dot{x}_2 = f(x_2, x_1, x_2) \qquad \dot{x}_3 = f(x_3, x_2, x_3). \tag{6}$$

Note that the arrows between the pairs $(1, 4), (2, 5)$ and $(3, 6)$ project to self-couplings in

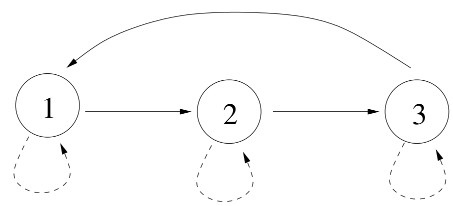

Fig. 5. The quotient system obtained from the network shown in Figure 3 with $K = \langle(1\ 4)(2\ 5)(3\ 6)\rangle$.

this network. This is reflected in the fact that x_i enters as an argument twice on the right hand side of each equation in (6). The resulting network is symmetric under the action of $G/K = \mathbb{Z}_3$.

6 Statement of the main results

Given a network architecture, what are the possible $K \subset H$ pairs, that is, what are the possible spatio-temporally symmetric periodic solutions that the network can support? Moreover, are there admissible equations for which such periodic solutions are attracting? As illustrated in Example 6, different network architectures may have admissible equations that are G-equivariant for the same group G, but may not support the same types of spatio-temporally symmetric solutions. It is therefore necessary to also consider network architecture, in addition to network symmetries to fully answer these questions.

We make two assumptions about the network dynamics: 1) the internal dynamics of each cell is at least two-dimensional, and 2) all cells in the network are oscillating:

Definition 11 *A solution $x(t)$ of an admissible ODE is called* fully oscillatory *if* all *cells in the coupled cell systems are oscillating, that is $x_i(t)$ is non-constant for any cell i in the network.*

The first assumption is made because a network of one dimensional cells supports very different types of spatiotemporally symmetric solutions than networks of higher dimensional cells. In particular, the case of phase oscillators will be considered elsewhere [12]. The second assumption is made to simplify the statement and proof of the theorems. While the results can be extended to the non-oscillatory case, this necessarily involves more intricate conditions about the action of the group G on the network.

As noted in the previous section, a coupled cell system with symmetries G is G-equivariant. Hence, the conditions of Theorem 5 are also necessary for the existence of spatio-temporally symmetric solutions in coupled cell systems. We therefore only need to provide sufficient conditions.

The result for strongly connected networks is stated in Theorem 13, and the general case in Theorem 17: for fully oscillatory hyperbolic solutions to exist there must be one strongly connected component that forces (maybe indirectly) each cell of the network. We thus obtain a complete characterizations of the types of *fully oscillatory hyperbolic* spatio-temporal patterns a particular network architecture can support.

We point out that under assumption 1), conditions (c) and (d) of Theorem 5 are always satisfied. Indeed, since the group G acts by permutation on the cells of the network and the internal dynamics of each cell is at least two-dimensional, the spaces $\mathrm{Fix}(\gamma)$ are of codimension two or higher for any nontrivial $\gamma \in G$. Therefore $\mathrm{Fix}(K) \setminus (\bigcup_{\gamma \notin K} \mathrm{Fix}(\gamma))$ is connected, which is condition (d). Similarly, $\dim \mathrm{Fix}(K) = 2$ implies that all cells are synchronous on the periodic solution of interest. Therefore $H = K$, and thus condition (c) holds as well.

To state our results it is convenient to consider the pattern of forcing between cells. Define the directed graph $\Gamma = \Gamma(\mathcal{G})$ derived from the coupled cell system $\mathcal{G}$ by placing each cell at a node, and connecting the nodes corresponding to cells c and d with a directed edge from c to d whenever there exists an edge e such that $\mathcal{H}(e) = c$ and $\mathcal{T}(e) = d$.

We remind the reader of a basic notion from graph theory:

Definition 12 *A directed graph is* strongly connected *if for any pair of nodes c and d there exists a directed path from c to d.*

Thus, if the directed graph $\Gamma(\mathcal{G})$ is strongly connected, then any cell is forced by any other cell, although this forcing may not be direct. The following is a special case of the main result, and provides sufficient conditions for the existence of spatiotemporally symmetric periodic solutions:

Theorem 13 *Assume that the internal dynamics of each cell in a coupled cell system $\mathcal{G}$ is at least two-dimensional, and that the associated directed graph Γ is strongly connected. Then, for any pair $K \subset H$ where K is an isotropy group, normal in H, with H/K cyclic, there is an admissible vector field supporting a* dynamically and structurally stable*, periodic, fully oscillatory solution with spatio-temporal symmetries given by $K \subset H$.*

A slight extension of the condition that the graph Γ be strongly connected is also necessary for the existence of such solutions. A similar condition has been examined in a different context in [30].

Definition 14 *A* strongly connected component *of a directed graph Γ is a maximal subgraph of Γ such that for every pair of nodes c and d, there is a directed path from c to d and a* *directed path from d to c.*

The strongly connected components partition the nodes into disjoint classes. We denote the strongly connected component containing a cell c by $\langle c\rangle$.

Construct the directed graph $\tilde{\Gamma} = \tilde{\Gamma}(\mathcal{G})$ in the following way: Each strongly connected component of Γ is a node in the graph $\tilde{\Gamma}$. If Γ contains a directed edge from node c in one strongly connected component to node d in another strongly connected component, then there is a directed edge from $\langle c\rangle$ to $\langle d\rangle$ in $\tilde{\Gamma}$.

The directed graph $\tilde{\Gamma}$ shows how the strongly connected components of the graph Γ interact. See Figure 7 for an example. The graph $\tilde{\Gamma}$ has no cycles (loops):

Proposition 15 *The graph $\tilde{\Gamma}$ is an acyclic, directed graph.*

Proof: Suppose that $\tilde{\Gamma}$ contains a directed cycle, so that there are two cells c and d belonging to two different connected components of Γ and a directed path from $\langle c\rangle$ to $\langle d\rangle$ and from $\langle d\rangle$ to $\langle c\rangle$. Since the components $\langle c\rangle$ and $\langle d\rangle$ are strongly connected, this implies that there is a directed path from c to d and a directed path from d to c in Γ. Therefore c and d must belong to the same connected component, a contradiction. $\square$

Definition 16 *A node which is not the endpoint of any directed arrow in a directed graph is called a* root node.

We can now state the sufficient and necessary conditions for the existence of fully oscillatory periodic solutions in a given network.

Theorem 17 *Assume that the internal dynamics of each cell in a coupled cell system $\mathcal{G}$ is at least two-dimensional. Then, for any pair $K \subset H$ where K is an isotropy group, normal in H, with H/K cyclic, there is an admissible vector field supporting a hyperbolic, fully oscillatory, periodic solution with spatio-temporal symmetries given by $K \subset H$* if and only if *the derived directed graph $\tilde{\Gamma}$ has* only one root node.

If these conditions are satisfied the solution can be exponentially attracting, *and therefore both dynamically and structurally stable.*

Remark 18 (Structural Stability) *In addition, under small H-equivariant perturbations of the vector field, hyperbolic solution with spatio-temporal symmetries $K \subset H$ are perturbed to solutions with the same symmetries [15].*

The proof of Theorem 13 will be given in Section 9, after a class of admissible vector fields is described in Section 8. Theorem 17 is proven in Section 10.

The next theorem provides the more general conditions under which spatio-temporal solutions can be stable under perturbations that preserve the network architecture.

Theorem 19 (General Existence) *Assume that the internal dynamics of each cell in a* *coupled cell system $\mathcal{G}$ is at least two-dimensional. Then, for any pair $K \subset H$ where K is an isotropy group, normal in H, with H/K cyclic, there is an admissible vector field supporting a periodic, fully oscillatory solution with spatio-temporal symmetries given by $K \subset H$.*

*If the "*only one root node*" condition is satisfied for the coupled cell system $\mathcal{G}/K$ determined by $\mathcal{G}$ on* $\mathrm{Fix}(K)$*, then this solution can be made exponentially attracting inside* $\mathrm{Fix}(K)$*. Such a solution will persist under perturbations of the vector field that preserve the network architecture, and, more generally, under all H-equivariant perturbations.*

7 Examples of spatio-temporally symmetric solutions

We start by presenting examples of coupled cell systems that support particular spatio-temporally symmetric solutions. In these examples network architecture and the structure of the solutions are closely related. We construct the solutions with the desired properties explicitly to illustrate the strategy that will be used in the proof of Theorem 13.

7.1 Ring of oscillators

The differential equations for a unidirectionally coupled ring of identical cells have the general form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_N) \\ \dot{x}_2 &= f(x_2, x_1) \\ &\vdots \\ \dot{x}_N &= f(x_N, x_{N-1}). \end{aligned} \tag{7}$$

Thus the set of equations is $\mathbb{Z}_N$-equivariant where $\mathbb{Z}_N$ permutes the variables x_i and is generated by the element that acts by moving cell i to cell $i+1 \mod N$. Models of this type can capture the essential dynamics of some biological central pattern generators (CPGs) [35,7,9]. A unidirectional ring of 3 cells is shown in Figure 6a).

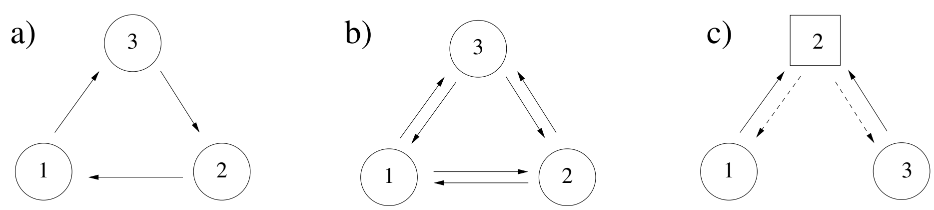

Fig. 6. The network architectures of the examples discussed in Section 7.

We will construct a system of differential equations that supports a periodic solution with spatio-temporal symmetry $H = G = \mathbb{Z}_N$ and spatial symmetry $K = \{1\}$. Such solutions represent a discrete rotating wave in the ring. Each of the cells in the ring follows the same periodic pattern, but is $\pm j\pi/N$ out of phase with its neighbors where j and N are mutually prime. Such solutions occur naturally in a number of examples, including Hopf bifurcations of systems with symmetries [15], and are frequently found to be stable [4,10].

We assume that $x_i \in \mathbb{R}^2$, and continue the construction in polar coordinates (r_i, θ_i). The dynamics of the radial variable is defined by

$$r_i' = (1 - r_i)r_i. \tag{8}$$

Therefore all solutions away from the origin approach the torus $\mathbb{T}^* = \{(r_i, \theta_i) | r_i = 1\}$ asymptotically. Moreover, the cells are assumed to interact only through the angular variables θ_i. Although this assumption may appear rather special, it is satisfied approximately if the coupling between the cells is weak [5,25].

If the phase variables satisfy the differential equation

$$\dot{\theta}_i = \omega + \sin(\theta_{i-1} - \theta_i + \frac{j\pi}{N}), \tag{9}$$

then it is easy to check that $\theta_i(t) = \omega t + (ji\pi)/N$ is a solution. Therefore the system of equations (8-9) supports a solution with the desired spatio-temporal symmetries.

We next determine the stability of the solution we have constructed. Since the torus $\mathbb{T}^*$ is asymptotically stable, it is sufficient to examine the dynamics restriction to the torus $\mathbb{T}^*$ which are determined by equations (9). If we let $\Delta(\theta_i, \theta_{i-1}) = \cos(\theta_{i-1} - \theta_i + j\pi/N)$, the Jacobian of the equations restricted to $\mathbb{T}^*$ takes the form

$$\begin{pmatrix} -\Delta(\theta_1, \theta_N) & 0 & \dots & 0 & \Delta(\theta_1, \theta_N) \\ \Delta(\theta_2, \theta_1) & -\Delta(\theta_2, \theta_1) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Delta(\theta_N, \theta_{N-1}) & -\Delta(\theta_N, \theta_{N-1}) \end{pmatrix}.$$

Evaluating this Jacobian on the orbit $\theta_i(t) = \omega t + (ji\pi)/N$ gives the constant, circulant matrix

$$\begin{pmatrix} -1 & 0 & \dots & 0 & 1 \\ 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\lambda_j = -1 + \exp(2\pi j/N),$$

where $\lambda_0 = 0$ is the Floquet multiplier corresponding to the direction of the flow. The remaining eigenvalues have negative real part, showing that the constructed periodic orbit is stable. In Section 9 we show that the same conclusion can be reached directly without computing the eigenvalues.

The equations for a unidirectionally coupled ring with nearest and next nearest neighbor coupling have the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_N, x_{N-1})\\ \dot{x}_2 &= f(x_2, x_1, x_N)\\ &\vdots\\ \dot{x}_N &= f(x_N, x_{N-1}, x_{N-2}).\end{aligned} \tag{10}$$

If $N > 3$ then these equations are also $\mathbb{Z}_N$-equivariant, but the two rings have very different structure. The construction described above can be used to produce stable solutions in this example as well.

7.2 Multifrequency solutions in bidirectionally coupled rings

In our next example we discuss the construction of a multifrequency solution in a ring of 3 bidirectionally coupled cells shown in Figure 6b). The construction can be extended directly to rings of an arbitrary odd number of bidirectionally coupled cells. The admissible differential equations have the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2, x_3)\\ \dot{x}_2 &= f(x_2, x_3, x_1)\\ \dot{x}_3 &= f(x_3, x_1, x_2)\end{aligned} \tag{11}$$

with the additional restriction that $f(x_2, x_1, x_3) = f(x_2, x_3, x_1)$. The symmetries of the system are given by $G = \mathbb{D}_3$. We let $K = \{1\}$ and H be the subgroup of G that interchanges cells 1 and 2, and fixes cell 3, so that $H = \mathbb{Z}_2$. The T-periodic solution with the desired spatio-temporal symmetries must satisfy $x_1(t) = x_2(t + T/2)$ and $x_3(t) = x_3(t + T/2)$. In other words, the projections $x_1(t)$ and $x_2(t)$ must be equal, up to a shift in time. However only the frequencies, but not the amplitudes of x_3 and x_1 are related. In the simplest case cell 3 evolves at twice the frequency of cells 1 and 2.

We again construct a system of admissible differential equations in polar coordinates that support a periodic solution with the desired characteristics. Let $f(r)$ be a cubic polynomial with roots $1, 2,$ and 3. We also require that $r = 1$ and $r = 3$ are hyperbolically stable solutions of the differential equation

$$r_i' = f(r_i) r_i \qquad i = 1, 2, 3. \tag{12}$$

It follows that the torus $\mathbb{T}^* = \{r_1 = r_2 = 1, r_3 = 3\}$ is normally hyperbolic and stable in $\mathbb{R}^6$, regardless of the dynamics of the phases.

We assume that the phases satisfy the following differential equations

$$\begin{aligned} \dot{\theta}_1 &= \omega(r_1) + \sin\left[a(r_1)\theta_1 + b(r_2)\theta_2 + b(r_3)\theta_3\right] \\ \dot{\theta}_2 &= \omega(r_2) + \sin\left[a(r_2)\theta_2 + b(r_3)\theta_3 + b(r_1)\theta_1\right] \\ \dot{\theta}_3 &= \omega(r_3) + \sin\left[a(r_3)\theta_3 + b(r_1)\theta_1 + b(r_2)\theta_2\right]. \end{aligned} \tag{13}$$

Note that equations (12-13) are admissible. Choose a smooth function $\omega(r)$ such that $\omega(1) = 1$ and $\omega(3) = 2$.

For the solutions with the desired spatio-temporal symmetries the phase variables need to satisfy

$$\theta_1(t) = t + \gamma_1, \theta_2(t) = t + \pi + \gamma_1, \theta_3 = 2t + \gamma_2. \tag{14}$$

Since γ_1 and γ_2 are arbitrary, we set them to 0. Next we choose the functions $a(r)$ and $b(r)$ so that (14) is a solution of (13) when $r_1 = r_2 = 1$ and $r_3 = 3$. This can be accomplished by setting

$$a(1) = -\beta - 2\alpha,\ b(1) = \beta,\ a(3) = -\beta,\ b(3) = \alpha \qquad\qquad \alpha, \beta \text{ integers},$$

and extending them to smooth functions of r. On the torus $\mathbb{T}^*$, the equations now take the form

$$\begin{aligned} \dot{\theta}_1 &= 1 + \sin\left[(-\beta - 2\alpha)\theta_1 + \beta\theta_2 + \alpha\theta_3\right] \\ \dot{\theta}_2 &= 1 + \sin\left[(-\beta - 2\alpha)\theta_2 + \alpha\theta_3 + \beta\theta_1\right] \\ \dot{\theta}_3 &= 2 + \sin\left[-\beta\theta_3 + \beta\theta_1 + \beta\theta_2\right]. \end{aligned}$$

The Jacobian evaluated on the periodic orbit (14) on $\mathbb{T}^*$ is again constant

$$\begin{pmatrix} -\beta - 2\alpha & \beta & \alpha \\ \beta & -\beta - 2\alpha & \alpha \\ \beta & \beta & -\beta \end{pmatrix}. \tag{15}$$

The eigenvalues of this matrix are easily computed to be $0, -2\alpha - \beta, -2(\alpha + \beta)$. Since the only restriction is that α and β are integers, simply choosing these constants to be positive guarantees that the solution is stable. The fact that α and β can be chosen so that (15) has one zero and two negative eigenvalues also follows from the more general results discussed in Section 9.

7.3 *Networks with several strongly connected components*

Consider the network $\mathcal{G}$ shown in Figure 7. The network contains 5 strongly connected components which are the nodes in the derived graph $\tilde{\Gamma}$ on the right of the figure. Consider the subgroup $H \approx \mathbb{Z}_6$ of the symmetries of the associated admissible equations which is generated by the permutation $(1\ \ 2\ \ 3)\ (4\ \ 6\ \ 8)\ (5\ \ 7\ \ 9)\ (10\ \ 11\ \ 12)\ (13\ \ 14)$.

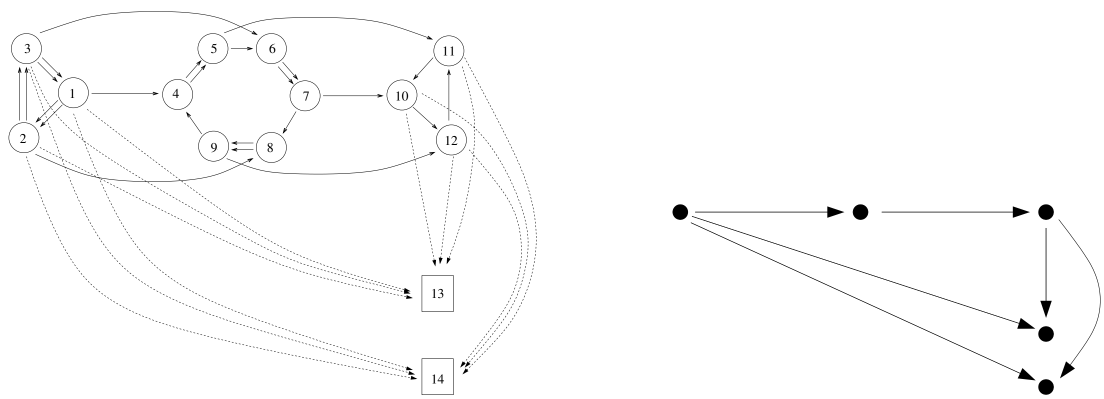

Fig. 7. The coupled cell system consisting of 5 strongly connected components discussed in §7.3 and the associated reduced graph.

Following the previous examples, we specify only the dynamics of the phases for the solution with the desired spatio-temporal symmetries:

$$\begin{aligned} \theta_1(t) &= \theta_4(t) = \theta_5(t) = \theta_{10}(t) = 4\pi t & \theta_{13}(t) &= 6\pi t \\ \theta_2(t) &= \theta_6(t) = \theta_7(t) = \theta_{11}(t) = 4\pi t + \frac{2\pi}{3} & \theta_{14}(t) &= 6\pi t + \pi \\ \theta_3(t) &= \theta_8(t) = \theta_9(t) = \theta_{12}(t) = 4\pi t + \frac{4\pi}{3}. \end{aligned} \tag{16}$$

Equations supporting this solution can be constructed explicitly using the ideas developed in Section 8. Since the derived graph $\tilde{\Gamma}$ has only a single root node, the conditions of Theorem 17 are satisfied, and this solution can also be made hyperbolic and asymptotically stable.

Consider the network obtained by *reversing* all the arrows on the left of Figure 7. The corresponding reduced network $\tilde{\Gamma}$ has the structure of the network on the right of Figure 7 with arrows reversed. Note that this modified network has the same symmetries as the network discussed above, but has multiple roots. Therefore, according to Theorem 17 it cannot support a hyperbolic, fully oscillatory periodic solution.

8 Construction of admissible vector fields. Stability

Using ideas illustrated in the examples in Section 7, we construct a class of admissible vector fields, and describe the condition for a vector field within this class to support a dynamically and structurally stable periodic solution of the desired type. The construction is accompanied by a concrete example.

8.1 *Standing assumptions*

Let $K \subset H$ be symmetry groups of $\mathcal{G}$, such that K is an isotropy group (that is $\mathrm{Fix}(K) \neq \{0\}$), K is normal in H, and H/K is cyclic. Moreover, we assume that the internal dynamics of each cell has dimension at least two.

8.2 *Outcome of the construction*

We summarize here the result of our construction. The details are given in §§8.4–8.12 below. Although in §§8.4–8.10 the notation refers to the groupoid $\mathcal{G}/K$, similar formulas hold for the groupoid $\mathcal{G}$ as well, because the differential equations are of the same type.

For each cell $c \in \mathcal{C}$ pick a two dimensional subspace in the phase space of c. We may arrange that the direct sum of these two-dimensional subspaces be a global attractor for the dynamics. In each of these subspaces we consider polar coordinates, (r_c, θ_c), and write the ODE as a skew-product over the radial variables. For each cell a number $r_c^* > 0$ will be selected, and we arrange that the torus $\{r_c = r_c^* \mid c \in \mathcal{C}\}$ be a local attractor. Near this torus the ODE becomes a "linear" ODE in the phase-variables, meaning that

$$\dot{\theta}_c = \omega_c + \sin\left(b_c\theta_c + \sum_{\mathcal{H}(e)=c} a_e\theta_{\mathcal{T}(e)} + \tau_c\right), \qquad c \in \mathcal{C} \tag{17}$$

where the sum is over all the *edges* that end in cell c, and b_c, a_e are integers, with $a_e > 0$ (see §8.6 and §8.7). We can visualize this data as assigning to each edge e the (positive) weight a_e, and to each cell c its radius r_c^*, its frequency ω_c, its self-drive weight b_c, and its "phase translation" τ_c. The assignment is made consistently with the constraints of the coupled cell system, so that the ODE is admissible. This system admits a solution (see §8.9)

$$\theta_c(t) = \omega_c t + \eta_c \quad \text{with } \omega_c > 0, \qquad c \in \mathcal{C} \tag{18}$$

that will have the symmetries imposed by K and H.

We represent the ODE (17) as (see §8.8)

$$\dot{\vec{\theta}} = \vec{\omega} + \vec{\sin}(A\vec{\theta} + \vec{\tau}).$$

Since $a_e > 0$, it follows that the off-diagonal entries of A are positive exactly when there is an edge of the graph associated to $\mathcal{G}$, and zero otherwise.

As we discuss in §8.12, the solution (18) is stable if and only if A has one zero eigenvalue (corresponding to the flow direction $\vec{\omega}$) and all the other eigenvalues have negative real part. To achieve this we need conditions on the connectivity of the coupled cell system, as described in Theorems 13 and 17.

8.3 Outline of the construction

We proceed as follows. In §8.4 we show that it is sufficient to only consider systems with 2-dimensional internal dynamics. Next, we construct the vector field on the space $\mathrm{Fix}(K)$, where the desired solution must lie. The admissible differential equations are introduced in §8.6, and in §8.7 the restrictions of the equations to an invariant torus is described. The equations are rewritten in a more manageable form in §8.8. The periodic solution with the desired spatio-temporal symmetries is described in §8.9, and in §8.10 we show that this is indeed a solution of the ODE introduced in §8.6 if the parameters are chosen appropriately. The equations on the quotient network are lifted to the full network in §8.11. The stability of the solution is discussed in §8.12. We accompany the construction with an illustrative example.

8.4 Two-dimensional internal dynamics suffice

If the internal phase space of the cell c is P_c, choose a two-dimensional linear subspace $P_c^0 \subset P_c$, such that $P_c^0 = P_d^0$ whenever $c \sim_{\mathcal{C}} d$ (recall that if $c \sim_{\mathcal{C}} d$ then $P_c = P_d$). Once we constructed an admissible ODE on the coupled cell network "restricted" to $\mathcal{P}^0 = \oplus P_c^0$, it is straightforward to extend it to an admissible ODE on the whole phase-space $\mathcal{P} = \oplus P_c$, such that $\mathcal{P}^0$ is attracting. Thus, from here on, we assume that the internal dynamics of each cell is two-dimensional.

8.5 The vector field on Fix(K): $K = \{1\}$, $H = \mathbb{Z}_p$

Recall the terminology introduced in §5.3. We begin by constructing, in §§8.6–8.10, the admissible vector field for the quotient network on $\mathrm{Fix}(K)$. Note that on $\mathrm{Fix}(K)$ the group K acts as the identity, and H acts as H/K.

Therefore, to construct the vector field on $\mathrm{Fix}(K)$ we have to consider only the case $K = \{1\}$ and $H \cong \mathbb{Z}_p$. We denote a generator of H by h.

8.6 The admissible differential equations

For the quotient coupled cell system on $\mathrm{Fix}(K)$ we assume the notation of Definition 7.

We start by introducing the equivalence relation $\sim_H$ induced by H on cells. Two cells c and d are H-related, $c \sim_H d$, if the the cells are on the same H-orbit.

Since the internal dynamics is two dimensional we can write the cell variables $x_c \in P_c^0$ in polar coordinates:

$$x_c = (r_c \cos\theta_c, r_c \sin\theta_c), \quad r_c \in [0, \infty), \quad \theta_c \in \mathbb{R}/2\pi\mathbb{Z}, \quad c \in \mathcal{C}.$$

For each cell $c \in \mathcal{C}$ pick a radius $r_c^* > 0$ such that $r_c^* = r_d^*$ if $c \sim_H d$, and $r_c^* \neq r_d^*$ otherwise.

The general class of admissible vector fields we consider is defined by

$$\begin{aligned} \dot{r}_c &= f_c(r_c) \\ \dot{\theta}_c &= \omega_c(r_c) + \alpha_c(r_c) \sin\left(b_c(r_c)\theta_c + \sum_{e \in \mathcal{E}, \mathcal{H}(e) = c} a_e(r_c, r_{\mathcal{T}(e)})\theta_{\mathcal{T}(e)} + \tau_c(r_c) \right) \end{aligned} \tag{19}$$

where:

1) the functions $f_c(r)$, $\omega_c(r)$, $\alpha_c(r)$, $a_e(r, q)$, and $\tau_c(r)$ are smooth; moreover, $\alpha_c(r)$, $\omega_c(r)$ and $f_c(r)$ are identically zero for small r so that the equations extend smoothly to the origin;
2) if $\alpha_c(r_c) \neq 0$ then $a_e(r_c, \,\cdot\,), b_c(r_c) \in \mathbb{Z}$;
3) $f_c = f_d$, $\omega_c = \omega_d$, $\alpha_c = \alpha_d$, $b_c = b_d$, and $\tau_c = \tau_d$ if c and d are input-equivalent, $c \sim_{\mathcal{I}} d$;
4) $a_u = a_v$ if u and v are edge-equivalent, $u \sim_{\mathcal{E}} v$.

Note that equations (19) together with conditions 1)–4) define an admissible vector field which is a skew-product over the radii.

8.7 An invariant torus

The following assumptions ensure that the torus $\mathbb{T}^* = \{r_c = r_c^* \mid c \in \mathcal{C}\}$ is attractive, and simplify the construction that will be presented subsequently:

5) f_c has stable equilibria at all values r_d^*, $d \in [c]_{\mathcal{I}}$;
6) $\omega_c(r)$ is a constant ω_d, and $\tau_c(r)$ is a constant τ_d, when r is in some neighborhood of r_d^*, $d \in [c]_{\mathcal{I}}$;
7) $\alpha_c(r) = 1$ when r is in some neighborhood of r_d^*, $d \in [c]_{\mathcal{I}}$.

Since we will consider only solutions in the vicinity of the torus $\mathbb{T}^*$, we introduce the notation $\omega_c(r_c^*) = \omega_c, b_c(r_c^*) = b_c, a_e(r_c^*, r_d^*) = a_e(c, d) = a_e$, $\tau_c(r_c^*) = \tau_c$. We spell out here conditions on these values which guarantee that the obtained vector field is admissible:

- the $a_e(c, d)$'s and b_c's are integers;

- ω_c, b_c, $a_e(c,d)$ and τ_c, are constant along h-orbits (recall that by Definition 9, the generator h of H extends to an edge-action);
- if two edges of the same type connect the same two cells, then their labels are equal: $a_e(c,d) = a_{e'}(c,d)$ if $e \sim_{\mathcal{E}} e'$.

These restrictions are implemented in the next subsection.

8.8 Simpler form for the equations

For each $\mathcal{E}$-equivalence class $\phi = [u]_{\mathcal{E}}$ of edges and each pair of H-equivalence classes $\rho = [a]_H$ and $\rho' = [a']_H$ of cells, introduce the matrices $A^{\phi,\rho,\rho'}$ which describe the connectivity of the graph, and B^ρ which describe H-orbits. Since H is a symmetry of the coupled cell system, these matrices commute with all elements of H. We let

$$\begin{aligned} A^{\phi,\rho,\rho'} &= (\delta^{\phi,\rho,\rho'}_{c,d})_{c,d\in\mathcal{C}} \\ \text{where} \quad \delta^{\phi,\rho,\rho'}_{c,d} &:= \begin{cases} \#\{e \in \phi \mid \mathcal{T}(e) = d, \mathcal{H}(e) = c\} \text{ if } c \in \rho \text{ and } d \in \rho', \\ 0 \text{ otherwise,} \end{cases} \end{aligned} \tag{20}$$

where # denotes the cardinality of a set. Therefore, the entry with index c, d of matrix $A^{\phi,\rho,\rho'}$ equals the number of edges of type ϕ from cell d to cell c. Let

$$B^\rho = (\beta^\rho_{c,d})_{c,d\in\mathcal{C}} \quad \text{where} \quad \beta^\rho_{c,d} := \begin{cases} 1 \text{ if } c = d \in \rho, \\ 0 \text{ otherwise.} \end{cases} \tag{21}$$

The matrices B^ρ are diagonal.

Example 20 *To illustrate the construction of the describe matrices consider the network shown in Figure 8. Admissible equations for this coupled cell system take the form*

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3, x_4, x_4, x_5, x_5) & \dot{x}_4 &= g(x_4, x_5, x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1, x_4, x_4, x_5, x_5) & \dot{x}_5 &= g(x_5, x_4, x_1, x_2, x_3) \\ \dot{x}_3 &= f(x_3, x_1, x_2, x_4, x_4, x_5, x_5) \end{aligned}$$

where $f(x,y,z,t,u,v,w)$ is symmetric separately in the variables (y,z) and (t,u,v,w), and $g(u,v,x,y,z)$ is symmetric in the variables (x,y,z). Denote by $\mathbf{S}_N$ the group of permutation of N elements. Then these equations are G-equivariant where $G = \mathbf{S}_3 \times \mathbf{S}_2$ and $\mathbf{S}_3$ acts by permuting cells 1, 2, and 3, while $\mathbf{S}_2$ acts by interchanging cells 4 and 5.

We next construct the matrix A defined in (20)*. Denote by ρ_1 the class consisting of cells 1, 2 and 3, and by ρ_2 the class consisting of cells 4 and 5. Following definitions* (20) *and* (21)*,*

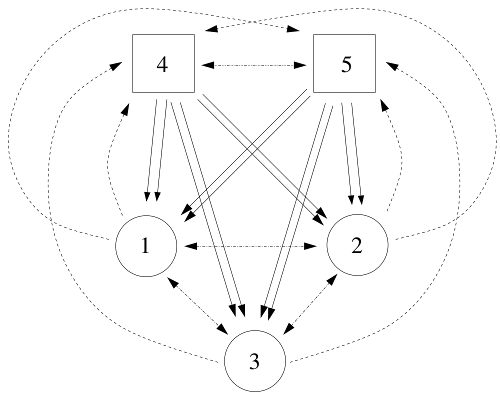

Fig. 8. A network composed of two bidirectionally coupled rings discussed in Example 20.

we have

$$A^{\rho_1,\rho_1} = \begin{pmatrix} 0\,1\,1\,0\,0 \\ 1\,0\,1\,0\,0 \\ 1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{pmatrix} \qquad A^{\rho_1,\rho_2} = \begin{pmatrix} 0\,0\,0\,2\,2 \\ 0\,0\,0\,2\,2 \\ 0\,0\,0\,2\,2 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \end{pmatrix}$$

$$A^{\rho_2,\rho_1} = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 1\,1\,1\,0\,0 \\ 1\,1\,1\,0\,0 \end{pmatrix} \qquad A^{\rho_2,\rho_2} = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,0\,1\,0 \end{pmatrix}$$

where we have suppressed the dependence on the edge type ϕ since each of the four matrices corresponds to exactly one edge type. Similarly we have

$$B^{\rho_1} = \mathrm{diag}(1,1,1,0,0) \qquad B^{\rho_2} = \mathrm{diag}(0,0,0,1,1).$$

$\diamondsuit$

Under assumptions 1)–7), in a neighborhood of $\mathbb{T}^*$ (that is, where r_c is close to r_c^* for each $c \in \mathcal{C}$), the differential equations (19) take the form

$$\begin{aligned} \dot{r}_c &= f_c(r_c) \\ \dot{\theta}_c &= \omega_c + \sin\left(b_c\theta_c + \sum_{\phi\in\mathcal{E}/\sim} \sum_{\rho'\in\mathcal{C}/H} a_\phi(\rho,\rho') \sum_{d\in\mathcal{C}} \delta_{c,d}^{\phi,\rho,\rho'} \theta_d + \tau_c \right) \text{ where } \rho = [c]_H. \end{aligned} \tag{22}$$

Assumption 5) implies that for these equations the torus $\mathbb{T}^*$ is attracting and invariant. We will henceforth consider the system restricted to this torus. Equations (22) restricted to $\mathbb{T}^*$

can be written more simply using vector notation. Let

$$\vec{\theta} = (\theta_c)_{c\in\mathcal{C}}, \quad \vec{\omega} = (\omega_c)_{c\in\mathcal{C}}, \quad \vec{\eta} = (\eta_c)_{c\in\mathcal{C}}, \text{ and } \vec{\tau} = (\tau_c)_{c\in\mathcal{C}}, \tag{23}$$

where θ_c, η_c, and τ_c are considered (mod 2π), and $\omega_c \in \mathbb{R}$. Equations (22) restricted to $\mathbb{T}^*$ then have the form

$$\begin{aligned}\dot{\vec{\theta}} &= \vec{\omega} + \vec{\sin}\left(\sum_{\rho\in\mathcal{C}/H} b_\rho B^\rho \vec{\theta} + \sum_{\phi\in\mathcal{E}/\sim}\ \sum_{\rho,\rho'\in\mathcal{C}/H} a_\phi(\rho,\rho') A^{\phi,\rho,\rho'}\vec{\theta} + \vec{\tau}\right) \\ &= \vec{\omega} + \vec{\sin}(A\vec{\theta} + \vec{\tau})\end{aligned} \tag{24}$$

where $\vec{\sin}(\vec{x}) = [\sin(x_1), \ldots, \sin(x_n)]^t$ and A is the matrix introduced in (22),

$$A = \sum_{\rho\in\mathcal{C}/H} b_\rho B^\rho + \sum_{\phi\in\mathcal{E}/\sim}\ \sum_{\rho,\rho'\in\mathcal{C}/H} a_\phi(\rho,\rho') A^{\phi,\rho,\rho'}. \tag{25}$$

Remark 21 Equations of the type (24) are a generalization of those modeling coupled identical oscillators discussed in [2]. Equations of this type are obtained from averaging or a normal form reduction in networks of weakly coupled oscillators [19,23].

8.9 The periodic solution

We show that one can choose parameters so that the system (24) admits a periodic solution with spatio-temporal symmetry H. To simplify the exposition we will consider periodic solutions of period $T = 1$. Solutions with other periods can be constructed by an appropriate scaling of the vector field. We denote equality modulo 2π by $=_{2\pi}$.

We will consider solutions of the form

$$\vec{\theta}(t) = t\vec{\omega} + \vec{\eta}. \tag{26}$$

For this function to be a periodic solution of the differential equation (24), and to be H-equivariant, it must satisfy the following conditions

8) $\vec{\theta}(t)$ has minimal period $T = 1$. This condition is satisfied if and only if $\vec{\omega} =_{2\pi} \vec{0}$, and $s\vec{\omega} \neq_{2\pi} \vec{0}$ for $s \in (0,1)$.
9) $\vec{\theta}(t)$ is a solution of (24), that is $\dot{\vec{\theta}} = \vec{\omega} + \vec{\sin}(A\vec{\theta} + \vec{\tau})$. This condition is satisfied if and only if $A\vec{\omega} = \vec{0}$ and $\vec{\tau} =_{2\pi} -A\vec{\eta}$.
10) For a generator h of the group H there exists a rational number s_h such that $h(\vec{\theta}(t)) =_{2\pi} \vec{\theta}(t+s_h)$ for all t. This condition is satisfied if and only if $(h-I)\vec{\omega} = \vec{0}$, and $(h-I)\vec{\eta} =_{2\pi} s_h\vec{\omega}$.

Without loss of generality we can assume that the frequencies ω_c are positive.

Example 20 (continued) We illustrate how a solution can be chosen to satisfy these conditions using the system in Figure 8. Consider the subgroup $H \subset G$ generated by the order 6 element h which is the product of the cycles (1 2 3) and (4 5) so that $H \approx \mathbb{Z}_6$. It can be checked directly that setting $s_h = 1/6$ and

$$\begin{aligned} \theta_1(t) &= 4\pi t & \theta_4(t) &= 6\pi t \\ \theta_2(t) &= 4\pi t + \frac{2\pi}{3} & \theta_5(t) &= 6\pi t + \pi \\ \theta_3(t) &= 4\pi t + \frac{4\pi}{3} \end{aligned} \tag{27}$$

gives a set of functions with the desired spatio-temporal symmetries.

Note that if the connections between cells of equal type in this network are removed, the network is still strongly connected. The admissible equations have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_4, x_4, x_5, x_5) & \dot{x}_4 &= g(x_4, x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_4, x_4, x_5, x_5) & \dot{x}_5 &= g(x_5, x_1, x_2, x_3) \\ \dot{x}_3 &= f(x_3, x_4, x_4, x_5, x_5) \end{aligned} \tag{28}$$

and are G-equivariant, where $G = \mathbf{S}_3 \times \mathbf{S}_2$, as above. If $H = \mathbb{Z}_6$ is chosen to act on the network as above, the solutions (27) again have the desired spatio-temporal symmetries. In the continuation of the discussion of this example we discuss the ODEs and the stability of the solution in both networks. ♢

8.10 Construction of ODEs supporting the periodic solutions

Our next goal is to illustrate how a set of periodic functions (26) with the desired spatio-temporal symmetries can be constructed. We then show how the matrix A and translations $\vec{\tau}$ can be chosen so that this set of functions corresponds to an actual solution of the equation (24), induced on the torus $\mathbb{T}^*$ by an admissible vector field. (The discussion in §8.7 implies that any H-invariant choice of A and $\vec{\tau}$ extend to an admissible vector field.)

Lemma 22 *The conditions 8) and 10) are equivalent to:*

(a) $s_h = q/p$ *where* q *is an integer that is relatively prime with* p*, the order of* H*.*
(b) $\omega_c \in 2\pi(p/n_{c,H})\mathbb{Z}$ *where* $n_{c,H} = \#([c]_H)$*,* $\omega_c = \omega_d$ *if* $c \in [d]_H$*, and* $\gcd(\{\omega_c/2\pi \mid c \in \mathcal{C}\}) = 1$*. That is,* ω_c *is "inversely proportional" to* $n_{c,H}$*, the length of the* H*-orbit of* c*.*
(c) $\vec{\eta}$ *is determined uniquely up to a choice of phase on each* H*-equivalence class of cells by the equation* $(h - I)\vec{\eta} = s_h\vec{\omega} + \vec{\Delta}$ *where* $\vec{\Delta} =_{2\pi} \vec{0}$ *is such that the sum of entries of* $s_h\vec{\omega} + \vec{\Delta}$ *is zero over each* H*-orbit of cells.*

Proof: Recall that p is the order of h, the generator of H. Then ps_h is an integer, and p is the smallest multiple of s_h for which this is true (otherwise, a lower power of h equals the identity). Any such choice of $s_h = q/p$ where p and q are relatively prime is acceptable.

The vector $\vec{\omega}$ is h-invariant, hence it is constant on each H-equivalence class of cells. These are exactly the cell-orbits of h. We choose its entries positive. By 8), the values $\omega_c \in 2\pi\mathbb{Z}$ must be relatively prime multiples of 2π. Their values are further constrained by the $\vec{\eta}$-equation in 10), as we describe next.

The vector $\vec{\eta}$ describes the phase-shift between different cells; it is determined up to a vector in the kernel of $h - I$, which corresponds to a change of phase for all the cells in a given h-orbit. To find $\vec{\eta}$ we must solve $(h - I)\vec{\eta} = s_h\vec{\omega} + \vec{\Delta}$, where $\vec{\Delta}$ is a vector with entries in $2\pi\mathbb{Z}$. This is possible provided $s_h\vec{\omega} + \vec{\Delta}$ is in the range of $h - I$, which is the orthogonal complement of the kernel of $(h - I)^t$ (here ^{t} denotes transposition). This kernel coincides with the kernel of $h - I$. Thus, the sum of the entries of $s_h\vec{\omega} + \vec{\Delta}$ over each h-orbit must be zero. This is possible if and only if $n_{c,H}s_h\omega_c =_{2\pi} 0$. For $s_h = q/p$, we obtain the conditions $n_{c,H}q\omega_c/p =_{2\pi} 0$. Since $\gcd(p, q) = 1$ and $n_{c,H}$ divides p, this means that $\omega_c/2\pi \in (p/n_{c,H})\mathbb{Z}$.

In conclusion: given a collection of relatively prime $\omega_c \in 2\pi(p/n_{c,H})\mathbb{Z}$, the phase-shift vector $\vec{\eta}$ is determined by the vector $\vec{\Delta}$. □

Note that modulo 2π and a vector in $\ker(h - I)$, for each $\vec{\omega}$ there are only finitely many possible vectors $\vec{\eta}$.

The following Lemma shows how condition 9) can be satisfied.

Lemma 23 *Let $\vec{\omega}$ and $\vec{\eta}$ have the values determined in Lemma 22. Given an arbitrary choice of positive integer values for $a_\phi(\rho, \rho')$, there are (unique) rational values for the b_ρ's such that the matrix $\tilde{A}$ constructed according to* (25) *satisfies $\tilde{A}\vec{\omega} = \vec{0}$. Pick a positive integer ℓ such that $A := \ell\tilde{A}$ has integer entries and let $\vec{\tau} := -A\vec{\eta}$.*

For this choice of A and $\vec{\tau}$ the periodic function (26) *is a solution of the equation* (24)*, determined by an admissible vector field.*

Proof: In order to satisfy $\tilde{A}\vec{\omega} = \vec{0}$ we pick $a_\phi(\rho, \rho')$ and compute b_ρ. Namely, for each $\phi \in \mathcal{E}/\sim$ and $\rho, \rho' \in \mathcal{C}/H$, choose an integer $a_\phi(\rho, \rho')$, and set $b_\rho = 0$. Denote by A_0 the corresponding matrix given by (25). We show that rational numbers b_ρ can be found such that

$$\Big(A_0 + \sum_{\rho\in\mathcal{C}/H} b_\rho B^\rho\Big)\vec{\omega} = \vec{0}.$$

Indeed, since A_0 commutes with h and $(h-I)\vec{\omega} = \vec{0}$, we conclude that $\vec{\xi} := A_0\vec{\omega} \in \ker(h-I)$. This means that $\vec{\xi}$ is constant on each h-orbit. But these orbits are exactly the nonzero diagonal entries of the B^ρ's. Therefore, the choice $b_\rho = -\xi_c/\omega_c$ for $c \in \rho$ yields the desired conclusion. The values ξ_c/ω_c are rational, because the ω_c's are all rationally related and A_0 is an integer matrix.

Let ℓ be a positive integer such that $A := \ell\tilde{A}$ has integer entries. According to §8.7 it only remains to check that the vector $\vec{\tau} := -A\vec{\eta}$ is constant (mod 2π) on H-orbits: since A commutes with h, $(h - I)A\vec{\eta} = A(h - I)\vec{\eta} =_{2\pi} A(s_h\vec{\omega}) =_{2\pi} \vec{0}$, as desired. □

Proposition 24 *A choice for s_h, $\vec{\omega}$ and $\vec{\eta}$ consistent with the conclusions of Lemma 22 is given by*

- $s_h = 1/p$,
- $\omega_c = \frac{p}{\#([c]_H)} 2\pi$,
- *For η_c choose an initial cell $c_{k,0}$ on each cycle k of H. Number the cells in the cycle sequentially, so that $c_{k,i} = h^i(c_{k,0})$. If l_k is the length of the cycle k, then set $\eta_{c_{k,i}} = i/l_k$.*

This proposition is proved by checking directly that these values satisfy the conclusions of Lemma 22. We illustrate the conclusions of this section by revisiting the coupled cell system introduced in Example 20.

Example 20 (continued) Consider the network described in Example 20. According to Proposition 24 we can choose $s_h = 1/6$, $\vec{\omega} = 2\pi(2, 2, 2, 3, 3)$, and $\vec{\eta} = 2\pi(0, 1/3, 2/3, 0, 1/2)$. This leads to the solution specified in (27).

Therefore the coupling matrix A is obtained as

$$\tilde{A} = \alpha A^{\rho_1,\rho_1} + \beta A^{\rho_1,\rho_2} + \delta A^{\rho_2,\rho_1} + \gamma A^{\rho_2,\rho_2} + b_1 B^{\rho_1} + b_2 B^{\rho_2} = \begin{pmatrix} b_1 & \alpha & \alpha & 2\beta & 2\beta \\ \alpha & b_1 & \alpha & 2\beta & 2\beta \\ \alpha & \alpha & b_1 & 2\beta & 2\beta \\ \delta & \delta & \delta & b_2 & \gamma \\ \delta & \delta & \delta & \gamma & b_2 \end{pmatrix} \tag{29}$$

The choice $b_1 = -2\alpha - 6\beta$ and $b_2 = -2\delta - \gamma$ is needed for $\tilde{A}\vec{\omega} = 0$. It is also immediate that there is an integer ℓ such that if b_1 and b_2 are determined by these equations, for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, then $A = \ell\tilde{A}$ satisfies $A\vec{\eta} =_{2\pi} \vec{0}$ hence one can choose $\vec{\tau} = \vec{0}$. By a slight abuse of notation, we will denote the entries of this new matrix A by $\alpha, \beta, \gamma, \delta$ so that it has the form given in (29).

In the case of the network whose evolution is given by equations (28), the matrix A has the same form as above with $\alpha = \gamma = 0$. $\diamondsuit$

8.11 Lifting from the quotient network

We extend now the vector field constructed on $\mathrm{Fix}(K)$ to the whole space, and conclude that all the notations and formulas lift in a natural way.

Assume given the coupled cell system $\mathcal{G}$, and $K \subset H$ as described in §8.1. To distinguish between $\mathcal{G}$ and $\mathcal{G}/K$, we denote the cells $\mathcal{C}_\mathcal{G}$ and edges $\mathcal{E}_\mathcal{G}$ of $\mathcal{G}$ by Greek letters, and keep the current notation for the cells $\mathcal{C}$ and edges $\mathcal{E}$ of $\mathcal{G}/K$. By §8.4 we may assume that each cell $\gamma \in \mathcal{C}_\mathcal{G}$ has two-dimensional internal dynamics; for each cell γ, let $(r_\gamma, \theta_\gamma)$ be the polar coordinates in that space. Recall (see §5.3) that there are maps $Q_\mathcal{C} : \mathcal{C}_\mathcal{G} \to \mathcal{C}$ and $Q_\mathcal{E} : \mathcal{E}_\mathcal{G} \to \mathcal{E}$ defined as follows: $Q_\mathcal{C}(\gamma)$ is the K-orbit of γ, and for edges $\varepsilon \in \mathcal{E}_\mathcal{G}$ with head γ, $Q_\mathcal{E}$ is an edge-type preserving bijection onto the edges $e \in \mathcal{E}$ having head $Q_\mathcal{C}(\gamma)$. If there is more than one such bijections, any one of them can be used.

Assume the vector field on $\mathrm{Fix}(K)$ is given by (19), which for $c \in \mathcal{C}$ has the form

$$\begin{aligned} \dot{r}_c &= f_c(r_c) \\ \dot{\theta}_c &= \omega_c(r_c) + \alpha_c(r_c) \sin\left(b_c(r_c)\theta_c + \sum_{e\in\mathcal{E}, \mathcal{H}(e)=c} a_e(r_c, r_{\mathcal{T}(e)})\theta_{\mathcal{T}(e)} + \tau_c(r_c) \right). \end{aligned}$$

Then the lifted vector field is

$$\begin{aligned} \dot{r}_\gamma &= f_c(r_\gamma) \\ \dot{\theta}_\gamma &= \omega_c(r_\gamma) + \alpha_c(r_\gamma) \sin\left(b_c(r_\gamma)\theta_\gamma + \sum_{\varepsilon\in\mathcal{E}_\mathcal{G}, \mathcal{H}(\varepsilon)=\gamma} a_{Q_\mathcal{E}(e)}(r_\gamma, r_{\mathcal{T}(\varepsilon)})\theta_{\mathcal{T}(e)} + \tau_c(r_\gamma) \right), \end{aligned} \tag{30}$$

where $\gamma \in \mathcal{C}_\mathcal{G}$ and $c = Q_\mathcal{C}(\gamma)$.

The results obtained so far carry over without any essential modification:

Lemma 25 *Assume the current conditions on the vector field on* $\mathrm{Fix}(K)$*, and consider the lifted vector field* (30)*. Then:*

(a) The torus $\mathbb{T}^*_\mathcal{G} := \{r_\gamma = r^*_{Q_\mathcal{C}(\gamma)} \mid \gamma \in \mathcal{G}_\mathcal{C}\}$ *is a local attractor.*
(b) On this torus the equations take the same form as (24)*, namely*

$$\begin{aligned} \dot{\vec{\theta}} &= \vec{\omega} + \vec{\sin}\left(\sum_{\rho\in\mathcal{C}_\mathcal{G}/H} b_\rho B^\rho_\mathcal{G}\vec{\theta} + \sum_{\phi\in\mathcal{E}_\mathcal{G}/\sim} \sum_{\rho,\rho'\in\mathcal{C}_\mathcal{G}/H} a_\phi(\rho,\rho') A^{\phi,\rho,\rho'}_\mathcal{G}\vec{\theta} + \vec{\tau} \right) \\ &= \vec{\omega} + \vec{\sin}(A_\mathcal{G}\vec{\theta} + \vec{\tau}) \end{aligned} \tag{31}$$

where the matrices $A^{\phi,\rho,\rho'}$ *and* B^ρ *introduced in* (20) *and* (21) *should be computed for* $\mathcal{G}$ *instead of* $\mathcal{G}/K$*, and the frequency and translation vectors lift via* $Q_\mathcal{C}$*:*

$$\omega_\gamma = \omega_{Q_\mathcal{C}(\gamma)}, \quad \tau_\gamma = \tau_{Q_\mathcal{C}(\gamma)}$$

(c) The periodic solution considered in §8.9 lifts to

$$\vec{\theta}(t) = t\vec{\omega} + \vec{\eta} \tag{32}$$

with the phase-shift lifted via $Q_\mathcal{C}$ *as well:* $\eta_\gamma = \eta_{Q_\mathcal{C}(\gamma)}$*.*

In conclusion:

- *The matrix* $A_\mathcal{G}$ *can have negative entries only on its diagonal.*

- *The off-diagonal entry $A_{\mathcal{G}}(\gamma, \delta)$ is positive exactly when there is an edge in $\mathcal{G}$ with head γ and tail δ (and zero otherwise).*
- *$\vec{\omega}$ is a positive vector in the kernel of $A_{\mathcal{G}}$.*

Proof: The claim in (a) follows from assumption 5) of §8.7.

For the other statements, we prefer to describe the lifted equations using the language introduced in §8.2. The equation on $\mathrm{Fix}(K)$ associated to each edge e an integer weight $a_\phi(\rho, \rho') > 0$ (see §8.8 and Lemma 23), and to each cell c a radius $r_c^* > 0$, a frequency $\omega_c > 0$, a self-drive weight $b_c \in \mathbb{Z}$, a phase-shift η_c, and a translation τ_c. When considering the lifted equations on the torus $\mathbb{T}_{\mathcal{G}}^*$, we only have to lift these values through $Q_{\mathcal{E}}$ and $Q_{\mathcal{C}}$. $\square$

Example 26 *To illustrate this lifting procedure, we give an example for the quotient described in Example 10 of the coupled cell system of Example 4. We describe the vector fields only in the vicinity of the attracting invariant tori $\mathbb{T}^*$ and $\mathbb{T}_{\mathcal{G}}^*$.*

On the quotient $\mathcal{G}/K$ number the cells $\mathcal{C} = \{1, 2, 3\}$ and consider the vector field determined by equation (19) after setting $r_i^* = 1$, $\omega_i = 2$, $\eta_i = 2\pi i/3$, $\tau_i = 2\pi/3$, $b_i = -6$, the strength of the interaction between cell i to $i+1$ to 4, so that $A_{i,i+1} = 4$, and the strength of drive from cell i to itself to 2, so that $A_{i,i} = 2$. Equation (19) then takes the form

$$\begin{aligned} \dot{r}_i =& f(r_i) \\ \dot{\theta}_i =& 2 + \sin(-6\theta_i + 2\theta_i + 4\theta_{i-1} + 2\pi/3) = 2 + \sin(-4\theta_i + 4\theta_{i-1} + 2\pi/3), \end{aligned}$$

where $f(1) = 0$, $f'(1) < 0$. It admits the solution $r_i = 1$, $\theta_i = 2t + 2\pi i/3$.

According to the discussion in this section, this vector field is lifted to

$$\begin{aligned} \dot{r}_i =& f(r_i), \qquad 1 \le i \le 6 \\ \dot{\theta}_1 =& 2 + \sin(-6\theta_1 + 2\theta_4 + 4\theta_3 + 2\pi/3) \qquad \dot{\theta}_4 = 2 + \sin(-6\theta_4 + 2\theta_1 + 4\theta_6 + 2\pi/3) \\ \dot{\theta}_2 =& 2 + \sin(-6\theta_2 + 2\theta_5 + 4\theta_1 + 2\pi/3) \qquad \dot{\theta}_5 = 2 + \sin(-6\theta_5 + 2\theta_2 + 4\theta_4 + 2\pi/3) \\ \dot{\theta}_3 =& 2 + \sin(-6\theta_3 + 2\theta_6 + 4\theta_2 + 2\pi/3) \qquad \dot{\theta}_6 = 2 + \sin(-6\theta_6 + 2\theta_3 + 4\theta_2 + 2\pi/3) \end{aligned}$$

and the solution lifts to $r_i = 1$, $\theta_1 = \theta_4 = 2t + 2\pi/3$, $\theta_2 = \theta_5 = 2t + 2 \cdot 2\pi/3$, $\theta_3 = \theta_6 = 2t + 3 \cdot 2\pi/3 =_{2\pi} 2t$. $\diamondsuit$

8.12 Stability of the periodic solution

We analyze now the stability of the solution (32) described in Lemma 25. In the vicinity of the attracting torus $\mathbb{T}_{\mathcal{G}}^*$, the system of equations (30) takes the form (22), where the latter equations should be considered for the full network $\mathcal{G}$ instead of the quotient $\mathcal{G}/K$.

Recall that the Floquet multipliers, describing the stability to variations $x(t)+\xi(t)$ along the solution $x(t)$ of the ODE $\dot{x} = f(x)$, are determined by the linear equation $\dot{\xi}(t) = (d_{x(t)}f)\xi(t)$.

Since the invariant torus $\mathbb{T}^*_{\mathcal{G}}$ is locally attracting, we only have to analyze stability on $\mathbb{T}^*_{\mathcal{G}}$.

If, for simplicity, we write the equation (31) as

$$\dot{\theta}_c = \omega_c + \sin(\sum_{d\in\mathcal{C}_{\mathcal{G}}} a_{c,d}\theta_d + \tau_c), \quad c \in \mathcal{C}_{\mathcal{G}} \tag{33}$$

then the linearization of the equation around the periodic solution has the form

$$\dot{\xi}_c = \sum_{d\in\mathcal{C}_{\mathcal{G}}} \cos(\sum_{d'\in\mathcal{C}_{\mathcal{G}}} a_{c,d}\theta_{d'} + \tau_c) a_{c,d}\xi_d, \quad c \in \mathcal{C}_{\mathcal{G}},$$

where $\vec{\theta}$ is a periodic solution of (33). By construction $\sum_{d'\in\mathcal{C}_{\mathcal{G}}} a_{c,d}\theta_{d'} + \tau_c$ is a multiple of 2π, so that

$$\dot{\xi}_c = \sum_{d\in\mathcal{C}_{\mathcal{G}}} a_{c,d}\xi_d, \quad c \in \mathcal{C}_{\mathcal{G}}.$$

Following our earlier convention, we write this equation as

$$\dot{\vec{\xi}}(t) = A_{\mathcal{G}}\vec{\xi}(t), \tag{34}$$

where

$$\dot{\vec{\theta}} = \vec{\omega} + \vec{\sin}(A_{\mathcal{G}}\vec{\theta} + \vec{\tau})$$

are the equations (31) written in vector form. The solution (32) is stable if the matrix $A_{\mathcal{G}}$ has all eigenvalues in the left half plane, except the one zero eigenvalue corresponding to the flow direction.

9 Proof of Theorem 13

In this section we use the results of the preceding construction to complete the proof of Theorem 13, and illustrate the result using Example 20.

In order to give more precise references, we use the notations introduced in Section 8 for the $\mathcal{G}/K$-admissible vector field (19), instead of its lift (30). It is easy to check (see Lemma 25) that the arguments remain valid for the lifted vector field.

Proof of Theorem 13: By Lemmas 22 and 23, one can find an admissible vector field (19) which restricts to (24) on $\mathbb{T}^*$. By construction $A\vec{\omega} = \vec{0}$, where $\vec{\omega}$ is the vector with positive entries described in Lemma 22.

From Lemma 23 it follows that the matrix $A = (a_{c,d})_{c,d\in\mathcal{C}}$ can be chosen to have negative entries only on the diagonal, and positive off-diagonal entries $a_{c,d}$ whenever there is an edge in $\mathcal{G}$ from d to c.

Pick a value $a > 0$ such that $M := A + aI$ has positive diagonal.

Because the diagonal of M is positive, the assumption that the graph Γ associated to $\mathcal{G}$ is strongly connected implies that M is a primitive matrix, that is, there is a power of M that has only positive entries.

Therefore, by the Perron-Frobenius Theorem, M has a unique positive eigenvector, and the corresponding (positive) eigenvalue is simple and strictly larger than the absolute value of all other eigenvalues of M.

But $\vec{\omega}$ is a vector with positive entries and $M\vec{\omega} = a\vec{\omega}$, hence a must be the leading eigenvalue of M.

This shows that $A = M - aI$ has eigenvalues inside the circle $\{z \in \mathbb{C} \mid |z + a| < a\} \subset \{z \in \mathbb{C} \mid \Re(z) < 0\}$, except for the eigenvalue zero that has multiplicity one. By §8.12, these are the properties needed for the stability of (26). $\square$

Example 20 (continued) To illustrate this result, we continue the construction in the case of the networks introduced in Example 20. The eigenvalues of the matrix (29) are given by $0, -3(\alpha+2\beta), -3(\alpha+2\beta), -2(3\beta+\delta)$, and $-2(\delta+\gamma)$, where the zero eigenvalue corresponds to the eigenvector parallel to the solution constructed in the example. The remaining eigenvalues are clearly negative if the off-diagonal elements α, β, γ and δ of the matrix A are chosen to be positive. Since these parameters can be freely specified, the constructed solution can always be stable.

In the second case, when there are no connections between cells of equal type, the coupling matrix A has the form given in (29) with $\alpha = \gamma = 0$. This matrix again has one zero eigenvalue, and 4 negative eigenvalues as long as β and δ are chosen to be negative. $\diamondsuit$

10 Proof of Theorem 17

We conclude with the proof of the more general Theorem 17. The proof also shows that the the spatio-temporally symmetric solution is stable whenever the off-diagonal entries of the coupling matrix A are positive. This is a surprisingly large set of parameters. Therefore, one could expect to observe such solutions in practice, when the admissible equations are obtained by averaging or other reduction methods.

We begin with a few preliminary lemmas. We recall the following result, related to the Perron-Frobenius Theorem, and we include a proof for completeness.

Lemma 27 *Assume that the square matrix A has non-negative entries, and admits an eigenvector v with positive entries, associated to the (non-negative) eigenvalue μ. Then the spectral radius of A is at most μ (that is, the absolute values of the eigenvalues of A are at most μ).*

Therefore, the spectrum of $A - \mu I$ lies in the left half plane, and it intersects the imaginary axis only at zero, which corresponds to the eigenvalue μ of A (note that the multiplicity of *zero in the spectrum may be larger than one).*

Proof: Let λ be an eigenvalue of A. Then λ is also an eigenvalue of A^t. Let $w \neq 0$ be such that $A^t w = \lambda w$. Denote by $|w|$ the vector whose entries are the absolute values of the entries of w. Then $|\lambda||w| \leq A^t|w|$, where $\leq$ between two vectors means that their corresponding entries are ordered. This implies that

$$\langle |\lambda||w|, v\rangle \leq \langle A^t|w|, v\rangle = \langle |w|, Av\rangle = \langle |w|, \mu v\rangle,$$

which shows that $|\lambda| \leq \mu$ because $\langle |w|, v\rangle > 0$. $\square$

Lemma 28 *Assume that the non-negative block lower triangular matrix*

$$M = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \tag{35}$$

has an eigenvector $(v^*, w^*)^t$ *with positive entries associated to the (positive) eigenvalue* μ*. If the matrix* C *is non-zero and the matrix* D *is primitive (or even irreducible), then* μ *is not an eigenvalue of* D*.*

Proof: For simplicity we use $R \geq 0$ to denote that the matrix or vector R has non-zero entries, and $R > 0$ to denote that all entries of R are positive.

Assume by contradiction that μ is an eigenvalue of D, and hence of D^t. By Lemma 27, M has no eigenvalue of absolute value larger than μ. Therefore μ is the leading positive eigenvalue of D and D^t, and, by the Perron-Frobenius Theorem, we can pick a vector $u > 0$ satisfying $D^t u = \mu u$. The relation $M(v^*, w^*)^t = \mu(v^*, w^*)^t$ shows that

$$Cv^* + Dw^* = \mu w^*.$$

Then

$$\langle \mu Dw^*, u\rangle = \langle \mu w^*, D^t u\rangle = \langle \mu w^*, \mu u\rangle = \langle Cv^* + Dw^*, \mu u\rangle = \langle Cv^*, \mu u\rangle + \langle Dw^*, \mu u\rangle,$$

which implies that

$$\langle Cv^*, u\rangle = 0.$$

Since $u > 0$ and $Cv^* \geq 0$ it follows that $Cv^* = 0$. But $v^* > 0$ and $C \geq 0$, hence $Cv^* = 0$ only if $C = 0$, a contradiction. $\square$

Proof of Theorem 17: From Remark 1 it follows that fully oscillatory *hyperbolic* periodic solutions cannot exist if the graph $\tilde{\Gamma}$ has more than one root node.

To prove sufficiency, we assume that a differential equation was constructed exactly as specified in Section 8. As in the preceding section, we need to show that the coupling coefficients can be chosen so that the matrix $A_{\mathcal{G}}$ of equation (34) has negative eigenvalues, except for the eigenvector $\vec{\omega}$. To this purpose we use Lemma 28.

The N nodes of the directed graph $\tilde{\Gamma}$ can be numbered so that i is a terminal node only for arrows that emanate from nodes $j < i$. This follows from the fact that the $\tilde{\Gamma}$ is acyclic and can be accomplished as follows: Let G_0 be the set consisting of the root node and denote the root node by 0. Let G_1 be the set of all nodes that receive inputs only from the root node. Number the nodes in G_1 sequentially, starting with 1. Continue inductively by letting G_i be the set of nodes that receive inputs only from nodes in $\cup_{j=0}^{i-1} G_j$ and numbering the nodes sequentially.

Once the nodes of $\tilde{\gamma}$ (that is, the strongly connected components of Γ) are enumerated, the theorem can be proved by induction. Let $\tilde{\Gamma}_i$ be the subgraph of $\tilde{\Gamma}$ containing nodes 0 through i and all directed arrows between them. The system of differential equations describing the evolution of the network can be restricted to this subgraph, since the entire system is a skew product over the cells contained in the nodes of $\tilde{\Gamma}_i$.

Since by assumption the root node corresponds to a strongly connected subgraph of the full connectivity graph Γ, the theorem holds for the subgraph $\tilde{\Gamma}_0$ as a direct consequence of Theorem 13. This proves the base case of the induction.

Assume next that we have shown that the theorem holds on the subgraph $\tilde{\Gamma}_i$. By construction the matrix A can be chosen to have non-negative entries everywhere except on the diagonal, and $A\vec{\omega} = 0$. Therefore, there exists an $a > 0$ such that $M := A + aI$ is a non-negative matrix with eigenvalue a and associated eigenvector $\vec{\omega}$. Denote by M_i and $\vec{\omega}_i$ the restriction of the matrix M and the vector $\vec{\omega}$ to the cells contained in the nodes of $\tilde{\Gamma}_i$. By our choice of the numbering of the nodes in $\tilde{\Gamma}$, the matrix M_{i+1} has the form

$$M_{i+1} = \begin{pmatrix} M_i & 0 \\ C_{i+1} & D_{i+1} \end{pmatrix}. \tag{36}$$

Clearly, M_{i+1} has a positive eigenvalue a, and an associated eigenvector $\vec{\omega}_{i+1}$ with positive entries. Moreover, by the induction hypothesis a is a simple eigenvalue of M_i. Since there is an arrow from some node of $\tilde{\Gamma}_i$ to the node $i+1$, the matrix C_{i+1} is non-zero. The matrix D_{i+1} is primitive since the node $i+1$ corresponds to a strongly connected component of the original directed graph Γ. Therefore the matrix M_{i+1} satisfies the conditions of Lemma 28 and the theorem is proved by induction. $\square$

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Figure 1. These networks support different types of nontrivial, structurally stable periodic orbits.

Figure 2. The networks shown in Figure 1 can support periodic solutions in which cells 1 and 2 (large amplitudes) are one half period out of phase, while cell 3 (small amplitude) oscillates at twice their frequency. For networks a) and b) this solution can be hyperbolic and attracting [12].

Figure 3. This $\mathbb{Z}_2 \times \mathbb{Z}_3$ symmetric network supports several types of spatio-temporally symmetric solutions. The figure on the right, top shows the timeseries of the phases of cells 1, 2 and 3 in a network of reduced Hodgkin-Huxley equations [13]. Cells 4, 5 and 6 evolve synchronously with cells 1, 2, and 3, respectively. At the bottom, the shift of the timeseries by one third of the period shows that all cells evolve identically up to a phase shift.

Figure 4. Both of these networks are $\mathbb{Z}_2$-symmetric. However, if these are networks of phase oscillators, then only the left network can support a solution with $H = \mathbb{Z}_2$.

Figure 5. The quotient system obtained from the network shown in Figure 3 with $K = \langle(1\ 4)(2\ 5)(3\ 6)\rangle$.

Figure 6. The network architectures of the examples discussed in Section 7.

Figure 7. The coupled cell system consisting of 5 strongly connected components discussed in §7.3 and the associated reduced graph.

Figure 8. A network composed of two bidirectionally coupled rings discussed in Example 20.