Let $T : U \rightarrow V$ be linear. We defined $\ker(T) = \{x | x \in U, T(x) = 0\}$ and $\text{im}(T) = \{y | T(x) = y \text{ for some } x \in U\}$. According to the dimension equality we have that $\dim(\ker(T)) + \dim(\text{im}(T)) = \dim(U) = n$. We have that the linear map $T$ is one-one or injective iff $\ker(T) = \{0\}$. If $U = V$ then $T$ is injective iff $T$ is surjective (onto). This is an easy consequence of the dimension equality.

For any linear map $T$ we have a matrix representation. The matrix $A$ of $T$ depends on the chosen bases $\alpha$ and $\beta$ of $U$ and $V$, respectively:

$$\text{Mat}(T; \alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_m) = (a_{ij}) \text{ where } T(\alpha_j) = \Sigma_{i=1}^{m} a_{ij} \beta_i; j = 1, \ldots, n; i = 1, \ldots, m$$

The matrix for $T$ is an $m \times n$ matrix where $\dim(V) = m, \dim(U) = n$. Each of the $n$ columns of $A$ contain the $m$ components of $T(\alpha_j)$ with respect to the basis $\beta_j$.

Let $x$ be a vector in $U$. If $x = \Sigma_{j=1}^{n} x_j \alpha_j$, then $T(x) = T(\Sigma_{j=1}^{n} x_j \alpha_j) = \Sigma_{j=1}^{n} x_j T(\alpha_j) = \Sigma_{j=1}^{n} x_j \Sigma_{i=1}^{m} a_{ij} \beta_i = \Sigma_{i=1}^{m} (\Sigma_{j=1}^{n} a_{ij} x_j) \beta_i = \Sigma_{j=1}^{m} y_j \beta_i$

where

$$y_i = \Sigma_{j=1}^{n} a_{ij} x_j$$

Thus

$$\begin{pmatrix}
  a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\
  \ldots & \ddots & \ldots & \ldots & \ldots \\
  a_{m1} & a_{m2} & \ldots & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \ldots \\
  \ldots \\
  x_n
\end{pmatrix}
= 
\begin{pmatrix}
  y_1 \\
  y_2 \\
  \ldots \\
  \ldots \\
  y_m
\end{pmatrix}$$

Thus the map $T(x) = y$ has a coordinate representation as $Ax = y$. With this in mind, we see that

$$\dim(\text{im}(T)) = \dim(\text{columnspace } A) = s, \dim(\ker(T)) = \dim(\text{solution space for } Ax = 0) = n - r$$

where $r$ is the row rank of $A$. By the dimension equality we have that $s + (n - r) = n$ which is $s = r$ that is row rank = column rank.

**Example 1:**

Let $U = \mathbb{R}^3, V = \mathbb{R}^4$ then the matrix $A = \begin{pmatrix}
  2 & 8 & -3 \\
  1 & 4 & 1 \\
  -5 & 2 & 2 \\
  1 & -3 & 8
\end{pmatrix}$ stands for the linear map $T$.
where \( T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 2 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \)

where we have chosen the unit vectors as bases for \( U = \mathbb{R}^3 \) and \( V = \mathbb{R}^4 \).

We have that

\[
\begin{pmatrix} 2 & 8 & -3 \\ 1 & 4 & 1 \\ -5 & 2 & 2 \\ 1 & -3 & 8 \end{pmatrix}
\]

has rank 3. This means that \( \ker(T) = \{0\} \), the map is injective and maps the 3 unit vectors to three linearly independent vectors. The image of the vector \( x \in \mathbb{R}^3 \) is a vector in \( \mathbb{R}^4 \):

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x + 8y - 3z \\ x + 4y + z \\ -5x + 2y + 2z \\ x - 3y + 8z \end{pmatrix}
\]

\( \ker(T) \) is the solution space for the homogeneous system which as we already saw consists only of the zero-vector of \( \mathbb{R}^3 \).

The map \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) is injective. We can find a linear map \( S : \mathbb{R}^4 \to \mathbb{R}^3 \) such that \( S \circ T : \mathbb{R}^3 \to \mathbb{R}^3 \) is the identity. Because \( T(e_1), T(e_2), T(e_3) \) are linearly independent, we can find a vector \( \beta_4 \) such that \( \{\beta_1 = T(e_1), \beta_2 = T(e_2), \beta_3 = T(e_3), \beta_4\} \) form a basis of \( \mathbb{R}^4 \). We define \( S \) on this basis by \( \beta_1 \mapsto e_1, \beta_2 \mapsto e_2, \beta_3 \mapsto e_3, \beta_4 \mapsto a \) where \( a \) is any vector in \( \mathbb{R}^3 \). For example \( a = 0 \in \mathbb{R}^3 \) is fine. Then \( S(T(e_1)) = S(\beta_1) = e_1 \) and similarly for the other unit vectors of \( \mathbb{R}^3 \). That is \( S(T(e_i)) = e_i \). That is, the composition \( S \circ T \) is the identity on the unit vectors of \( \mathbb{R}^3 \) and therefore the identity on \( \mathbb{R}^3 \). Because of \( S \circ T = id_{\mathbb{R}^3} \) we have for general reasons that \( S \) is surjective and \( T \) injective. Something we showed for arbitrary maps. We also see that \( S \) is not uniquely determined by \( T \). First \( \beta_4 \) is not unique and if we have found some \( \beta_4 \) we can assign any vector \( \alpha \) in \( \mathbb{R}^3 \) as its image under \( S \). The map \( S \) is unique only on \( im(S) \) by assigning to \( T(\alpha) \), the vector \( \alpha \).

**Example 2.** This is a simple example which makes the logic quite transparent. Let

\( T : \mathbb{R}^2 \to \mathbb{R}^3, e_1^2 \mapsto e_1^3, e_2^2 \mapsto e_2^3. \) The matrix of \( T \) is

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

We add the third unit vector \( e_3^3 \) to \( T(e_1^2) = e_1^3, T(e_2^2) = e_2^3 \) and define the map \( S \) as

\( S(T(e_1^2)) = e_1^2, S(T(e_2^2)) = e_2^3, S(e_3^3) = a = \begin{pmatrix} a \\ b \end{pmatrix}, a, b \) arbitrarily chosen. Then
This is an example where the product of two non-square matrices is square and invertible. Notice that
\[
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b
\end{pmatrix}
\begin{pmatrix}
1 & 0 & a \\
0 & 1 & b
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
is also square but not the identity.

Matrices and linear maps can be identified. If $A$ is an $m \times n$ -matrix, then
$L_A : \mathbb{R}^n \to \mathbb{R}^m, X \mapsto AX = Y$ is linear and $Mat(L_A) = A$. This is because $Ae_j^n = A_j$ where $A_j$ is the $j^{th}$ -column vector of $A$.

**Example 3.**
The matrix
\[
\begin{pmatrix}
2 & 3 & 1 \\
1 & 1 & 4
\end{pmatrix}
\]
stands for a map, $A$, from $\mathbb{R}^3$ into $\mathbb{R}^2$. The map is onto (why?), and therefore $\ker(A) + 2 = 3$, which gives us a one-dimensional null-space. How can we compute the kernel, that is find a basis? We have for
\[
\begin{pmatrix}
2 & 3 & 1 \\
1 & 1 & 4
\end{pmatrix}
\]
as row echelon form:
\[
\begin{pmatrix}
1 & 0 & 11 \\
0 & 1 & -7
\end{pmatrix}
This is $x = -11z, y = 7z$ or
\[
\begin{pmatrix}
-11 \\
7 \\
1
\end{pmatrix}
\]
ker($A$) is the span of the vector

We have that the matrix of the composition of maps corresponds to the product of the matrices
If $S : U \to V, T : V \to W, A = Mat(S; a_1, a_2, \ldots, a_n; \beta_1, \beta_2, \ldots \beta_m), B = Mat(T; \beta_1, \beta_2, \ldots \beta_m; \gamma_1, \gamma_2, \ldots, \gamma_l)$
then

\[ \text{Mat}(T \circ S; \alpha_1, \alpha_2, \ldots, \alpha_n; \gamma_1, \gamma_2, \ldots, \gamma_l) = BA \]

In particular, \( \text{Mat}(LBA) = BA : LBA(X) = (BA)X = B(AX) = L_B(L_A(X)) \). Thus \( L_{BA} = L_B \circ L_A \) and therefore \( \text{Mat}(L_{BA}) = \text{Mat}(L_B \circ L_A) = \text{Mat}(L_B)\text{Mat}(L_A) = BA \)

The linear map \( T : U \to V \) is invertible if there is a linear map \( S : V \to U \) such that \( S \circ T = \text{id}_U \) and \( T \circ S = \text{id}_V \). For a linear map \( T : U \to V \) to have an inverse, \( T^{-1} \), it is necessary that \( \dim U = \dim V \).

We have the following important result:

For an \( n \times n \) matrix the following are equivalent:

- \( Ax = 0 \) has only the trivial solution;
- \( Ax = y \) has for every \( y \) exactly one solution \( x \)
- \( A \) has an inverse.

All of this follows from the theorem that a linear map on a finite dimensional vector space is injective if and only if it is surjective.

Now, how can we find the inverse of a matrix? While the book postpones this up to a later chapter, see p.100, Example 2, using our current knowledge on linear maps this is actually quite trivial to do. Let us explain this on that example:

\[
A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}
\]

stands for the linear map \( T = L_A : \mathbb{R}^2 \to \mathbb{R}^2 \) where

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 7 \\ 3 \end{pmatrix}.
\]

That is \( A = \text{Mat}(T; e_1, e_2; e_1, e_2) \) where the \( e_i \) are the unit vectors in \( \mathbb{R}^2 \). We have

\[
T^{-1} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad T^{-1} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then obviously,

\[
\text{Mat}(T^{-1}; Te_1, Te_2; e_1, e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

and this is not what we want. We want

\[
\text{Mat}(T^{-1}, e_1, e_2; e_1, e_2) = A^{-1}.
\]

For this we need to find

\[
T^{-1}(e_1) = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \quad T^{-1}(e_2) = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}
\]

But for this we need to express the unit vectors as linear combinations of \( \begin{pmatrix} 5 \\ 2 \end{pmatrix} \) and

\[
\begin{pmatrix} 7 \\ 3 \end{pmatrix} = x \begin{pmatrix} 5 \\ 2 \end{pmatrix} + y \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

This is the inhomogeneous linear system of
2 equations in 2 unknowns with augmented matrix
\[
\begin{pmatrix}
5 & 7 & 1 \\
2 & 3 & 0
\end{pmatrix}
\] which has the row echelon form:
\[
\begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -2
\end{pmatrix}
\]. Hence:
\[x = 3, y = -2.\]
And from:
\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
= 3 \begin{pmatrix}
5 \\
2
\end{pmatrix} - 2 \begin{pmatrix}
7 \\
3
\end{pmatrix}
\]
we get
\[
T^{-1} \begin{pmatrix}
1 \\
0
\end{pmatrix} = 3T^{-1} \begin{pmatrix}
5 \\
2
\end{pmatrix} - 2T^{-1} \begin{pmatrix}
7 \\
3
\end{pmatrix} = 3 \begin{pmatrix}
1 \\
0
\end{pmatrix} - 2 \begin{pmatrix}
0 \\
1
\end{pmatrix} = \begin{pmatrix}
3 \\
-2
\end{pmatrix}
\]
and similarly for the second column of the inverse. Actually we can work out simultaneously both inhomogeneous systems where the right hand sides are the unit vectors
\[
\begin{pmatrix}
5 & 7 & 1 & 0 \\
2 & 3 & 0 & 1
\end{pmatrix}, \text{ row echelon form: } \begin{pmatrix}
1 & 0 & 3 & -7 \\
0 & 1 & -2 & 5
\end{pmatrix}
\]
Thus, \(\text{Mat}(T^{-1}; e_1, e_2; e_1, e_2) = \begin{pmatrix}
3 & -7 \\
-2 & 5
\end{pmatrix}\).

Let \(A\) be an \(n \times n\) matrix where \(L_A\) has an inverse. Then \(A\) has an inverse \(A^{-1}\). Then if \(A_j\) is the \(j^{th}\) column of \(A\) then \(L_A(e_j) = A_j\). Then if
\[x_1A_1 + x_2A_2 + \ldots + x_nA_n = e_j,\]
where \(e_j\) is the \(j\)th unit vector
then applying \(L_A^{-1}\) to this equation gives:
\[x_1e_1 + x_2e_2 + \ldots + x_ne_n = L_A^{-1}(e_j)\]
This is \(L_A^{-1}(e_j) = \begin{pmatrix}
x_{1j} \\
x_{2j} \\
\vdots \\
x_{nj}
\end{pmatrix} = X_j\) and the matrix with columns \(X_j\) is the matrix of \(L_A^{-1}\) and the inverse of \(A\).

Let \((A \mid I_n)\) be the matrix \(A\) augmented by the \(n\) columns of unit vectors \(e_1, e_2, \ldots, e_n\). Then using the elementary row operations transforms \(A\) into \(I_n\) and \(I_n\) into \(A^{-1}\).

\[(A \mid I_n) \overset{\text{elementary row operations}}{\Rightarrow} (I_n \mid A^{-1})\]
Actually, \(AA^{-1} = I_n\). This tells us that \(L_A^{-1}\) is injective. But then it must be also surjective, that is \(AA^{-1}\) also.