

Problems and Comments for Section 11, 12, 13

~

Problems: 12.1, 12.2, 12.3, 12.4 (a)-(c), 12.29, 12.30; 13.1, 13.5, 13.7, 13.8; 11.3, 11.4, 11.5, 11.6, 11.14, 11.15, 11.16, 11.26 (optional)

Comments: These three sections should be read simultaneously. It's best to start with section 12 on Homomorphisms. You may skip everything after example 4 on page 121.

A homomorphism is a map between similar algebras that preserves all the operations. Say, if $f = f^{\mathbf{A}}$ is some binary operation on the algebra \mathbf{A} and $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ a map from \mathbf{A} to a similar algebra \mathbf{B} , where f is the operation $f^{\mathbf{B}}$, then φ is called a *homomorphism* if

$$\varphi(f^{\mathbf{A}}(a_1, a_2)) = f^{\mathbf{B}}(\varphi(a_1), \varphi(a_2))$$

We may drop the superscripts and talk about the operation f on \mathbf{A} and \mathbf{B} , respectively. A homomorphism between groups preserves the binary operation $*$, the unary operation $^{-1}$ and the constant e . Thus for a group homomorphism we stipulate the conditions

$$\begin{aligned}\varphi(a_1 * a_2) &= \varphi(a_1) * \varphi(a_2) \\ \varphi(a^{-1}) &= \varphi(a)^{-1} \\ \varphi(e) &= e\end{aligned}$$

As it is shown in the text (Theorem 12.4), a map between groups is already a homomorphism if the multiplication $*$ is preserved.

Recall that for any map f between sets A and B the equivalence relation $\ker(f)$ is defined on A by:

$$a_1 \sim a_2 \text{ iff } f(a_1) = f(a_2)$$

The set of equivalence classes is called the quotient set $C = A/\ker(f)$ and $q_f : A \rightarrow C, a \mapsto [a]$ is called the *quotient map*. Of course, q_f is surjective. Given any element $c \in C$, it is of the form $c = [a]$ for some $a \in A$. But then $q_f^{-1}(c) = [a]$.

Because f is constant on every class, we can define a map

$$\dot{f} : C \rightarrow B, [a] \mapsto f(a)$$

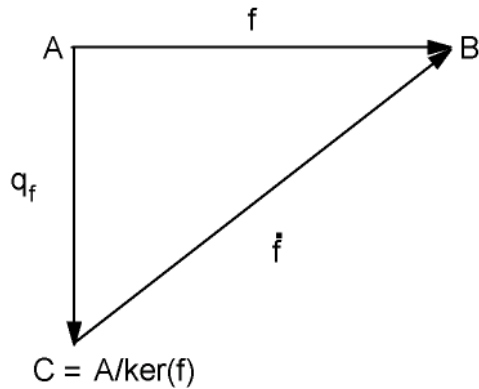
If we take any element of C , then it is of the form $[a]$ for some $a \in A$. If a' is any other element in $[a]$ then by definition of the equivalence class of $[a]$ we have

$f(a) = f(a')$. Thus, \dot{f} is a *well defined* map.

We have that \dot{f} is always injective, and bijective if and only if f is surjective. We have that:

$$\dot{f} \circ q_f = f$$

and this can be expressed by a *commutative diagram*:



It also says: Any map whatsoever is the composition of a surjective map, followed by an injective map.

For a homomorphism $f = \varphi$ and groups \mathbf{A} and \mathbf{B} one defines

$\ker(\varphi) = [e] = \{a | \varphi(a) = e\} = N_\varphi$ where N_φ is a *normal* subgroup of \mathbf{A} and the equivalence classes turn out to be co-sets:

$$[a] = aN = Na$$

These cosets can be made to a group \mathbf{C} by defining:

$$[a] * [b] = [a * b]$$

$$[a]^{-1} = [a^{-1}]$$

$$e = [e] = N$$

and with these definitions $\mathbf{C} = \mathbf{A}/N$ is a group and q_φ is (trivially) a homomorphism, and $\bar{\varphi}$ is a homomorphism. In particular, if φ is surjective then $\bar{\varphi}$ is an isomorphism. This is the **Fundamental Theorem on group homomorphisms (Theorem 13.2)**.