Nonlinear seismic imaging via reduced order model backprojection

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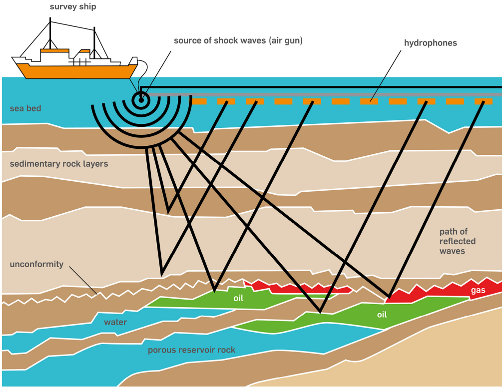
Motivation: seismic oil and gas exploration



- **Seismic exploration**
- Seismic waves in the subsurface induced by sources (shots)
- Measurements of seismic signals on the surface or in a well bore
- Determine the acoustic or elastic parameters of the subsurface

Acoustic wave equation

- Consider an acoustic wave equation in the **time domain**

$$u_{tt} = \mathbf{A}u \quad \text{in } \Omega, \quad t \in [0, T]$$

  with initial conditions

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = 0$$

- The spatial operator $\mathbf{A} \in \mathbb{R}^{N \times N}$ is a fine grid discretization of

$$A(c) = c^2 \Delta$$

  with the appropriate boundary conditions
- The solution is

$$u(t) = \cos(t\sqrt{-\mathbf{A}})u_0$$

Source model

- We stack all p sources in a single tall skinny matrix $\mathbf{S} \in \mathbb{R}^{N \times p}$ and introduce them in the initial condition

$$\mathbf{u}|_{t=0} = \mathbf{S}, \quad \mathbf{u}_t|_{t=0} = 0$$

- The solution matrix $\mathbf{u}(t) \in \mathbb{R}^{N \times p}$ is

$$\mathbf{u}(t) = \cos(t\sqrt{-\mathbf{A}})\mathbf{S}$$

- We assume the form of the source matrix

$$\mathbf{S} = q^2(\mathbf{A})\mathbf{C}\mathbf{E},$$

  where $\mathbf{E}$ are p point sources supported on the surface, $q^2(\omega)$ is the Fourier transform of the source wavelet and $\mathbf{C} = \text{diag}(\mathbf{c})$
- Here we take $q^2(\omega) = e^{\sigma\omega}$ with small σ so that $\mathbf{S}$ is localized near $\mathbf{E}$, only assumes the knowledge of $\mathbf{c}$ and thus $\mathbf{A}$ near the surface

Receiver and data model

- For simplicity assume that the sources and receivers are collocated
- Then the receiver matrix $\mathbf{R} \in \mathbb{R}^{N \times p}$ is

$$\mathbf{R} = \mathbf{C}^{-1}\mathbf{E}$$

- Combining the source and receiver we get the **data model**

$$\mathbf{F}(t; \mathbf{c}) = \mathbf{R}^T \cos(t\sqrt{-\mathbf{A}(\mathbf{c})})\mathbf{S},$$

  a $p \times p$ matrix function of time
- The data model can be fully symmetrized

$$\mathbf{F}(t) = \widehat{\mathbf{B}}^T \cos\left(t\sqrt{-\widehat{\mathbf{A}}}\right)\widehat{\mathbf{B}},$$

  with $\widehat{\mathbf{A}} = \mathbf{C}\mathbf{\Delta}\mathbf{C}$ and $\widehat{\mathbf{B}} = q(\widehat{\mathbf{A}})\mathbf{E}$

Seismic inversion and imaging

1. **Seismic inversion**: determine $\mathbf{c}$ from the knowledge of measured data $\mathbf{F}^\star(t)$ (full waveform inversion, FWI); highly nonlinear since $\mathbf{F}(\,\cdot\,;\mathbf{c})$ is nonlinear in $\mathbf{c}$
   - Conventional approach: non-linear least squares (output least squares, OLS)
   $$\underset{\mathbf{c}}{\text{minimize}}\,\|\mathbf{F}^\star - \mathbf{F}(\,\cdot\,;\mathbf{c})\|_2^2$$
   - Abundant local minima
   - Slow convergence
   - Low frequency data needed
2. **Seismic imaging**: estimate $\mathbf{c}$ or its discontinuities given $\mathbf{F}(t)$ and also a smooth kinematic model $\mathbf{c}_0$
   - Conventional approach: linear migration (Kirchhoff, reverse time migration - RTM)
   - Major difficulty: multiple reflections

Reduced order models

- The data is always discretely sampled, say uniformly at $t_k = k\tau$
- The choice of τ is very important, optimally we want τ around Nyquist rate
- The discrete data samples are

$$\mathbf{F}_k = \mathbf{F}(k\tau) = \widehat{\mathbf{B}}^T \cos\left(k\tau\sqrt{-\widehat{\mathbf{A}}}\right)\widehat{\mathbf{B}} =$$

$$= \widehat{\mathbf{B}}^T \cos\left(k \arccos\left(\cos \tau\sqrt{-\widehat{\mathbf{A}}}\right)\right)\widehat{\mathbf{B}} = \widehat{\mathbf{B}}^T T_k(\widehat{\mathbf{P}})\widehat{\mathbf{B}},$$

  where T_k is Chebyshev polynomial and the **propagator** is

$$\widehat{\mathbf{P}} = \cos\left(\tau\sqrt{-\widehat{\mathbf{A}}}\right)$$

- We want a **reduced order model** (ROM) $\widetilde{\mathbf{P}}$, $\widetilde{\mathbf{B}}$ that fits the measured data

$$\mathbf{F}_k = \widehat{\mathbf{B}}^T T_k(\widehat{\mathbf{P}})\widehat{\mathbf{B}} = \widetilde{\mathbf{B}}^T T_k(\widetilde{\mathbf{P}})\widetilde{\mathbf{B}}, \quad k = 0, \ldots, 2n-1$$

Projection ROMs

- Projection ROMs are obtained from

$$\widetilde{\mathbf{P}} = \mathbf{V}^T\widehat{\mathbf{P}}\mathbf{V}, \quad \widetilde{\mathbf{B}} = \mathbf{V}^T\widehat{\mathbf{B}},$$

  where **V** is an orthonormal basis for some subspace
- How do we get a ROM that fits the data?
- Consider a matrix of **solution snapshots**

$$\mathbf{U} = [\widehat{\mathbf{u}}_0, \widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_{n-1}] \in \mathbb{R}^{N\times np}, \quad \widehat{\mathbf{u}}_k = T_k(\widehat{\mathbf{P}})\widehat{\mathbf{B}}$$

Theorem (ROM data interpolation)

If $span(\mathbf{V}) = span(\mathbf{U})$ *and* $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ *then*

$$\mathbf{F}_k = \widehat{\mathbf{B}}^T T_k(\widehat{\mathbf{P}})\widehat{\mathbf{B}} = \widetilde{\mathbf{B}}^T T_k(\widetilde{\mathbf{P}})\widetilde{\mathbf{B}}, \quad k = 1, \ldots, 2n-1,$$

where $\widetilde{\mathbf{P}} = \mathbf{V}^T\widehat{\mathbf{P}}\mathbf{V} \in \mathbb{R}^{np\times np}$ *and* $\widetilde{\mathbf{B}} = \mathbf{V}^T\widehat{\mathbf{B}} \in \mathbb{R}^{np\times p}$.

Obtaining the ROM from the data

- We do not know the solutions in the whole domain **U** and thus **V** is unknown
- How do we obtain the ROM from just the data $\mathbf{F}_k$?
- The data does not give us **U**, but it gives us the **inner products**!
- A basic property of Chebyshev polynomials is

$$T_i(x)T_j(x) = \frac{1}{2}(T_{i+j}(x) + T_{|i-j|}(x))$$

- Then we can obtain

$$(\mathbf{U}^T\mathbf{U})_{i,j} = \mathbf{u}_i^T\mathbf{u}_j = \frac{1}{2}(\mathbf{F}_{i+j} + \mathbf{F}_{i-j}),$$
$$(\mathbf{U}^T\widehat{\mathbf{P}}\mathbf{U})_{i,j} = \mathbf{u}_i^T\widehat{\mathbf{P}}\mathbf{u}_j = \frac{1}{4}(\mathbf{F}_{j+i+1} + \mathbf{F}_{j-i+1} + \mathbf{F}_{j+i-1} + \mathbf{F}_{j-i-1})$$

Obtaining the ROM from the data

- Suppose $\mathbf{U}$ is orthogonalized by a **block QR** procedure

$$\mathbf{U} = \mathbf{V}\mathbf{L}^T,$$

so $\mathbf{V} = \mathbf{U}\mathbf{L}^{-T}$, where $\mathbf{L}$ is a **block Cholesky** factor of the Gramian $\mathbf{U}^T\mathbf{U}$ known from the data

$$\mathbf{U}^T\mathbf{U} = \mathbf{L}\mathbf{L}^T$$

- The projection is given by

$$\widetilde{\mathbf{P}} = \mathbf{V}^T\widehat{\mathbf{P}}\mathbf{V} = \mathbf{L}^{-1}\left(\mathbf{U}^T\widehat{\mathbf{P}}\mathbf{U}\right)\mathbf{L}^{-T},$$

where $\mathbf{U}^T\widehat{\mathbf{P}}\mathbf{U}$ is also known from the data

- The use of Cholesky for orthogonalization is essential, (block) lower triangular structure is the linear algebraic equivalent of **causality**

Use of ROMs

- Once we have the ROM $\widetilde{\mathbf{P}} = \mathbf{V}^T\widehat{\mathbf{P}}\mathbf{V}$, $\widetilde{\mathbf{B}} = \mathbf{V}^T\widehat{\mathbf{B}}$ how do we estimate $\mathbf{c}$ from it?

- The ROM for the operator $\mathbf{A}$ itself is

$$\widetilde{\mathbf{A}} = \frac{2}{\tau^2}(\widetilde{\mathbf{P}} - \mathbf{I})$$

  from truncated Taylor's expansion

- **Inversion**: transform $\widetilde{\mathbf{A}}$ to a **block finite difference** (bFD) scheme, use the bFD coefficients in optimization

- **Imaging**: Using a smooth kinematic model $\mathbf{c}_0$ **backproject** $\widetilde{\mathbf{A}}$ to get the coefficient $\mathbf{c}$ directly

Seismic inversion: optimization preconditioning

- Recall the conventional FWI (OLS)

$$\underset{\mathbf{c}}{\text{minimize}} \, \|\mathbf{F}^\star - \mathbf{F}(\,\cdot\,;\mathbf{c})\|_2^2$$

- Replace the objective with a **"nonlinearly preconditioned"** functional

$$\underset{\mathbf{c}}{\text{minimize}} \, \|\widetilde{\mathbf{A}}^\star - \widetilde{\mathbf{A}}(\mathbf{c})\|_F^2,$$

where $\widetilde{\mathbf{A}}^\star$ is computed from the data $\mathbf{F}^\star$ and $\widetilde{\mathbf{A}}(\mathbf{c})$ is a (highly) nonlinear mapping

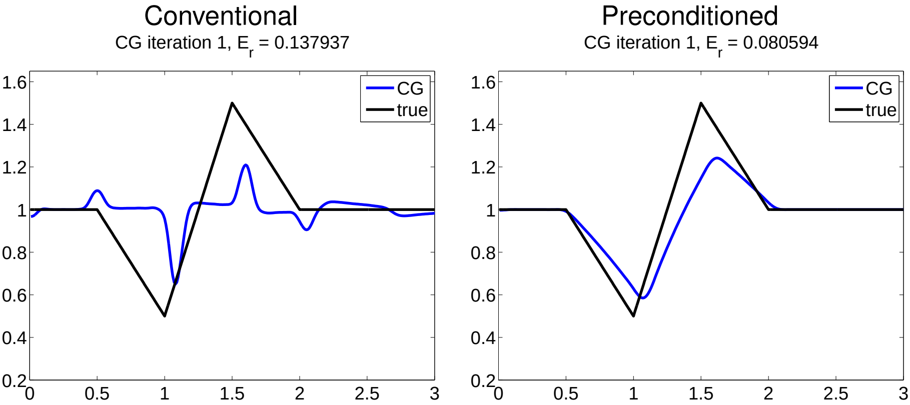
$$\widetilde{\mathbf{A}} : \mathbf{c} \to \mathbf{A}(\mathbf{c}) \to \mathbf{U} \to \mathbf{V} \to \widetilde{\mathbf{P}} \to \widetilde{\mathbf{A}}$$

- Why does this have a preconditioning effect?

Advantages of ROM-preconditioned optimization

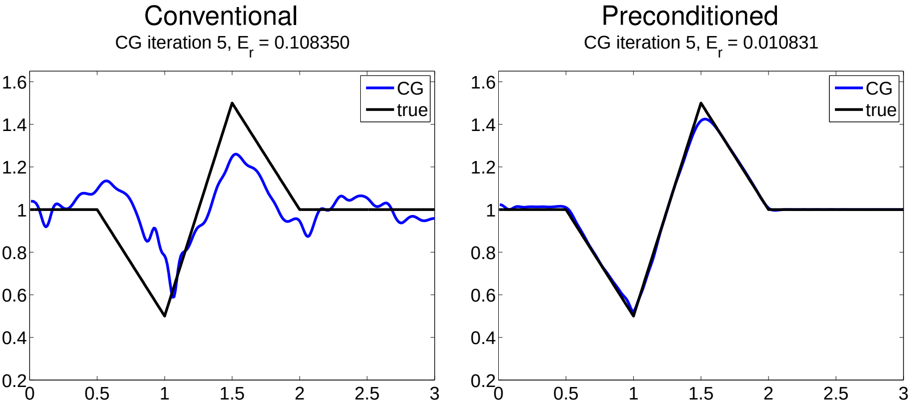
- The biggest issue of conventional OLS FWI is the abundance of **local minima** (cycle skipping)
- The dependency of $\mathbf{A}(\mathbf{c}) = \mathbf{c}^2\boldsymbol{\Delta}$ on $\mathbf{c}^2$ is linear
- In a certain parametrization the dependency of $\widetilde{\mathbf{A}}$ on $\mathbf{c}^2$ should be close to linear
- The preconditioned objective functional is close to quadratic, thus **close to convex**
- Approximate convexity leads to faster, more robust convergence
- Implicit orthogonalization of solution snapshots $\mathbf{V} = \mathbf{U}\mathbf{L}^{-T}$ removes the multiple reflections

Conventional vs. preconditioned in 1D



Faster convergence.

Conventional vs. preconditioned in 1D



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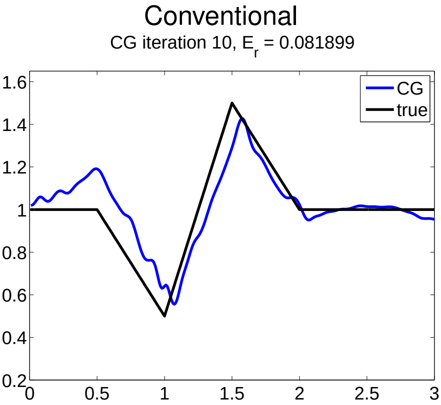

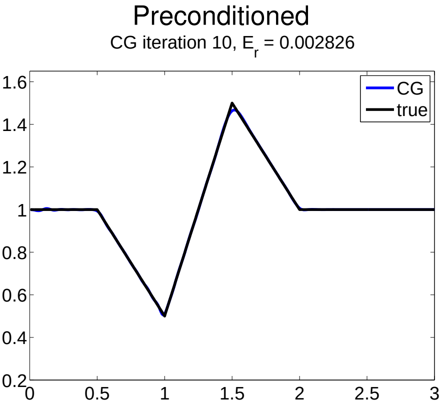

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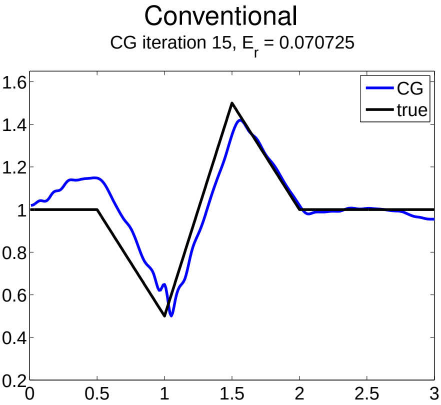
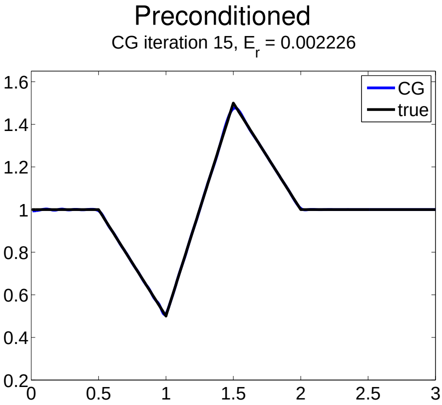
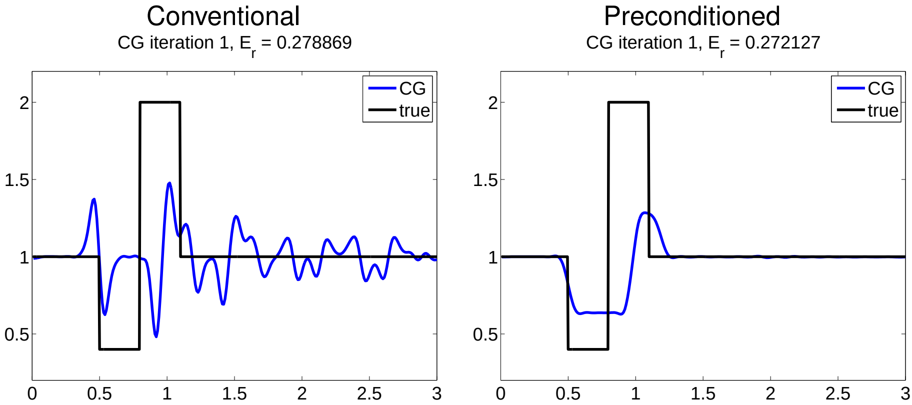




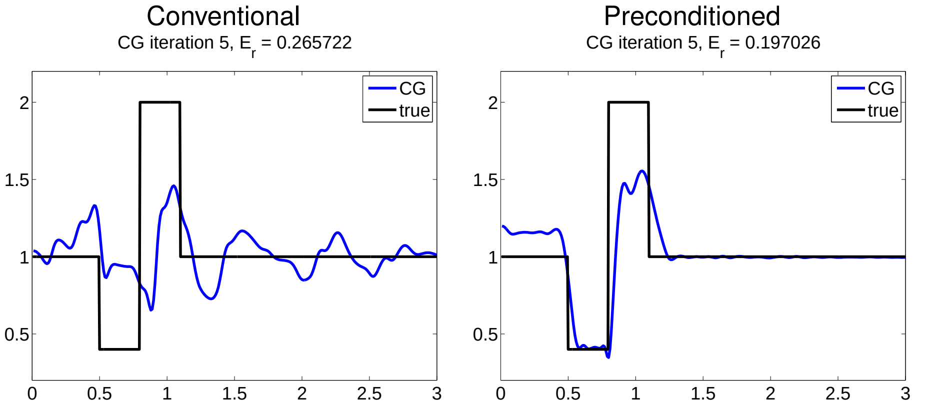
Faster convergence.

Conventional vs. preconditioned in 1D



Automatic removal of multiple reflections.

Conventional vs. preconditioned in 1D



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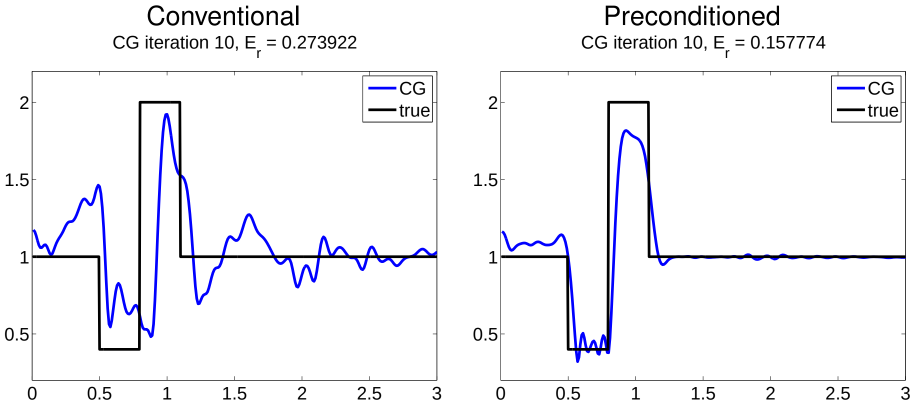

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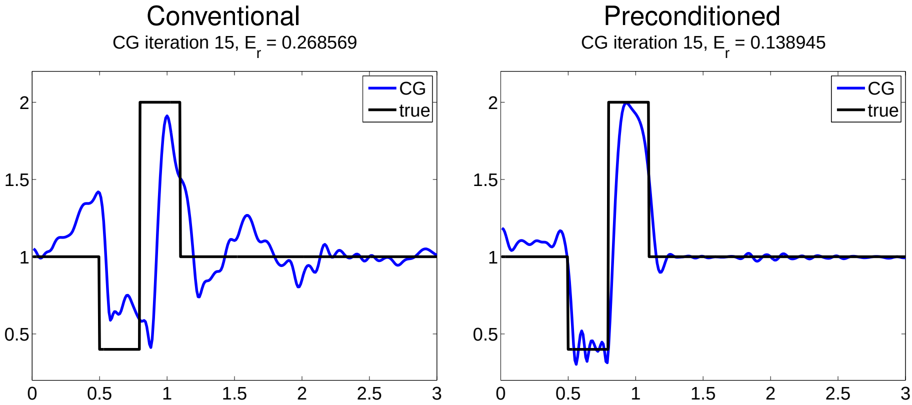

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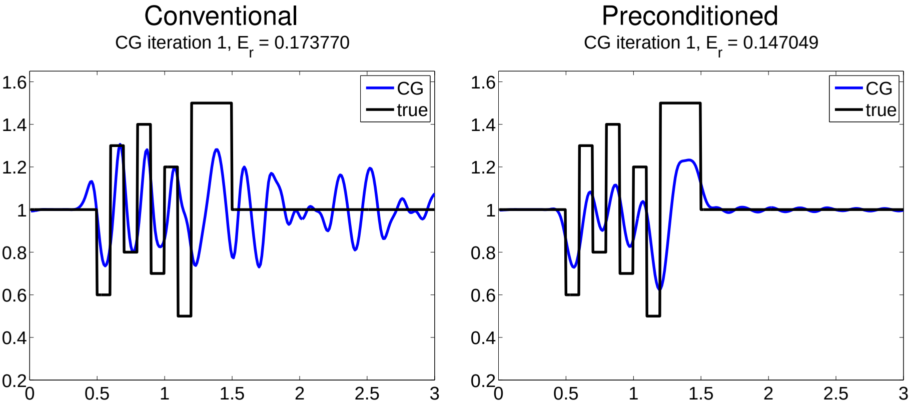
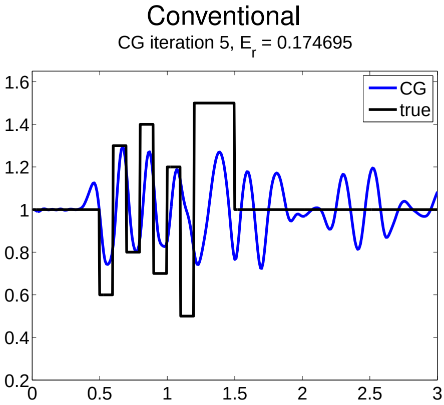
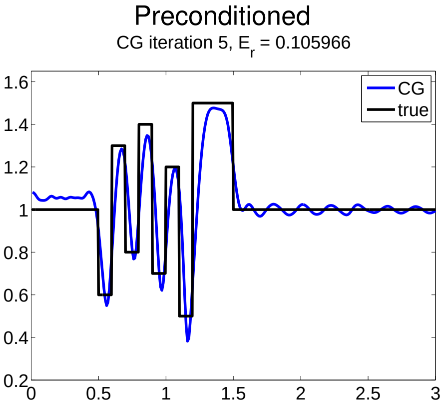


Avoiding the cycle skipping.

Conventional vs. preconditioned in 1D





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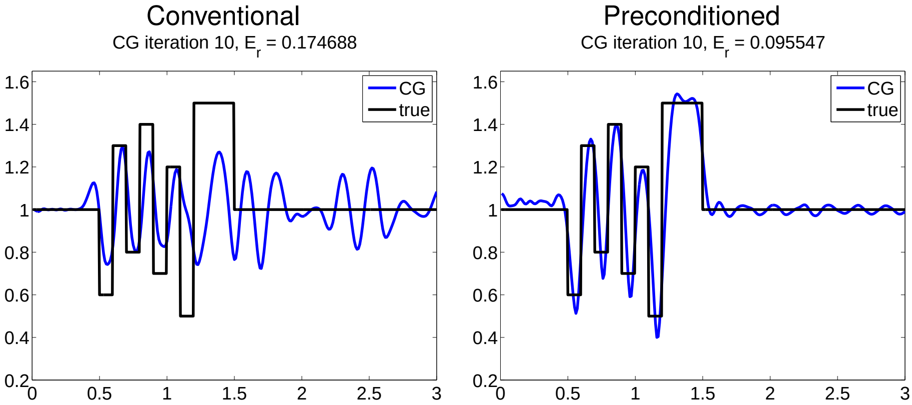
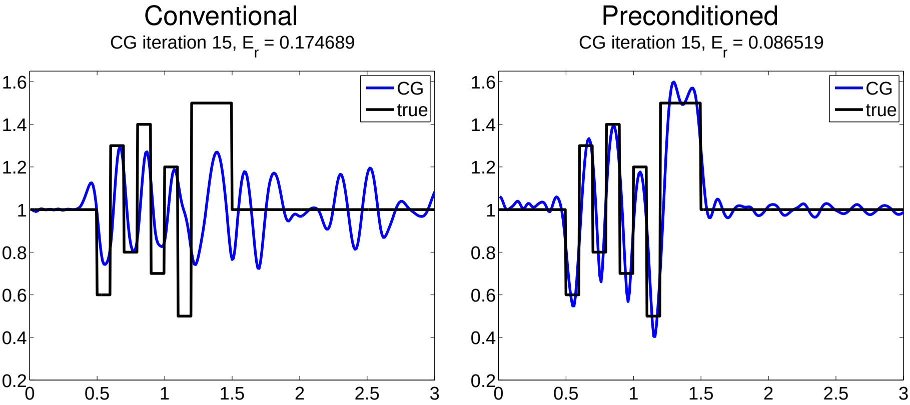


Avoiding the cycle skipping.

Conventional vs. preconditioned in 1D



Avoiding the cycle skipping.

Imaging: backprojection

- The ROM for $\widetilde{\mathbf{A}}$ approximately satisfies
$$\widetilde{\mathbf{A}} \approx \mathbf{V}^T \widehat{\mathbf{A}} \mathbf{V}$$
- If the subspace spanned by $\mathbf{V}$ is sufficiently rich, then
$$\mathbf{V}\mathbf{V}^T \approx \mathbf{I},$$
so we can **backproject** the ROM to the fine grid space
$$\widehat{\mathbf{A}} \approx \mathbf{V}\widetilde{\mathbf{A}}\mathbf{V}^T \approx \mathbf{V}\mathbf{V}^T \widehat{\mathbf{A}} \mathbf{V}\mathbf{V}^T$$
- **Problem**: we do not know $\mathbf{V}$, since the snapshots $\mathbf{U}$ are unknown to us in the whole domain
- Known smooth **kinematic model** $\mathbf{c}_0$ is needed
- From c_0 we can explicitly compute everything: $\widehat{\mathbf{A}}_0$, $\widetilde{\mathbf{A}}$, $\mathbf{U}_0$ and, most important, $\mathbf{V}_0$
- Replace the unknown true $\mathbf{V}$ by known $\mathbf{V}_0$
$$\widehat{\mathbf{A}} \approx \mathbf{V}_0 \widetilde{\mathbf{A}} \mathbf{V}_0^T$$

Backprojection: extracting the PDE coefficient

- We do not need the whole operator $\mathbf{A}$ or $\widehat{\mathbf{A}}$, just the fine grid coefficient $\mathbf{c}^2$
- Recall that $\widehat{\mathbf{A}} = \mathbf{C}\boldsymbol{\Delta}\mathbf{C}$, thus

$$\mathbf{c}^2 \propto \text{diag}(\widehat{\mathbf{A}})$$

- Similarly for the difference we have

$$\delta\mathbf{c}^2 = \mathbf{c}^2 - \mathbf{c}_0^2 \propto \text{diag}(\widehat{\mathbf{A}} - \widehat{\mathbf{A}}_0)$$

- Approximate $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{A}}_0$ by their backprojections to obtain an **imaging relation**

$$\delta\mathbf{c}^2 \propto \text{diag}\left(\mathbf{V}_0(\widetilde{\mathbf{A}} - \widetilde{\mathbf{A}}_0)\mathbf{V}_0^T\right)$$

- Choosing different proportionality factors leads to various **imaging formulae**, for example a multiplicative

$$\mathbf{c}^\star = \mathbf{c}_0\sqrt{1 + \alpha\delta\mathbf{c}^2}$$

Backprojection imaging: features

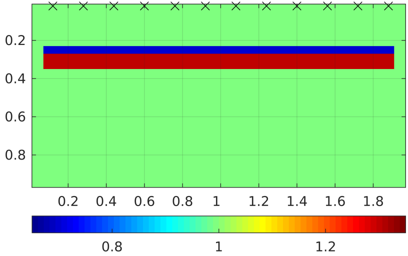
- Conventional imaging techniques (Kirchhoff, RTM) are **linear** in the data
- Our approach is **non-linear** because of **implicit orthogonalization**

$$\widetilde{\mathbf{P}} = \mathbf{L}^{-1} \left(\mathbf{U}^T \widehat{\mathbf{P}} \mathbf{U} \right) \mathbf{L}^{-T}, \quad \mathbf{U}^T \mathbf{U} = \mathbf{L} \mathbf{L}^T$$

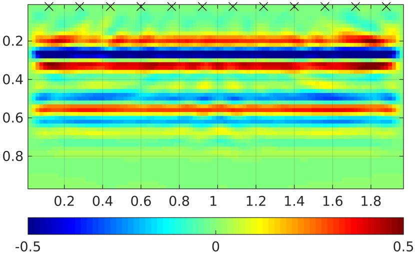
- Block Cholesky: **causal orthogonalization**, removes the “tail”, only the wavefront survives
- Thus, **multiple reflection** artifacts are removed
- We image correctly not only the locations of reflectors, but also their strength: **true amplitude imaging**
- Computationally cheap: we need a forward solution (same as RTM) and an extra orthogonalization step

Removal of multiple reflection artifacts

True sound speed **c**

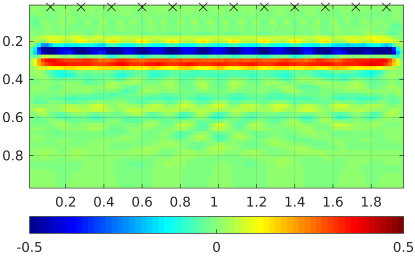


RTM image



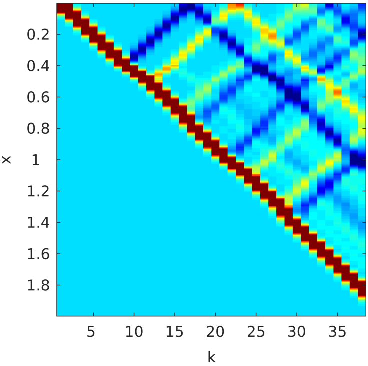
- A simple layered model, $p = 12$ sources/receivers (black $\times$)
- Multiple reflections from waves bouncing between layers and surface
- Each multiple creates an RTM artifact below actual layers

Backprojection image

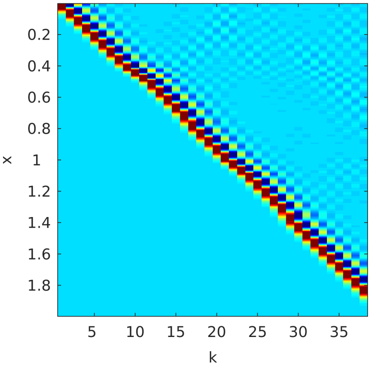


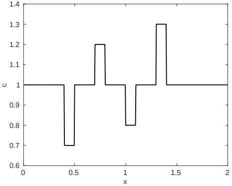
Solution snapshot orthogonalization

Solution snapshots **U**



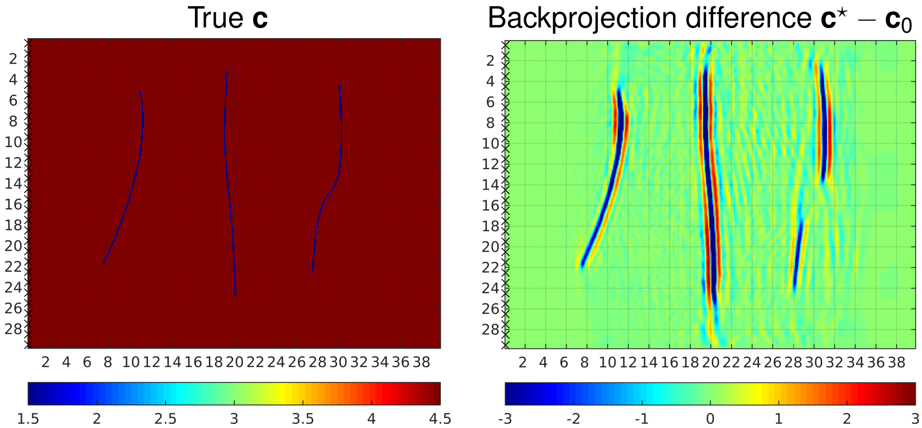
Orthogonalized basis **V**





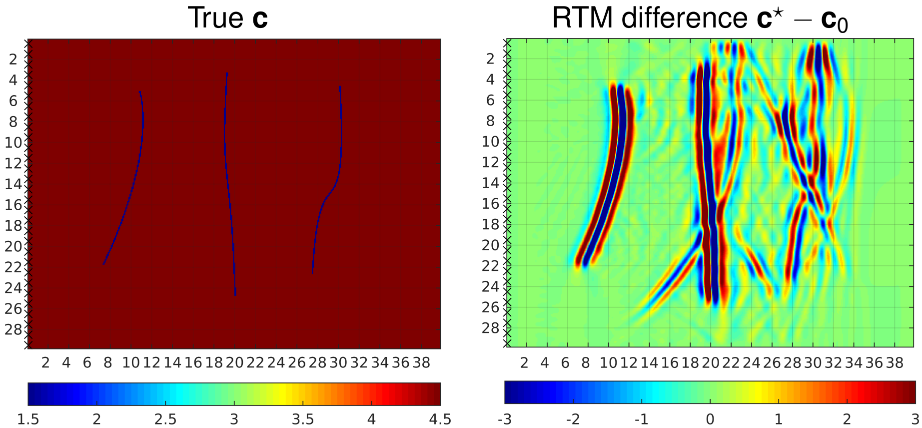
- A 1D analogue of the previous example
- Strong primaries/multiples in **U**, almost none in **V**
- The operator $\widehat{\mathbf{A}}$ is probed with **V** that is mostly a single propagating wavefront

High contrast imaging: hydraulic fractures



- Important application: seismic monitoring of hydraulic fracturing
- Multiple thin fractures (down to 1cm in width, here 10cm)
- Very high contrasts: $c = 4500 m/s$ in the surrounding rock, $c = 1500 m/s$ in the fluid inside fractures

High contrast imaging: hydraulic fractures



- Important application: seismic monitoring of hydraulic fracturing
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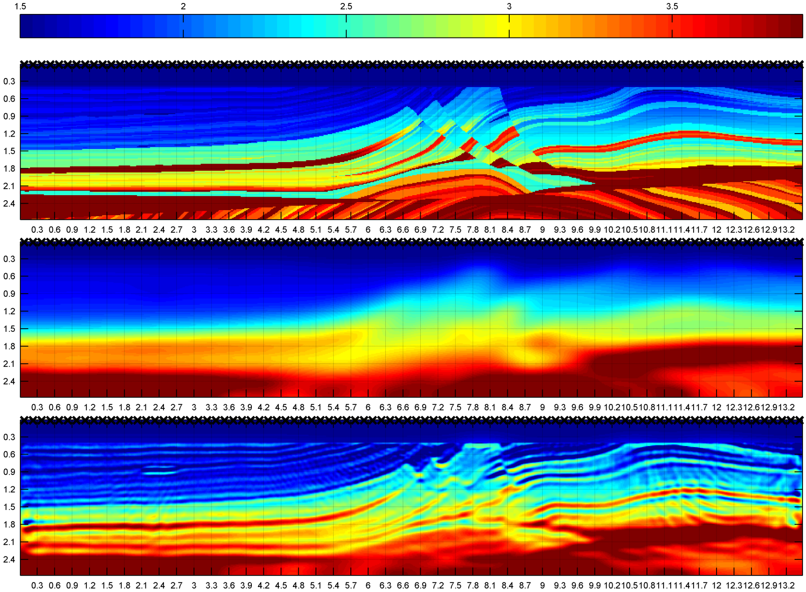
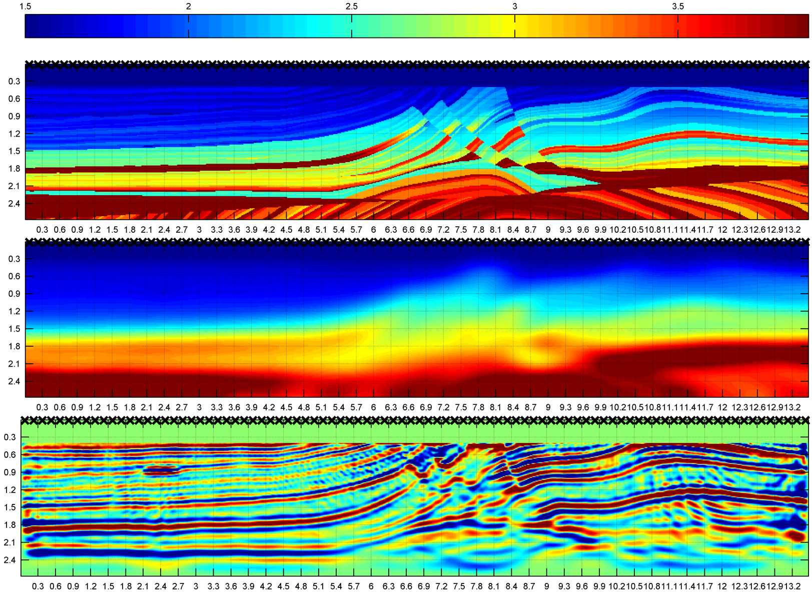
Numerical example: Marmousi model

- Classical Marmousi model, $13.5km \times 2.7km$
- Forward problem is discretized on a $15m$ grid with $N = 900 \times 180 = 162,000$ nodes
- Kinematic model $\mathbf{c}_0$: smoothed out true $\mathbf{c}$ ($465m$ horizontally, $315m$ vertically)
- Time domain data sample rate $\tau = 33.5ms$, source frequency about $15Hz$, $n = 35$ data samples measured
- Number of sources/receivers $p = 90$ uniformly distributed with spacing $150m$
- Data is split into 17 overlapping windows of 10 sources/receivers each ($1.5km$ max offset)
- Reflecting boundary conditions
- No data filtering, everything used as is (surface wave, reflections from the boundaries, multiples)

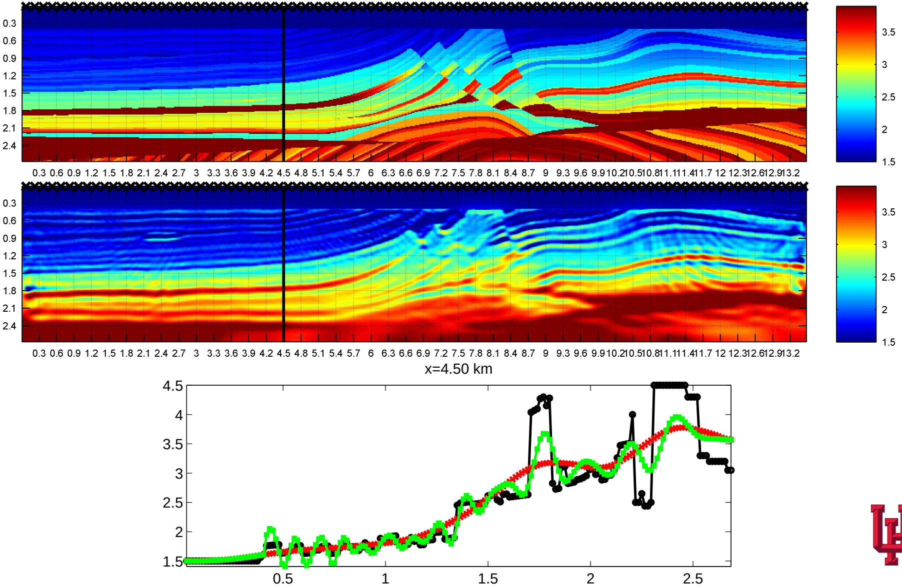
Backprojection imaging: Marmousi model



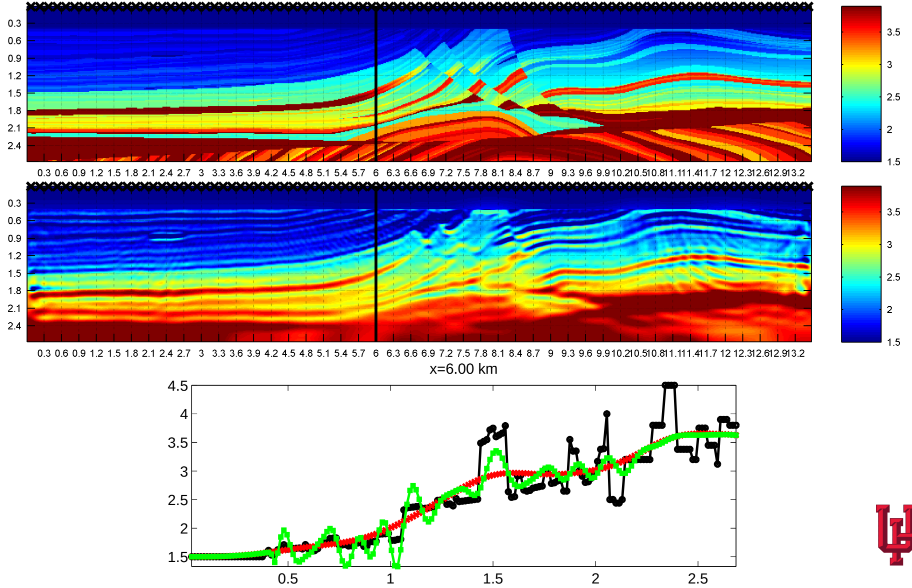
Backprojection imaging: Marmousi model



Marmousi backprojection image: well log



Marmousi backprojection image: well log



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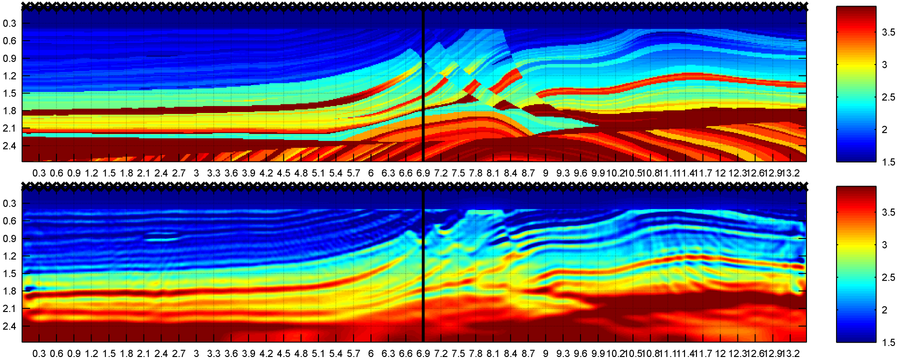

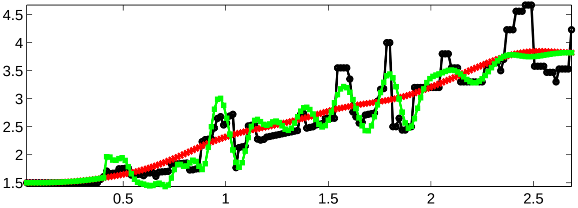

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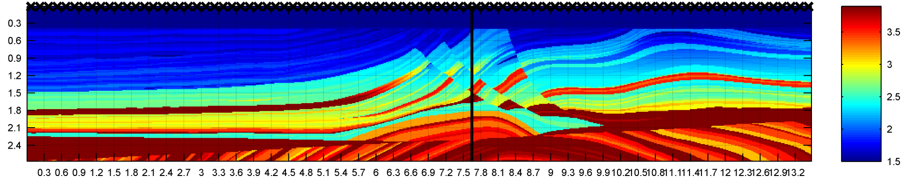

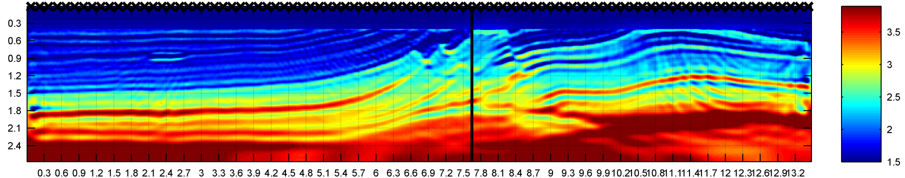

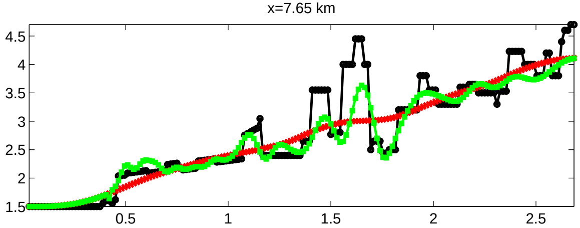

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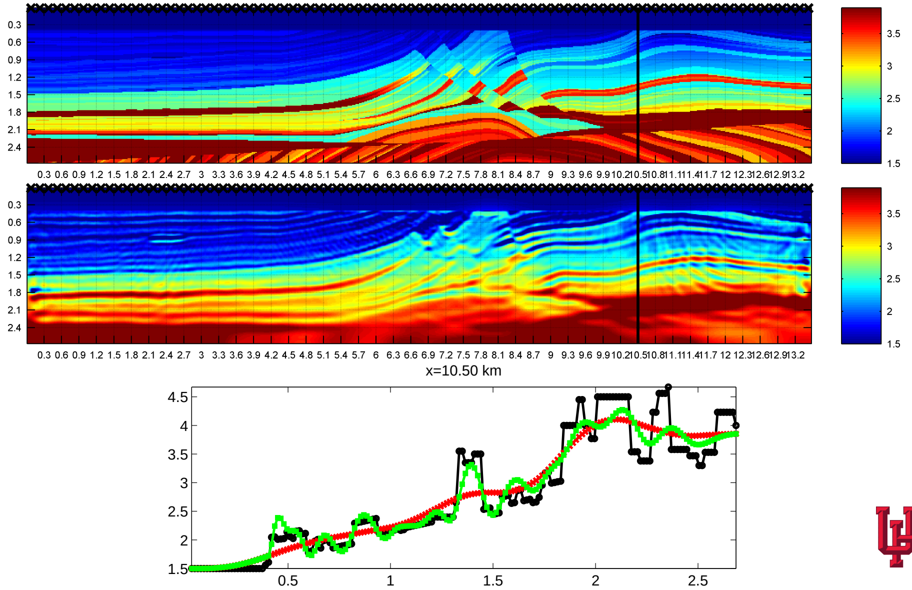

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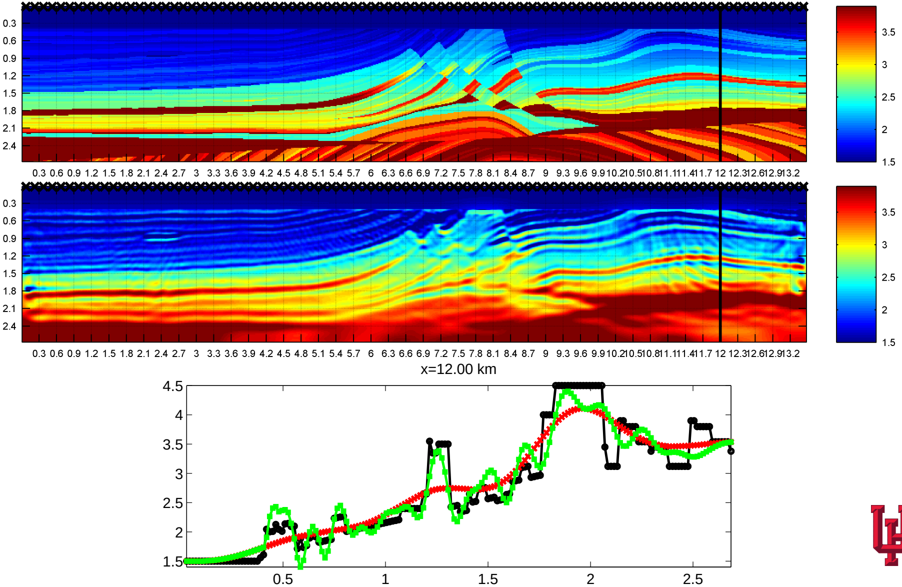

Conclusions and future work

- Novel approach to seismic imaging using reduced order models
- Time domain formulation is essential, makes use of causality (linear algebraic analogue - Cholesky decomposition)
- Nonlinear construction of ROM via implicit causal orthogonalization of solution snapshots
- Strong suppression of multiple reflection artifacts

Future work:

- Non-symmetric setting (non-collocated sources/receivers)
- Full waveform inversion in higher dimensions
- Better theoretical understanding

References:

[1] A.V. Mamonov, V. Druskin, M. Zaslavsky, *Nonlinear seismic imaging via reduced order model backprojection*, SEG Technical Program Expanded Abstracts 2015: pp. 4375–4379.

[2] V. Druskin, A. Mamonov, A.E. Thaler and M. Zaslavsky, *Direct, nonlinear inversion algorithm for hyperbolic problems via projection-based model reduction*. arXiv:1509.06603 [math.NA], 2015.