Section 1.3
Place Value Systems of Numeration

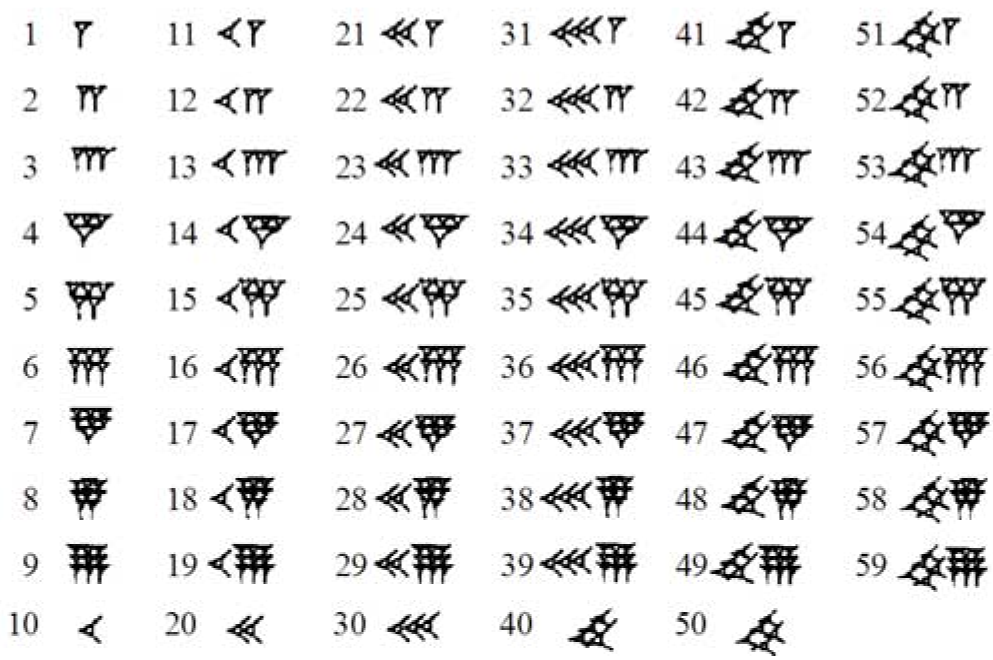
Place Value Systems of Numeration

The fourth type of system of numeration is called a place value system. A place value system consists of a base (a natural number greater than one) and a set of symbols representing the numbers from zero to one less than the base. The numbers from zero to the base minus one are called the digits in the system. Any number is represented by a string of digits, and the value of each digit depends on both which digit it represents and on its place in the string. Hence the name place value system of numeration.

The Hindu-Arabic system that is used in most of the world today is a place value system with a base of ten. However, the idea of a place value system goes back as far as the Babylonian system from 2500 BCE. The Mayan civilization in Central America also developed a place value system of numeration in the first millennium CE. This text will examine both of these systems and compare them with the Hindu-Arabic System we use today.

Babylonian Numerals

The Babylonians had a system with a base of 60. It is not, strictly speaking, a true place value system because the system did not have a symbol for zero. This caused a great deal of ambiguity in the separation of the places in large numbers. Also, there are only two symbols used in the system. A wedge pointing down represented one and a wedge pointing left represented ten. The numbers from 1-59 were made from the appropriate number of wedges of each type. The following picture shows how these numbers appear on tablets found by archeologists.



Notice that the ones and tens were grouped by specific patterns to make them easier to count. Since this form of grouping the ones and tens is not a part of our typesetting system, we will write Babylonian numbers in a horizontal line from left to right and use the symbol "|" for one and "<" for ten.

The place values in the Babylonian system are

$$\ldots, 60^3 = 216{,}000, 60^2 = 3600, 60, 1$$

Each place in a Babylonian number was separated simply by a space. To express a number in Babylonian, it was divided into groups of powers of 60 and written with the largest denomination on the left and decreasing by one factor of 60 in each successive place. There was no symbol for zero, so the places could become confused without more context. The number "|" could represent one or 60*1 without extra reference information.

Example 1: The Babylonian number || <<<|| <<<<||||| is interpreted as $2 \times 60^2 + 32 \times 60 + 45$. To translate this to a Hindu-Arabic number, perform the multiplication.

$$2 \times 60^2 + 32 \times 60 + 45 = 7200 + 1920 + 45 = 9165.$$

Example 2: The Babylonian number <   <|||| <<<||||||||| is interpreted as $10 \times 60^2 + 14 \times 60 + 39$. To translate this number to a Hindu-Arabic number, perform the indicated multiplication:

$$10 \times 60^2 + 14 \times 60 + 39 = 36{,}000 + 840 + 39 = 36{,}879$$

To translate a Hindu-Arabic number into Babylonian, the number must be regrouped into groups of 60. Regrouping into groups of 60 is dividing the number by 60 or a power of 60. There is a simple procedure for performing this division.

Translating a Hindu-Arabic Number into Babylonian

1. Determine the highest power of 60 which is smaller than the number of interest. Let's call this power "n".
2. Divide the number by 60 to the nth power.
3. Find the quotient and remainder. The quotient is the number that goes in that "n+1" place from the right in our Babylonian number.
4. Divide the remainder by the next smaller power of 60. This quotient is the number that goes in the nth place from the right in our Babylonian number.
5. Continue in the same pattern by dividing by the next smaller power of 60.
6. Once the division by 60 has been performed, the division is done. Arrange the quotients in order in Babylonian symbols followed by the last remainder to obtain the final answer.

Example 3: Convert the number 8,404 to Babylonian numerals.

Solution:

The largest power of 60 that is smaller than 8404 is $3600 = 60^2$. Divide by 3600.

$$\begin{array}{r} 2 \\ 3600\overline{)8404} \\ \underline{7200} \\ 1204 \end{array}$$

Next, divide 1204 by $60 = 60^1$.

$$\begin{array}{r} 20 \\ 60\overline{)1204} \\ \underline{1200} \\ 4 \end{array}$$

The remainder is 4, which is less than 60.
$8404 = 2\times 60^2 + 20\times 60 + 4 =$ || << ||||.

Example 4: Write the number 14,589 in Babylonian numerals.

Solution:

The largest power of 60 which is smaller than 14,589 is $3600 = 60^2$. Start by dividing by 3600, and continue by dividing the remainder by 60.

$$\begin{array}{r} 4 \\ 3600\overline{)14,589} \\ \underline{14,400} \\ 189 \end{array} \qquad \begin{array}{r} 3 \\ 60\overline{)189} \\ \underline{180} \\ 9 \end{array}$$

This shows $14,589 = 4\times 3600 + 3\times 60 + 9$.
In Babylonian numerals this is |||| ||| |||||||||.

The base 60 system of the Babylonians is called a sexagesimal system. We are not sure why they developed a base 60 system, and a base of 60 seems very unnatural to most modern peoples using base 10. However, the sexagesimal system has had a large influence on modern mathematics. How many seconds are in a minute? How many minutes are in an hour? How many degrees in a circle? One reason that the base 60 is convenient is that 60 divides evenly by 1,2,3,4,5,6,10,12,15,20 and 30. This makes breaking one unit of 60 into parts or fractions work out nicely. One-third of 60 is 20, a nice counting number; compared with the fact that one-third of 10 is not a natural number.

Mayan Numerals

The Mayan civilization of Central America was very advanced in many ways in mathematics. The Mayans were very interested in astronomy, and developed mathematical language to describe the movements of the sun, moon and stars.

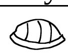
The Mayans used a base 20 system. The earliest system, or “ordinary” system was a strictly base 20 system which used a dot for one and a line for 5. The ordinary system did not function well in astronomy or calendar dates, so the Mayans also had another system that was not strictly base 20. Records of this astronomical system are what have been preserved in the historical record, so this is the system we will learn.

The place values in the Mayan system are

$$...18\times 20^3, 18\times 20^2, 18\times 20, 20, 1.$$

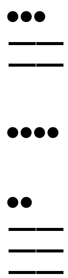
Why 18 times 20? The explanation is that the Mayan calendar was made up of 18 months with 20 days each plus five extra “ghost days” to complete the 365 day year. The “ghost days” were nameless days and were considered very unlucky. The “ghost days” were put into the calendar year because the Mayans knew that the solar year was a little more than 365 days long. The fact that one of the place values in the Mayan system of numeration was 360 made the counting of days easy to do, in fact easier than it is in our system. For example, the calendar date for 1×10^3 days from now is not a trivial calculation in our system. However, in the Mayan system $1\times(18\times 20^2)$ days from now would be the same date 20 years from now on the Mayan calendar (do not count the “ghost days”).

Mayan numerals were written vertically, with the units on the bottom and the place values increasing from bottom to top. The Mayan system also had a symbol for zero which was a fully functioning number. The symbols in the Mayan system are listed in the following table.

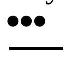
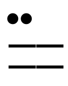
Number	Mayan	Number	Mayan	Number	Mayan	Number	Mayan
0		5	—	10	— —	15	— — —
1	•	6	• —	11	• — —	16	• — — —
2	••	7	•• —	12	•• — —	17	•• — — —
3	•••	8	••• —	13	••• — —	18	••• — — —
4	••••	9	•••• —	14	•••• — —	19	•••• — — —

To translate a Mayan number into Hindu-Arabic numerals, multiply the value of the digit in each place times its place value and then add these numbers.

Example 5: Write the following Mayan number in Hindu-Arabic numerals.

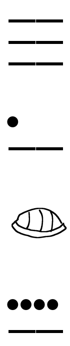


Solution:

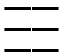
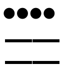
Mayan Numeral	Symbol	Place Value
	13	(18×20)
	4	20
	17	1

The number is $[13 \times (18 \times 20)] + [4 \times 20] + 17 = 4777$.

Example 6: Write the following Mayan number in Hindu-Arabic numerals.



Solution:

Mayan Numeral	Symbol	Place Value
	15	18×20^2
	6	18×20
	0	20
	14	1

The number is $[15 \times (18 \times 20^2)] + [6 \times (18 \times 20)] + [0 \times 20] + 14 = 110{,}174$

To translate a Hindu-Arabic number into Mayan numerals, the number must be grouped into groups the size of each place value. Just as with Babylonian numbers, this grouping procedure is mathematically accomplished by division. The procedure is:

1. Determine the highest place value which is smaller than the number of interest.
2. Divide the number by this place value.
3. Find the quotient and remainder. The quotient is the number that goes in the corresponding "place" in the answer, which will be the top digit in the vertical representation of the number.
4. Divide the remainder by the next smaller place value. This quotient is the number that goes in the next place down in the Mayan number.
5. Continue in the same pattern by dividing by the next place value until you divide by 20. The quotient goes in the 20 place and the remainder goes in the digit (ones) place.

Arrange the quotients from top to bottom and then add the last remainder as the final digit.

Example 7: Write the number 9,235 in Mayan numerals.

Solution:

The place values in the Mayan system are …, 144,000, 7200, 360, 20, 1. The largest place value smaller than 9235 is 7200. So, we will start by dividing by 7200.

$$\begin{array}{r} 1 \\ 7200 \overline{)9235} \\ \underline{7200} \\ 2035 \end{array}$$

Next we will continue by dividing the remainder of each division problem by the next smaller place value until we have finished dividing by 20.

$$\begin{array}{r} 5 \\ 360\overline{)2035} \\ \underline{1800} \\ 235 \end{array} \qquad\qquad \begin{array}{r} 11 \\ 20\overline{)235} \\ \underline{220} \\ 15 \end{array}$$

The quotients, plus the last remainder, in order are 1, 5, 11 and 15.
The following chart helps us to organize our division results and translate to Mayan numerals easily. Note that the last remainder goes in the quotient column.

Place Value	Quotient	Mayan Numeral
7200	1	•
360	5	—
20	11	• — —
1	15	— — —

The right column is 9235 as a Mayan numeral.

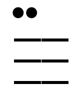
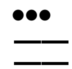
Example 8: Write the number 6,133 in Mayan numerals.

Solution: The Mayan place values are …, 7200, 360, 20, 1. The largest place value smaller than 6133 is 360. Therefore, we will start the division process with dividing by 360. Then we will divide each remainder by the next smaller place value.

$$\begin{array}{r} 17 \\ 360\overline{)6133} \\ \underline{6120} \\ 13 \end{array} \qquad\qquad \begin{array}{r} 0 \\ 20\overline{)13} \\ \underline{0} \\ 13 \end{array}$$

Note that the remainder after dividing by 360 was smaller than 20. However, the division is not finished until we divide by 20. So, the next division problem is needed, and the quotient of 0 means that there should be a 0 in the 20 place in our number. The final remainder is 13, which goes in the 1's place in our Mayan number.

To write the Mayan number, it is easiest to construct it by means of the table as in the last example.

Place Value	Quotient	Mayn Numeral
360	17	
20	0	
1	13	

The right column is the Mayan representation of 6133.

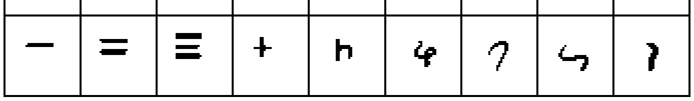
The major advantage of the Mayan system over the earlier Babylonian system was the inclusion of a zero. This eliminates the ambiguity problem if one of the “places” in the representation of a number was empty. The Mayan system also served their needs for counting days and months and years with ease. However, the base 20 system is awkward, and especially so when the place values are not even simple powers of 20.

Hindu-Arabic Numerals

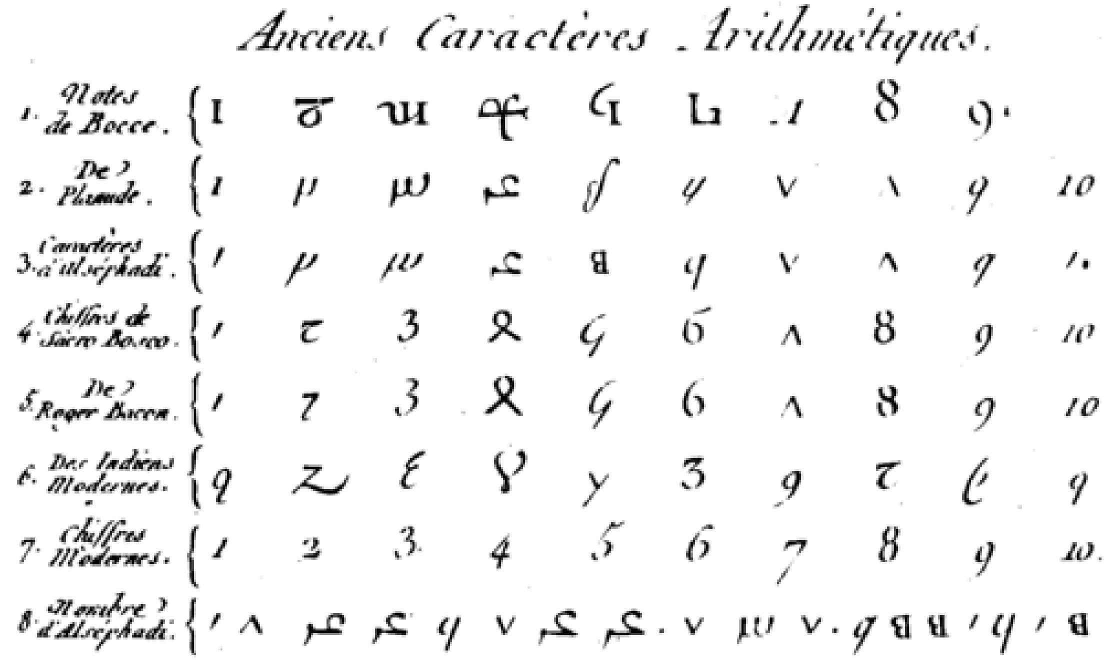
The system of numeration used in the United States and in most of the modern world is called the Hindu-Arabic system of numeration. It is a place value system of numeration with a base of ten. The system was developed in India by the ninth century CE, hence the “Hindu” part of the title. The “Arabic” part of the name of the system is because the system was introduced into Europe by Arabic traders and scholars as early as the 12th century CE. The Hindu-Arabic system was not widely adopted in Europe until the 15th or 16th century. Before this time, most of Europe was using Roman numerals. Can you imagine a scientist working in Roman numerals? Imagine how excited he would be able to do simple arithmetic in Hindu-Arabic numerals instead of Roman numerals!

The symbols for the digits 1-9 are descended from the Brahmi numerals in India. The following table shows the symbols as they were written in 100 CE.

1	2	3	4	5	6	7	8	9



The symbols evolved over the centuries as written language evolved. The following table was created by the French scholar J. E. Montucla in his *Histoire de la Mathematique*, which was published in 1757.



Further modifications of the symbols have been made by the advent of modern typesetting and computers.

The Hindu-Arabic system also has a fully functioning zero. This is a critical to the ability to distinguish numbers like 1, 10, 100, 1000 and 10,000. The real number zero also is an integral part of our understanding of the concept of quantity, so a simple symbol for zero is absolutely necessary in modern mathematics.

Why base ten? What makes the quantity ten so special? Most of us just assume that base ten is the "perfect base" because that is what we learned. Why not base 12? The simple answer is that most human beings have ten fingers, so a base of ten IS the natural base for a place value system.

Summary

The examples of the different types of systems of numeration show that the simple exercise of writing the numbers one through ten in modern symbols is actually a very sophisticated process. It has taken millennia to develop the base ten system with a zero. With this system we can devise relatively simple algorithms for arithmetic. We can also represent very large numbers and very small numbers with scientific notation. The system is even flexible enough to represent fractions in decimal notation with a decimal point and places with values 1/10, 1/100, etc. The advances of modern mathematics would not have been possible without this sophisticated system for representing numbers in written form.