Section 1.5
Arithmetic in Other Bases

Arithmetic in Other Bases

The operations of addition, subtraction, multiplication and division are defined for counting numbers independent of the system of numeration used to express the numbers. However, the algorithms we use to perform these operations are heavily dependent on the properties of system of numeration used. The basic features of the Hindu-Arabic system of numeration are implicit in the basic algorithms we teach elementary school students for the arithmetic operations. In this section we will learn some of these algorithms in other bases. The challenge is to pinpoint the features of the algorithms that are directly dependent on a place value system of numeration.

Addition

Adding 6 plus 7 is the mathematical language representation of putting together a collection of six things and a collection of seven things and asking how many things are in the total collection. This definition of addition does not really depend on the system of numeration used, since 6 and 7 could be written in any system of numeration without changing the question. The answer of thirteen things is also not dependent on the system of numeration. However, as soon as the answer is given as 6 + 7 = 13, the fact that 6+7 is one group of ten plus three extras uses a base ten system of numeration.

Our algorithm for adding two or more digit numbers in base ten is even more dependent on place values. When we "carry a 1" we are adding one group of ten to the next higher place.

Example 1: Add in base ten 639 + 683

Solution:

First, add the 9 and the 3 in the units column, giving 12. Twelve is one set of ten plus 2, so place the 2 in the answer line in the units column and carry the 1 for one set of ten to the tens column.

$$\begin{array}{cccc} & & ^{1} & \\ & 6 & 3 & 9 \\ + & 6 & 8 & 3 \\ \hline & & & 2 \end{array}$$

Then, add the numbers in the tens column, giving 12. This is 10 sets of 10 plus 2 sets of ten. Place the 2 in the answer line in the tens column and carry a one to the hundreds column since 10 sets of 10 is one set of 100.

$$\begin{array}{cccc} & \overset{1}{6} & \overset{1}{3} & 9 \\ + & 6 & 8 & 3 \\ \hline & & 2 & 2 \end{array}$$

Finally, add the numbers in the hundreds column, giving 13. This is 10 sets of 100 plus 3 sets of 100. The 3 sets of 100 is a 3 in the hundreds column in the answer line. The 10 sets of 100 is one set of 1000, so this 1 goes in the thousands column in the answer.

$$\begin{array}{cccc} & \overset{1}{6} & \overset{1}{3} & 9 \\ + & 6 & 8 & 3 \\ \hline 1 & 3 & 2 & 2 \end{array}$$

The final answer is 1322.

The same "carrying" technique can be used to add multi-digit numbers in a place value system with any base. However, remember that in a base b, we carry one group of b over to the next place instead of one group of ten. Considering our example of 6 plus 7, $6_8 + 7_8$ is one group of 8 with 5 left over, or $6_8 + 7_8 = 15_8$.

Example 2: Add in base 7: $153_7 + 216_7$

Solution:

First, write the addition problem in column form. Adding starts with the units column on the right just as in base ten. Since 3+6=9 and 9 is one set of 7 plus 2 extras units, the 2 goes in the units column of the answer line and the one set of 7 is "carried over" to the next column of sets of 7.

$$\begin{array}{cccc} & 1 & \overset{1}{5} & 3_7 \\ + & 2 & 1 & 6_7 \\ \hline & & & 2_7 \end{array}$$

Next, add the numbers in the sevens column, 1+1+5=7. Since seven sets of seven is one set of 49 with nothing left over, carry the 1 to the next column over and put a 0 in the answer line in the second column.

$$\begin{array}{cccc} & \overset{1}{1} & \overset{1}{5} & 3_7 \\ + & 2 & 1 & 6_7 \\ \hline & & 0 & 2_7 \end{array}$$

Then add the numbers in the left column, 1+1+2=4.

$$\begin{array}{cccc} & \overset{1}{1} & \overset{1}{5} & 3_7 \\ + & 2 & 1 & 6_7 \\ \hline & 4 & 0 & 2_7 \end{array}$$

Since this number is less than 7, there is no need to carry anything to the next column, and our answer is 402_7.

Example 3: Add in base 16: $A28_{16} + 739_{16}$.

Solution:

First, write the addition problem in column form. Then add the numbers in the rightmost column. Since 8 + 9 is 17, which is one set of 16 plus 1 left over, write the remainder 1 in the answer line under the right column and add the 1 set of 16 to the next column.

$$\begin{array}{cccc} & A & \overset{1}{2} & 8 \\ + & 7 & 3 & 9 \\ \hline & & & 1 \end{array}$$

Next, add the second column. Since 1 + 2 + 3 is 6 which is less than 16, simply put the 6 in the answer line for the second column.

$$\begin{array}{cccc} & A & \overset{1}{2} & 8 \\ + & 7 & 3 & 9 \\ \hline & & 6 & 1 \end{array}$$

Finally, add A + 7. From the base 16 chart earlier in this section, we know that A represents the Hindu-Arabic number 10. Since 10 + 7 = 17 which is one set of 16 plus 1, put the remainder 1 under the A and 7 column. Place the one set of 16 as the first digit in the answer.

$$\begin{array}{cccc} & A & \overset{1}{2} & 8 \\ + & 7 & 3 & 9 \\ \hline 1 & 1 & 6 & 1 \end{array}$$

Thus, $A28_{16} + 739_{16} = 1161_{16}$.

Example 4: Add in base 2: $11011_2 + 101101_2$.

Solution:

Adding column by column staring with the units column on the right, and carrying a set of two to the next column, the addition is shown below.

$$\begin{array}{ccccccc}
{\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\
 & & 1 & 1 & 0 & 1 & 1_2 \\
+ & 1 & 0 & 1 & 1 & 0 & 1_2 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0_2
\end{array}$$

Thus, $11011_2 + 101101_2 = 1001000_2$

Subtraction

The algorithm used for subtraction of multi-digit numbers is the borrowing algorithm that reverses the "carrying" procedure for addition.

Example 5: Subtract 223 – 186.

Solution:

First write the problem in column form. Remember that we start by subtracting the numbers in the units column. Since 3 is smaller than 6, we cannot directly subtract and get a positive answer. Therefore, we "borrow" one set of ten from the ten's column and break it up into 10 units. Adding those ten units to the three already there gives thirteen. Then, we can easily subtract 13 – 6.

$$\begin{array}{cccc}
 & & {\scriptstyle 1} & {\scriptstyle 13} \\
 & 2 & \not{2} & \not{3} \\
- & 1 & 8 & 6 \\
\hline
 & & & 7
\end{array}$$

Since we borrowed one of the two sets of ten in the ten's column, we only have one set left. This is why the 2 in the middle column is scratched out and the one left is written above it. The next step is to subtract the numbers in the ten's column. As before, we cannot subtract 1 – 8 and get a positive answer. So, we borrow one set of 100 from the hundred's column and break it up into ten sets of 10. This gives 10 + 1 = 11 sets of ten for the top number in the ten's column. Now we subtract 11 – 8 = 3. Since we borrowed one set of 100, we now only have 1 set of 100 in the hundred's column for the top number. This is why the 2 is crossed out with a one on top of it.

$$\begin{array}{cccc} & \overset{1}{\cancel{2}} & \overset{11}{\cancel{2}} & \overset{13}{\cancel{3}} \\ - & 1 & 8 & 6 \\ \hline & & 3 & 7 \end{array}$$

Finally, subtract the hundred's column.

$$\begin{array}{cccc} & \overset{1}{\cancel{2}} & \overset{11}{\cancel{2}} & \overset{13}{\cancel{3}} \\ - & 1 & 8 & 6 \\ \hline & 0 & 3 & 7 \end{array}$$

Leading zeroes do not change a number, so the answer is simple 37.

The basic concept is to "regroup" the larger number so that the subtraction is reduced to subtraction between the digits in each place of the two numbers. This same procedure works for subtracting numbers in any place value system.

Example 6: Subtract in base 7: $536_7 - 245_7$.

Solution:

Before starting the subtraction problem, remember that the place values in base 7 are $...7^2, 7^1 = 7, 1$. Thus, $536_7 = 5 \times 7^2 + 3 \times 7 + 6$. The number is grouped into groups of 7 and powers of 7. Therefore, if we need to "borrow" from the next higher place, it is group of 7, not 10, that we are borrowing.

To subtract, first write the problem in column form.

$$\begin{array}{cccc} & 5 & 3 & 6_7 \\ - & 2 & 4 & 5_7 \\ \hline \end{array}$$

Begin subtracting with the units place. Since 6 is bigger thatn 5, we can simply subtract and record our answer in the answer line.

$$\begin{array}{cccc} & 5 & 3 & 6_7 \\ - & 2 & 4 & 5_7 \\ \hline & & & 1 \end{array}$$

Next, try subtracting in the 7^1 place. Since 3 is smaller than 4, we cannot subtract without borrowing from the next larger place. When borrowing from the next larger place, we have $5 \times 7^2 = 5 \times 49$ things to borrow from, NOT 500! Borrowing one set of 49 from the

5 present, we have 4 sets of 49 left. This is why the 5 is marked out with a 4 above it in the largest valued place. So, we have borrowed one set of 49 for the 7's place. One set of 49 is 7 sets of seven, so we add 7 sets of seven to the 3 already present in the 7's place. The notation is written as "7+3" instead of "10" to emphasize the fact that we added 7 extra sets.

$$\begin{array}{cccc} & \overset{4}{\cancel{5}} & \overset{7+3}{\cancel{3}} & 6_7 \\ - & 2 & 4 & 5_7 \\ \hline & & 6 & 1 \end{array}$$

Finally, subtract the numbers in the 49's place. Since 4 is larger than 2, simply subtract the digits as normal.

$$\begin{array}{cccc} & \overset{4}{\cancel{5}} & \overset{7+3}{\cancel{3}} & 6_7 \\ - & 2 & 4 & 5_7 \\ \hline & 2 & 6 & 1 \end{array}$$

The final answer is 261_7.

Example 7: Subtract in base 5: $1423_5 - 424_5$.

Solution:

First, write the problem in column form:

$$\begin{array}{cccc} 1 & 4 & 2 & 3_5 \\ - & 4 & 2 & 4_5 \\ \hline \end{array}$$

To begin subtracting, we must borrow one set of 5 from the 5^1 column and break it into 5 units. Since we have 2 sets of 5 in the 5^1 column, borrowing one leaves 1 left.

$$\begin{array}{cccc} 1 & 4 & \overset{1}{\cancel{2}} & \overset{5+3}{3}_5 \\ - & 4 & 2 & 4_5 \\ \hline & & & 4 \end{array}$$

Next, we need to borrow from the 5^2 column to be able to subtract in the 5^1 column. Taking one of our 4 groups of 5^2 leaves three left and adds 5 more sets of 5 to the 5^1 column. Then we can subtract in that column.

$$\begin{array}{rcccc} & & ^{3} & ^{1+5} & ^{5+3} \\ & 1 & \cancel{4} & \cancel{2} & 3_5 \\ - & & 4 & 2 & 4_5 \\ \hline & & & 4 & 4 \end{array}$$

Finally, since the subtraction in the 5^2 column cannot be performed directly, we must borrow the 1 in the 5^3 column and break it up into 5 sets of 25 and add those to the 3 in the 5^2 column. Then, we simply subtract 8 – 4.

$$\begin{array}{rcccc} & & ^{3+5} & ^{1+5} & ^{5+3} \\ & \cancel{1} & \cancel{4} & \cancel{2} & 3_5 \\ - & & 4 & 2 & 4_5 \\ \hline & 0 & 4 & 4 & 4 \end{array}$$

The final answer is 444_5.

Example 8: Subtract in base 16: $5B2_{16} - 17C_{16}$.

Solution:

Write the problem in column form. Also look up B and C on the base 16 chart and find B=11 and C=12.

$$\begin{array}{rccc} & 5 & B & 2_{16} \\ - & 1 & 7 & C_{16} \\ \hline \end{array}$$

Since 2 is smaller than C, we must borrow one set of 16 from the 16^1 column to be able to subtract in the units column. Borrowing 1 from B leaves 11-1=10=A sets of 16 left. Adding the 16 to the 2 in the units column gives 18-12=6.

$$\begin{array}{rccc} & & ^{A} & ^{2+16} \\ & 5 & \cancel{B} & 2_{16} \\ - & 1 & 7 & C_{16} \\ \hline & & & 6 \end{array}$$

The remainder of the subtraction may be performed without any borrowing.

$$\begin{array}{rccc} & & ^{A} & ^{2+16} \\ & 5 & \cancel{B} & 2_{16} \\ - & 1 & 7 & C_{16} \\ \hline & 4 & 3 & 6 \end{array}$$

The final answer is 436_{16}.

Example 9: Subtract in base 2: $110111_2 - 11001_2$.

Solution: First, write the problem in column form.

$$\begin{array}{rcccccc} & 1 & 1 & 0 & 1 & 1 & 1_2 \\ - & & 1 & 1 & 0 & 0 & 1_2 \\ \hline \end{array}$$

Next, begin subtracting from the units place on the right. If the top digit is larger than the bottom digit, simply subtract.

$$\begin{array}{rcccccc} & 1 & 1 & 0 & 1 & 1 & 1_2 \\ - & & 1 & 1 & 0 & 0 & 1_2 \\ \hline & & & & 1 & 1 & 0 \end{array}$$

The procedure works the same way in any base, so at this point we need to borrow from the next column. Just remember that in base 2 we are borrowing sets of 2 and multiples of 2. The rest of the subtraction is below.

$$\begin{array}{rcccccc} & & 2+0 & 2+0 & & & \\ & \not{1} & \not{1} & 0 & 1 & 1 & 1_2 \\ - & & 1 & 1 & 0 & 0 & 1_2 \\ \hline & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

The final answer is 11110_2.

Multiplication

One of the challenges of elementary school is to learn the multiplication table up to ten times ten. This is actually a complicated task. First, the student must understand that multiplication is faster addition and 8×9 is the number of total objects in 8 sets of 9 things each. Then, the student has to express this quantity as a base ten number. Since 8 sets of 9 things do not “naturally” separate themselves into 7 sets of ten plus 2 left over, learning these facts takes a significant amount of memorization. Fortunately, once these facts are mastered, the algorithm for multiplying multi-digit numbers is not much harder than the procedure for addition.

Multiplication in other bases uses a similar algorithm. The first step is to determine the products of all the single digits express as numbers in the base.

Example 10: Fill in the multiplication table for multiplication in base 8.

Solution:

The empty table is:

*	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6				???				
7								

The number that goes in each box is the product, in base 8, of the digit at the beginning of that row and the digit at the top of that column. So the number that goes in the box with the question marks is $6_8 \times 3_8$. This is the number of items in six sets of three items each. In base ten, this would be 18 items. Regrouping the total of 18 into groups of 8 items each, 18 is 2 groups of 8 plus 2 extra. Thus, $6_8 \times 3_8 = 22_8$. The completed table is below. All numbers are in base 8, the base subscript has been left off for simplicity.

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Example 11: Construct the multiplication table for digits in base three.

Solution:
The table is completed below. The subscript “2” has been omitted for simplicity. The answer of $11_3 = 2_3 \times 2_3$ follows from the fact that 2*2 = 4 and 4 is one group of 3 with one left over.

*	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

The multiplication table for a particular base is needed to perform the algorithm for multiplication of multi-digit numbers in that base.

Example 12: Multiply $32_8 \times 5_8$.

Solution: First write the multiplication problem in column form.

$$\begin{array}{rrr} & 3 & 2_8 \\ \times & & 5_8 \\ \hline \end{array}$$

Begin by multiplying the units digits. By the base 8 multiplication table, $5_8 \times 2_8 = 12_8$. This is one set of 8 plus 2 extra. The 2 goes in the units column of the answer. The 1 set of 8 is placed above the 8^1 column.

$$\begin{array}{rrr} & \overset{1}{3} & 2_8 \\ \times & & 5_8 \\ \hline & & 2 \end{array}$$

The next step is to multiply $3_8 \times 5_8 = 17_8$. Adding the one from the units multiplication gives $17_8 + 1_8 = 20_8$. The 20 is what goes in the answer line.

$$\begin{array}{rrr} & \overset{1}{3} & 2_8 \\ \times & & 5_8 \\ \hline 2 & 0 & 2 \end{array}$$

The final answer is 202_8.

Example 13: Multiply $221_3 \times 2_3$.

Solution:

Write the problem in column form and begin by multiplying the units digits

$$\begin{array}{rrrr} & 2 & 2 & 1_3 \\ \times & & & 2_3 \\ \hline & & & 2 \end{array}$$

Next, multiplying in the 3^1 column gives $2_3 \times 2_3 = 11_3$. The units digit of this answer goes in the 3^1 column place in the answer, the 3's digit gets carried over to the next column and added to the multiplication result.

$$\begin{array}{rrrr} & \overset{1}{2} & 2 & 1_3 \\ \times & & & 2_3 \\ \hline & & 1 & 2 \end{array}$$

Multiplying in the 3^2 column and adding the extra 1 is $2_3 \times 2_3 + 1_3 = 11_3 + 1_3 = 12_3$. The 12 is what finishes the answer line.

$$\begin{array}{rrrr} & \overset{1}{2} & 2 & 1_3 \\ \times & & & 2_3 \\ \hline 1 & 2 & 1 & 2 \end{array}$$

The final answer is 1212_3.

Division

The operation of division is the inverse of multiplication. The question $12 \div 3 = ?$ asks what number, when multiplied by 3, gives 12. Therefore, the multiplication table in a particular base can be used to solve some simple division problems.

Example 14: Divide $43_8 \div 7_8$.

Solution:

The multiplication table for base 8 shows that $7_8 \times 5_8 = 43_8$. Therefore, $43_8 \div 7_8 = 5_8$.

Example 15: Divide $11_3 \div 2_3$.

Solution:

The multiplication table for base 2 shows that $2_3 \times 2_3 = 11_3$. Therefore, $11_3 \div 2_3 = 2_3$.