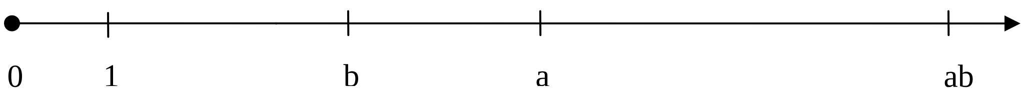
2.7 Greatest Common Factor and Least Common Multiple

also known as: GCF and LCM

Suppose you have two different natural numbers, a and b, and you know that a > b > 1 (i.e., a is bigger than b and they're both bigger than 1). You may then arrange them on a number line and notice that you can figure out quite a bit about these numbers.



Note that 1 divides each of a and b evenly.
Note that both divide the product ab evenly and that ab is bigger than either a or b.
Note that a – b > 0 (i.e. is a positive number).
Note that the ratio of b and a, written $\frac{b}{a}$, is a fraction less than 1.

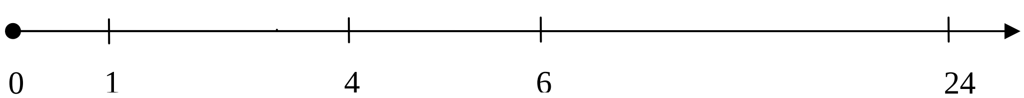
Sometimes there are numbers bigger than 1 that divide both a and b; the largest of the numbers that divide both a and b is called the Greatest Common Factor (GCF). It can happen that b is the GCF. It is also called the Greatest Common Divisor (GCD) in some books. The terms will be used interchangeably in this text.

Sometimes there are numbers smaller than ab that are divisible by both a and b; the smallest of these numbers is called the Least Common Multiple (LCM). We will use the LCM to add and subtract fractions. It can happen that a is the LCM.

Let's look at some examples of pairs of numbers to see if we can find the GCD and the LCM.

Example

Suppose we have b = 4, a = 6, and ab = 24. Put them on a number line with 1:



One divides both 4 and 6; they each divide 24 evenly.

Is there a number bigger than one that divides them both? That will be the GCF. If not, then 1 = GCF and we say that the two numbers are "relatively prime".

Is there a number smaller than 24 that they each divide? That will be the LCM.

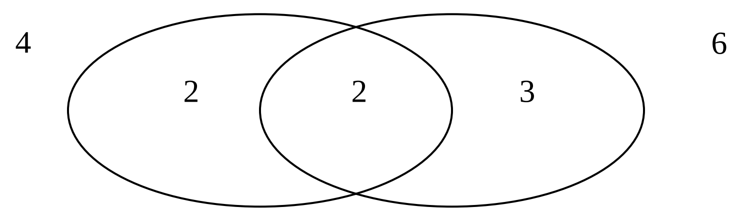
Let’s explore two ways to find these important numbers. We will look at the Venn Diagram Approach and the Table and Arrows Approach. You may use whichever method you like the best.

The first step to either approach is factoring each number to primes.

Looking at 4, note that 4 factors to 2 (2) .
We also have that 6 = 2 (3).

The Venn Diagram Approach:

You can work this out by making a modified Venn diagram: a circle for the 4 and another circle for the 6 and noting that 4 and 6 share a single prime factor of 2. Put this in the space for the intersection of the factor circles. 4 has an additional 2 that 6 lacks while 6 has a prime factor of 3 that 4 lacks. Put the non-shared factors in the crescents outside the intersection space.



The GCF is in the intersection of the two sets of prime factors. It is 2.

If you multiply across the Venn diagram you get (2)(2)(3) = 12. This is the LCM.

Now it happens that there are 4 factors if you count the numbers in the section where we factored 4 and 6 to primes and only 3 numbers in the Venn Diagram…the intersection space means that the 2 is used TWICE…once to multiply to 4 and again to multiply to 6. We only list it once, though.

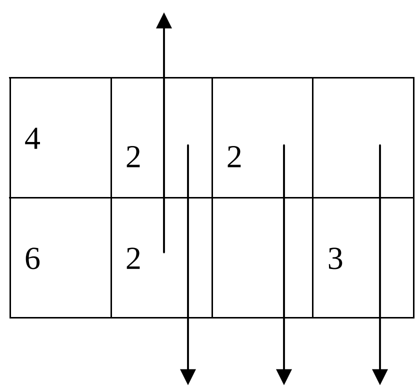
Remembering to list shared factors only once is the key to using the Venn Diagram Approach effectively.

The Table with Arrows Approach:

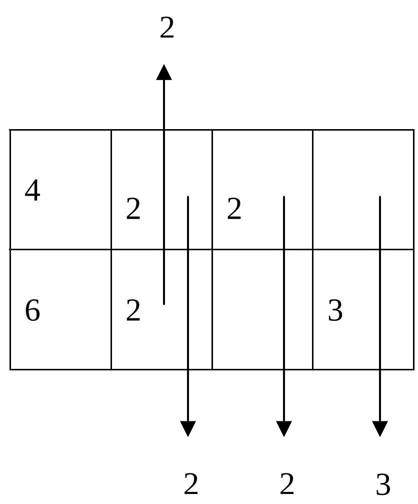
Write out a table with a row for 4 and a row for 6, put in as many columns as you need to put one factor in each column for both numbers. Stack the factors that can be packed into any one column…don't make a new column unless you absolutely must.

4	2	2	
6	2		3

Put an arrow pointing up for each full column. And arrows down in each column.

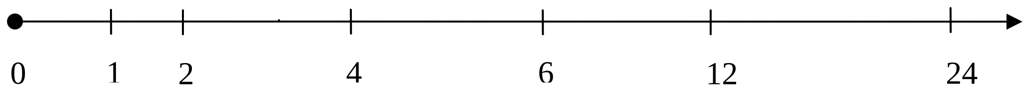


Put the numbers in the columns at the tips of the arrows.



Multiply across the top of the arrows to get the GCD. In this case: 2.
Multiply across the bottoms of the arrows to get the LCM. In this case: 12.

Let's look at our number line again, and put the GCF and LCM on it.

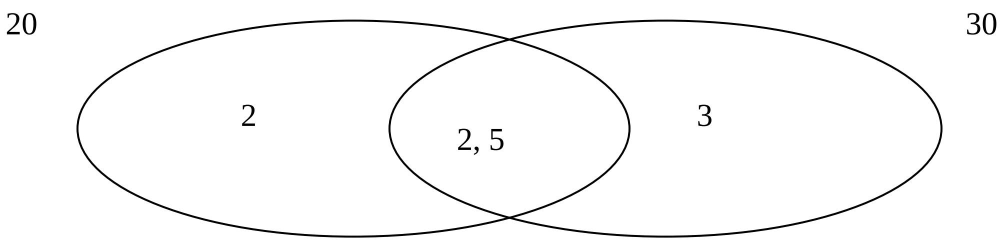


Notice that 2 is greater than 1 while 12 is less than 24.
We use the LCM to add fractions. We use the GCF to simplify fractions.

Another example:

Find the GCD and LCM for 20 and 30 using Venn Diagrams.
First factor both numbers to primes:

20 = (2)(2)(5)
30 = (2)(3)(5)



Fill in the ellipses with the factors – be sure to put the shared factors in the overlapping part of the ellipses (intersections space) listing them only once. Note that when you count the factors you have 6 multiplied numbers BUT putting them in the Venn Diagram…where you don't list a number twice, you have 4 numbers to multiply.

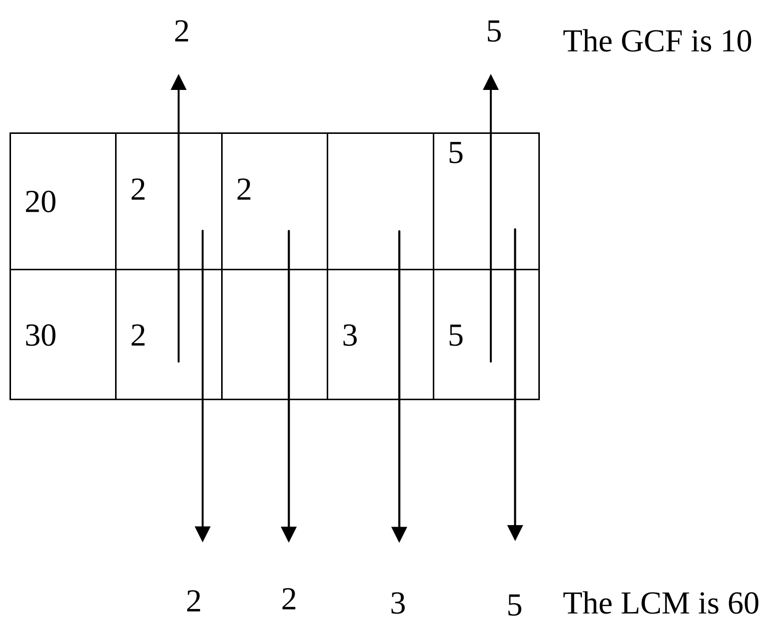
You should have found that 10 is the GCD (GCF). And by multiplying across the diagram gives 60 as the LCM.

Find the GCD and LCM for 20 and 30 using a Table with Arrows.

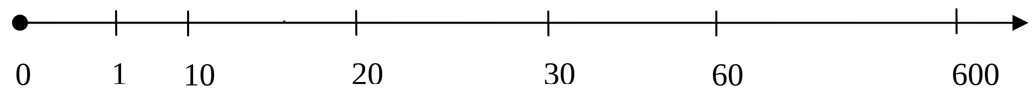
Now fill in the following table – note that the table will have 2 rows: one for 20 and one for 30. The number of columns is the number of numbers written across the ellipses above: a 2, a shared 2, a shared 5 and a 3.

20	2	2		5
30	2		3	5

Now put on the arrows and write numbers at the tips:



Here's the whole problem on a number line:



Another example:

Find the GCD and LCM for 36 and 84.

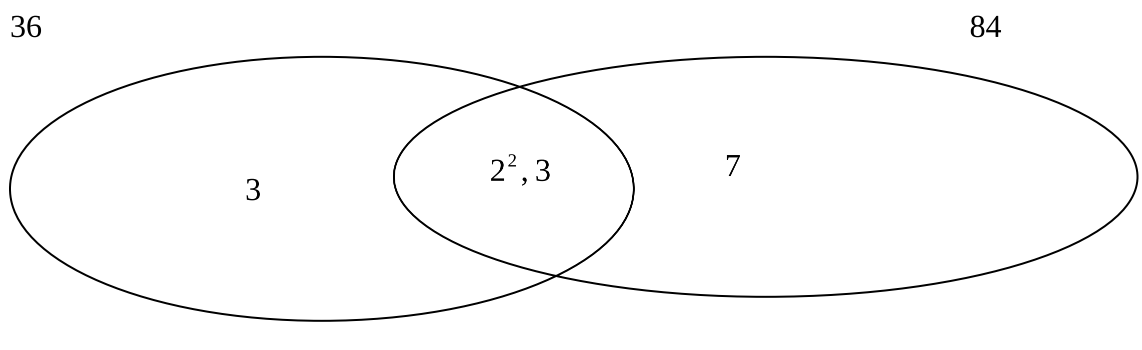
First factor each number to primes:

36 = $2^2 \cdot 3^2$

84 = $2^2 \cdot 3 \cdot 7$

The Venn Diagram Approach:

Next fill in the shapes below putting all shared factors in the overlapped space:



Calculate the GCD: 4(3) = 12

Calculate the LCM: 3(4)(3)(7) = 252

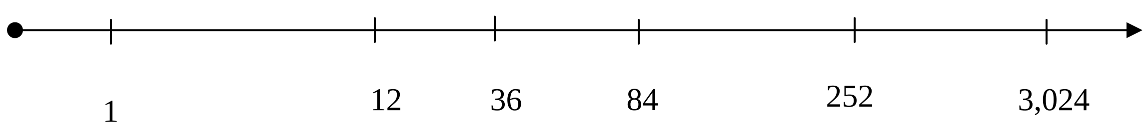
The Table and Arrows Approach:

2 2 3 The GCF is 12

36	2	2	3	3	
84	2	2	3		7

2 2 3 3 7 The LCM is 252

Fill in the number line:



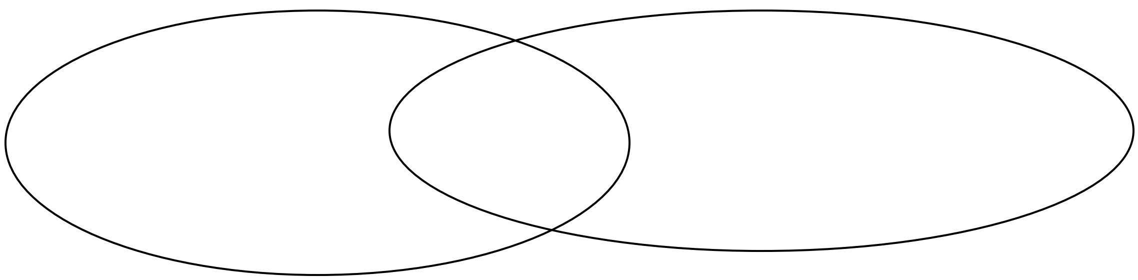
In class example:

Find the GCD and LCM for 66 and 90.

First factor each number to primes:

66 = 2(3)(11) 90 = $2(3^2)(5)$

Next fill in the shapes below putting all shared factors in the overlapped space:



You should find that the GCF is 6 and the LCM is 990.

Next – fill in the correct number of rows and columns – the table is larger than it needs to be so be careful to do it correctly. It's ok to have the 11 is a column closer to the left than the column with the 5, too.

Note that the Venn Diagram Approach works only for finding the GCF and LCM of 2 numbers. While the Table and Arrows Approach will work for more than 2 numbers.

Example:

Find the GCF and LCM for 12, 20, and 30.

2 is the GCF

12	2	2	3	
20	2	2		5
30	2		3	5

2 2 3 5 60 is the LCM

Notice that (12)(20)(30) = 7,200 – which is really too big to be fun to work with.

Uses for the GCF and the LCM:

Add the following fractions:

$$\frac{1}{12}+\frac{2}{20}-\frac{7}{30}$$

First you need a common denominator. We use the LCM for this. The LCM is 60.

12	2	2	3	
20	2	2		5
30	2		3	5

Note that 60 = 12 (5) = 20 (3) = 30 (2). We will use these numbers to arrive at fractions that can be combined easily by multiplying each term by a carefully chosen "1". We'll use 5/5, 3/3, and 2/2. Note that what you use is the numbers in the columns that are NOT already taken by a denominator. It's pretty easy to see what to use if you've organized things with a table.

$$\frac{1}{12}\cdot\frac{5}{5}+\frac{2}{20}\cdot\frac{3}{3}-\frac{7}{30}\cdot\frac{2}{2}=\frac{5}{60}+\frac{6}{60}+\frac{-14}{60}=\frac{5+6-14}{60}=-\frac{3}{60}=-\frac{1}{20}$$

Notice that the subtract got changed to adding a negative number. This is so that students don't forget the subtraction in all the work with the common denominator!

Simplify the following fraction:

$\frac{66}{90}$ The GCF for 66 and 90 is 6. $\frac{66}{90}=\frac{6(11)}{6(15)}=\frac{11}{15}$.

This is much more efficient than factoring all the way to primes. Simply factor to the GCF and cancel it out. The numbers that remain are in simplest form.

2.7 Exercises. Those with asterisks have complete solutions attached.

1. Find the LCM and the GCF for 6 and 22.

2. Find the LCM and the GCD for 60 and 210.

3. Find the LCM and the GCF for 8 and 20.

4. Find the LCM and the GCD for 18 and 30.

5. Find the LCM and GCF for 16 and 100.

6. Find the LCM and GCD for 24 and 96.

7.* Find the LCM and GCF for 66 and 165.

8. Find the LCM and GCD for 7 and 11.

9. Find the LCM and GCF for 81 and 90.

10. Find the LCM and GCD for 15 and 45.

11. Find the LCM and GCF for 6, 22, 35.

12. Find the LCM and GCD for 15, 42, and 66.

13.* Find the LCM and GCF for 18, 30, and 42.

14. Find the LCM and GCD for 14, 21, and 42.

15. Find the LCM and GCF for 30, 66, 390.

Perform the following operations

16. $\frac{2}{7}-\frac{3}{14}+\frac{5}{42}$

17. $-\frac{1}{12}+\frac{4}{15}-\frac{3}{21}$

18. $\frac{5}{18}-\frac{1}{20}+\frac{5}{12}$

19.* $\frac{5}{42}-\frac{7}{8}+\frac{3}{10}$

20. $\frac{3}{16}+\frac{7}{66}-\frac{5}{33}$

21. $\frac{1}{30}-\frac{5}{66}+\frac{1}{390}$

22. $\frac{1}{5}-\frac{6}{55}+\frac{3}{22}$

23.* $\frac{1}{80}+\frac{4}{33}-\frac{1}{6}$

24. $-\frac{2}{15}-\frac{1}{66}-\frac{5}{42}$

25. $-\frac{1}{30}-\frac{5}{66}-\frac{7}{20}$

Simplify the following fractions by canceling the GCF.

26. $\frac{165}{396}$

27. $\frac{42}{98}$

28. $\frac{130}{182}$

29.* $\frac{72}{84}$

30. $\frac{90}{165}$

31. $\frac{85}{102}$

32. $\frac{136}{153}$

33. $\frac{650}{850}$

34. $\frac{315}{385}$

35. $\frac{336}{462}$

Solutions:

7. Find the LCM and GCF for 66 and 165.

66	2	3	11	
165		3	11	5

GCF is 33
LCM is 330

13. Find the LCM and GCF for 18, 30, and 42.

18	2	3	3		
30	2	3		5	
42	2	3			7

GCF is 6
LCM is 630

19. $\frac{5}{42} - \frac{7}{8} + \frac{3}{10}$

42	2	3	7			
8	2			2	2	
10	2					5

LCM is 840 = $2^3(3)(5)(7)$

$$\frac{5}{42}-\frac{7}{8}+\frac{3}{10}=\frac{5}{2(3)(7)}\cdot\frac{2^2\cdot 5}{2^2\cdot 5}+\frac{-7}{8}\cdot\frac{3(5)(7)}{3(5)(7)}+\frac{3}{10}\cdot\frac{2^2(3)(7)}{2^2(3)(7)}$$

$$\frac{100-735+252}{840}=-\frac{383}{840}$$

383 is a prime number so the calculation is complete.
A really quick way to check to see if a number up to 10,000 is prime is to go to Number Gossip (google it) and insert it in the gossip box.

23. $\frac{1}{80}+\frac{4}{33}-\frac{1}{6}$

80	2	2	3	5	
33			3		11
6	2		3		

LCM is 600

$$\frac{1}{80}+\frac{4}{33}-\frac{1}{6}=\frac{1}{80}\cdot\frac{11}{11}+\frac{4}{33}\cdot\frac{40}{40}+\frac{-1}{6}\cdot\frac{110}{110}=\frac{11+160-110}{600}=\frac{81}{600}=\frac{3\cdot 27}{3\cdot 200}=\frac{27}{200}$$

29. $\frac{72}{84}=\frac{12\cdot 6}{12\cdot 7}=\frac{6}{7}$