2.8 Real Numbers and their Properties

We return to the Real Numbers now to discuss some properties of numbers and sets as well as operations on the numbers. One very important set property is to be "well-defined". If a set is well-defined, then a reasonable observer can tell if an object belongs in the set or not.

Well-defined Sets:

We have the set of Natural numbers:

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

If someone asks if 5^{-1} is an element of the Natural numbers simply convert it to it's rational or decimal form:

$$5^{-1} = \frac{1}{5} = 0.2$$

and note that none of the natural numbers have a fractional part. So, no, it's not a set element.

The Whole numbers are $W = N \cup \{0\} = \{0, 1, 2, 3, 4, \ldots\}$.

The Integers are the negative Natural numbers union the Whole numbers. A negative natural number is – 1 times the natural number in question. For example 2 is a natural number and $-1 \cdot 2 = -2$ is a negative natural number and each is an integer.

Is -2×10^2 an integer? Yes, because $-2 \times 10^2 = -200$ which is a negative natural number.

If we take the Integers and calculate the reciprocals of each one except zero (it has no reciprocal; it's reciprocal is undefined) by putting an exponent of – 1 on each, and then make a set of the reciprocals. Taking all reciprocals and all the sums and products of them with the integers, we get the Rational numbers. To say this "in Math":

$$Q = \left\{ \frac{a}{b} \,\middle|\, \text{a and b are integers and b is not zero} \right\} =$$
$$\left\{ a \cdot b^{-1} \,\middle|\, \text{a and b are integers and b is not zero} \right\}$$

We can check to see if a number is a Rational number by rewriting the number in it's rational form $\frac{a}{b}$.

- 5 is an element of Q because $5 = \frac{10}{2}$

- $-.01 = -1 \times 10^{-2}$ is an element of Q because it can be rewritten $\frac{-1}{100}$

- $.3\bar{3} = \frac{1}{3}$ is an element of Q because it can be written in the proper form.

How about .2020020002000020000002….

No, it is NOT a rational number. It is irrational because it cannot be written in the proper form. Ditto for pi and $\frac{\sqrt{3}}{2}$.

Now this brings us to a problem. What if you know the number is rational because it is a repeating decimal but you need to write in the fraction form to demonstrate that it is a rational number? How do you get it to the proper form?

Converting repeating decimals to fraction form:

The process is:

1. Set the decimal equal to x (This is Equation 1).
2. Adjust Equation 1 so the repeat comes right after the decimal if necessary. Count the number of digits under the repeat bar.
3. Multiply both side of the fraction by 1 with the same number of zeros following as repeating digits (This is Equation 2).
4. Subtract Equation 1 from Equation 2.
5. Divide the coefficient of x on both sides.

Example:

Convert $.\overline{235}$ to its rational form.

1. $x = .\overline{235}$ Equation 1

   The repeat comes immediately after the decimal; no need to adjust.

2. There are 3 digits in the repeat. Multiply both sides by 1,000.

3. $1000x = 235.\overline{235}$ Equation 2

4. Subtract Equation 1 from Equation 2

$$\begin{array}{rl} 1000x &= 235.\overline{235} \\ -\quad x &= .\overline{235} \\ \hline \end{array}$$

note: the repeated part is subtracted away!

$$999x = 235$$

5.

$$x = \frac{235}{999}$$

Example:

Convert $-3.\overline{09}$ to it's rational form.

1. $x = -3.\overline{09}$ Equation 1

The repeat comes immediately after the decimal; no need to adjust.

2. There are 2 digits after the decimal so multiply both sides by 100.

3. $100x = 309.\overline{09}$ Equation 2

4. Subtract Equation 1 from Equation 2

$$\begin{array}{rl} 100x &= -309.\overline{09} \\ -\quad x &= -3.\overline{09} \\ \hline 99x &= -306 \end{array}$$

5. Divide both sides by 99

$$x = -\frac{306}{99}$$

Be careful with subtracting negatives: $--3.\overline{09}$ is a positive number.

Example:

Convert $.01\overline{23}$ to it's rational form.

1. $x = .01\overline{23}$ Equation 1

The repeat comes two digits after the decimal. Multiply both sides of Equation 1 by 100…2 zeros, one for each digit.

$100x = 1.\overline{23}$ new Equation1 –
Note leading zeros are not usually written out

2. There 2 digits under the bar. Multiply both sides by 100.

3. $10000x = 123.\overline{23}$ Equation 2

4. Subtract new Equation 1 from Equation 2

$$\begin{array}{rl} 10000\,x = & 123.\overline{23} \\ -\quad 100\,x = & 1.\overline{23} \\ \hline 9900\,x = & 122 \end{array}$$

5. $x = \dfrac{122}{9900}$

Sets that are closed under an operation:

Another important set property is "closure under an operation". If a set is "closed under an operation", then when you combine 2 set elements with the given operation, you get an answer that is a set element in the same set as the 2 that were combined. It is a requirement that the operation to combine set elements be given as part of the question.

Example: Is the set N closed under subtraction?

We will take 2 natural numbers: 7 and 9. 9 – 7 is 2, so this looks ok…BUT
7 – 9 = –2, which is not a set element of N. So, no, the Natural numbers are not closed under subtraction. It only takes ONE time for the whole set to be "not closed".

Example: Take the proper subset E, the even natural numbers, of N, the natural numbers. Is the set E closed under addition?

Take 2 even natural numbers and add them: 122 + 242 = 364. Good so far.

In fact, this is true and can be proved in the following manner:

Let x and y be Natural numbers. 2x and 2y are even natural numbers. If we add them together we get 2x + 2y = 2 (x + y). This means that the sum of the 2 numbers factors to a multiple of 2 which is the definition of an even number.

E = { 2x │ x is a set element of N}

The Odd natural numbers are defined to be O = {2x + 1 │ x is a set element of N}.
(In the homework this definition will come in handy!)

Example: Is the set of Irrational numbers closed under multiplication?

Look at $3^{\frac{1}{2}}$ and $3^{-\frac{1}{2}}$. Both are Irrational numbers. If we multiply them, we get

$$3^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}} = 3^{\frac{1}{2}-\frac{1}{2}} = 3^0 = 1$$

and 3 is a Natural number NOT an Irrational one.

Let's look at this again in traditional notation:

$$3^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}} = \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 3^0 = 1$$

If we want to work with the Real Numbers instead of subsets of them, we will find that the Real numbers are closed under addition, subtraction, and multiplication. Division by zero results in an undefined number that is not a set element of the Real numbers so the set of Real numbers is not closed under division.. In fact, the Real numbers have been studied so much that there is a standard list of properties that apply to most arithmetic operations with them.

Arithmetic Properties of Real Numbers:

We will use a, b, and c to illustrate these properties. Each is a real number.

1. Commutative property of addition and multiplication.

   The order in which you add them or multiply them does not matter.

   $a + b = b + a$ $\qquad$ $ab = ba$

2. Associative Property of Real numbers.

   If you have 3 or more numbers to combine using the same operation, you may combine them pair-wise in any order to add and multiply.

   $a + (b + c) = (a + b) + c$ $\qquad$ $a(bc) = (ab)c$

3. Distributive Property of Multiplication over Addition.

   $a(b + c) = ab + ac$ $\qquad$ You may add first then multiply **–** or **–** multiply first then add.

Examples:

<u>To illustrate the Commutative Property:</u>

Addition $\qquad$ Multiplication

$2\sqrt{3} + -\sqrt{3} = -\sqrt{3} + 2\sqrt{3} = \sqrt{3}$ $\qquad$ $3^{-1}(9) = 9(3^{-1}) = 3$

Neither subtraction nor division are commutative so it is best to rewrite subtraction as adding a negative and division as multiplication by a reciprocal.

To illustrate the Associative Property:

Addition

$$\frac{1}{3}+(\frac{1}{2}+\frac{2}{3})=\frac{1}{3}+(\frac{1}{2}\cdot\frac{3}{3}+\frac{2}{3}\cdot\frac{2}{2})=\frac{1}{3}+\frac{7}{6}=\frac{1}{3}\cdot\frac{2}{2}+\frac{7}{6}=\frac{9}{6}=\frac{3}{2}=1.5$$

$$(\frac{1}{3}+\frac{1}{2})+\frac{2}{3}=(\frac{1}{3}\cdot\frac{2}{2}+\frac{1}{2}\cdot\frac{3}{3})+\frac{2}{3}=\frac{5}{6}+\frac{2}{3}\cdot\frac{2}{2}=\frac{9}{6}=1.5$$

Multiplication

$$2^2(6\cdot 3^{-1})=4(2)=8$$

$$\left(2^2\cdot 6\right)\cdot 3^{-1}=(24)(\frac{1}{3})=8$$

To illustrate Distribution of Multiplication over Subtraction:

$$5^{-1}(25+10)=\frac{35}{5}=7$$

$$5^{-1}(25+10)=(5^{-1}\cdot 25)+(5^{-1}\cdot 10)=5+2=7$$

Example:

A student has worked the illustration of the Associative Property of addition (above) in a quicker way by combining properties. Below are his steps. Tell which properties the student used to solve the problem.

$$\frac{1}{3}+(\frac{1}{2}+\frac{2}{3})=\frac{1}{3}+(\frac{2}{3}+\frac{1}{2})=(\frac{1}{3}+\frac{2}{3})+\frac{1}{2}=1+\frac{1}{2}=1.5$$

First the student used the Commutative Property inside the parenthesis. Then the student used the Associative Property to add the fractions with the same denominator. This was MUCH quicker!

Example:

Use the Distributive Property to solve the following problems:

$6(2^{-1}+3^{-1})=6(2^{-1})+6(3^{-1})=3+2=5$ Note that adding first would have resulted in quite a few more steps.

$2^{\frac{1}{2}}(8^{\frac{1}{2}}-2^{\frac{1}{2}})=(16^{\frac{1}{2}})-2^{\frac{1}{2}+\frac{1}{2}}=4-2=2$ Note that adding first isn't "doable" here.

So the Real numbers are well-defined and closed under addition, subtraction, and division. We build on these basics in the next chapter as we move into Algebra.