Chapter 2

Real Numbers and Their Properties

Section 2.1 and 2.2

Developing the Real Number System from Counting Numbers and the Relationships Between Subsets of the Real Number System

The very earliest civilizations probably counted in 3 terms: one, few, and many. The issue with the difference between “few” and “many” is what must have forced the intellectual development of the Counting numbers.

At what point does few become many and how does a whole group of people agree on that point?

If you look at a table for this kind of counting you can see the difficulty:

adding	none	few	many
none	none	few	many
few	few	?	many
many	many	many	many

The Counting numbers originated with counting body parts and counting trade goods, not to mention counting soldiers and counting how many children a couple had to feed.

For example: one nose, 2 eyes, 5 fingers on one hand (10 all together), 32 adult teeth.

> My family has 3 children and yours has 5 children. Let’s agree that 3 is few and 5 is many. What’s the decision on 4?

We generally call these Counting numbers the Natural numbers and denote the whole set of them with “N”. In set builder notation:

N = { 1, 2, 3, 4, …}

> [aside: ellipses “$\cdots$” can be read “and so on in this pattern”]

Graphically: a series of labeled dots on a horizontal number line is often used for a representation of the Natural numbers.

Put your number line here with the first 6 natural numbers on it:

Did some of you put a double-headed arrow? Did others start with a dot at 1? There are many different ways to display the Natural numbers and having them on a solid number line is slightly misleading though a normal way to show them. Why might it be most accurate to put a dot labeled 1 and a space, then a dot labeled 2 and a space, $\cdots$?

A decimal representation of any Natural number is "#.0" . If you pick any Natural number there is NO fractional part to make a decimal with.

- 4 is 4.0 and
- 1,225 = 1,225.0

We rarely use these following zeros unless we're doing calculations with number that have a non-zero fractional part so the point sometimes gets lost – no Natural numbers have a fractional part.

Accounting for trade goods and counting money led directly to the notion of ZERO, a representation for "none" of a quantity. And we account for this new number with a new set and a new set name: W, the Whole Numbers.

W = { 0, 1, 2, 3, 4, …} = N $\cup$ { 0 }

Note that the Whole numbers are the Natural Numbers unioned with a set of one more number, 0. In set notation: $N \subset W$.

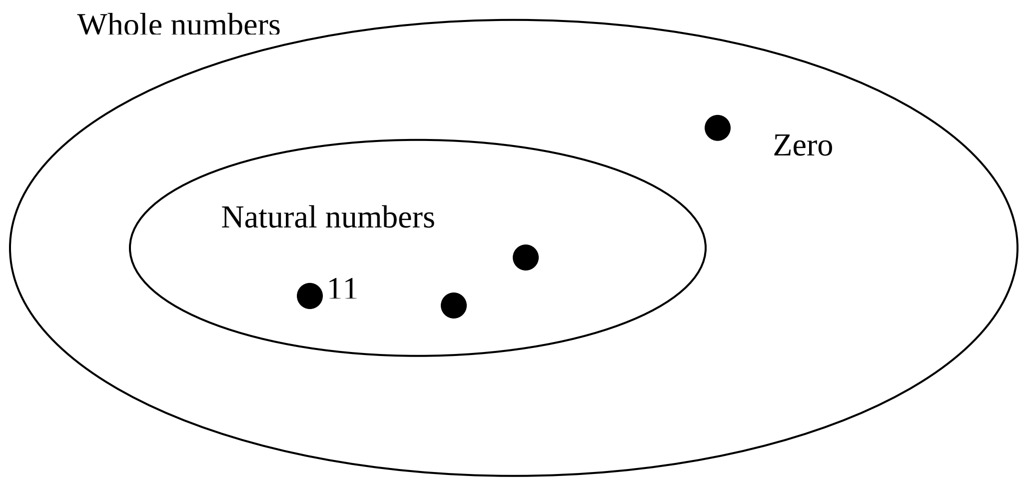
This says N is a PROPER subset of W which means that there is at least one set element in W that is definitely not in N. This set element is 0. We say $0 \in W$ and $0 \notin N$. If we actually didn't know that 0 was not an element of N, we might write $N \subseteq W$ (N is a proper subset or equal to W…note the similarity to $x \leq 5$).

Graphically, we extend our number line one unit to the left and put on a labeled dot for 0. Or, if we had a double-headed arrow line, we'd just add a big dot to the left of 1).

Do this here:

Zero has no fractional or decimal part. You may write 0 or, if you're doing calculations where you need to match a decimal string, you may write 0.0000.

In Venn Diagrams, we have



Mathematicians discovered negative numbers and for a long while they were known but not used much. Once merchants and tradesmen found a use for them, then they became quite accepted. We can look at – 1 and we know it's not a Natural number nor is it a Whole number so if we want to put it in a set of numbers, we need a new set. It is convenient to take the Whole numbers and drop them into another set that contains these new negative numbers.

This new set of number is called the Integers denoted: I. We can show it in set notation in the following way:

$I = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} = \{\ldots -3, -2, -1\} \cup W$

If we put these on a number line our line is now double-headed with big dots at intervals of length 1 in BOTH directions.

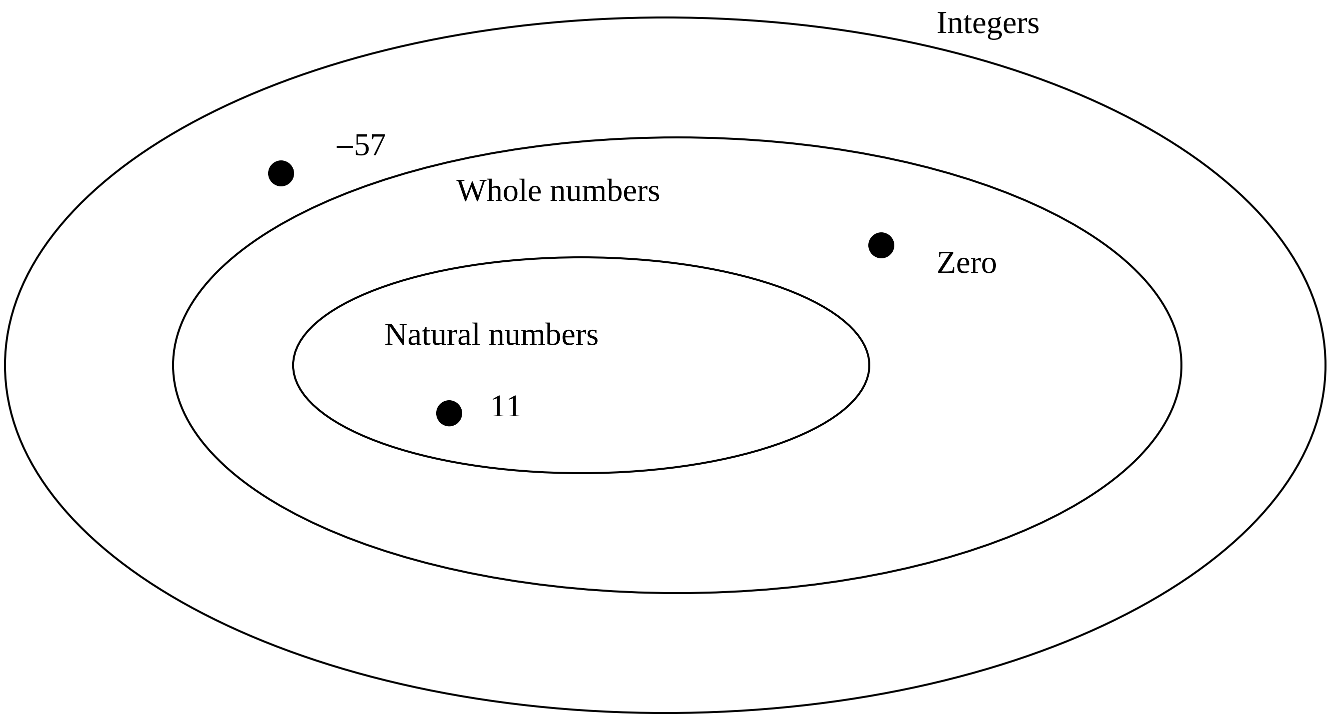
Draw a number line with 13 integers: from –6 to 6; use arrows to indicate that there are more numbers than we are showing.

Note that 5 is 5 steps to the right of zero and –5 is 5 steps to the left of zero.

Note also that –0 = 0, so zero is not paired with a number that is the identical distance from zero in the "opposite" direction. Each Natural number, though, now has an "almost-twin" the same distance from 0 but to the left.

Now the decimal representation for Integers is just like that of the Whole numbers: "#.0". Note that we've included all the Natural numbers in this statement implicitly because every Natural number IS a Whole number. So, if we need to, we can write – 57.0 but usually, if our computations are all in the Integers, we drop the decimal and zero.

A Venn Diagram of the new set is:



In set notation we now have: $N \subset W \subset I$. This means that every Natural number is also a Whole number and is also an Integer.

What's missing from these nested sets as far as numbers go?

If we took out the number line and put a dot for each number in I on a sheet of blank paper…there would be spaces between the dots, wouldn't there? The fractions and mixed numbers are definitely missing.

Luckily, we include them in the same way we included the negative natural numbers, we drop what we've already got into another set that includes fractions and mixed numbers. This new containing set is called the Rational Numbers, Q. (We want to reserve R for the Real numbers so Q is the most common choice, but it can vary from author to author).

The set builder notation for the Rational Numbers is

$$Q = \{\frac{a}{b} \mid \text{ a and b are integers and b is not zero}\}$$

The set inclusion on all the sets we've looked at so far is still that of a proper subset. We have $N \subset W \subset I \subset Q$.

Now, for the first time, we encounter an "undefined number". In every set up until the Rational numbers, there is some physical application that can be applied to each number and it really exists as something physical (even negative numbers – if you overdraw your checking account, you'll have a negative balance and owe the people to whom your wrote the hot check!). Now, we specify that the "b" in the denominator is NOT zero. If the denominator is zero, we get an undefined number. The "why" of this is Exercise 33 at the end of the section.

Students often have a terrible time remembering this. Here's a mnemonic that will help you remember what is going on with 0 and fractions:

- If you are dividing INTO zero it is $\frac{0}{k}$.
- If you are dividing BY zero, you've got a $\frac{n}{0}$.

The number $\frac{1}{2}$ is a rational number that is not in any of the subsets discussed so far, so I (the integers) is a proper subset of Q (the rational numbers). Note that we now have a number with a fractional part and it's decimal representation is 0.5 and we can say $0.5 \in$ Q. We will call this a "strictly" Rational number because it is not contained in the number subsets of the Rational numbers.

Now, each element in each subset is not only in the given subset, it is IN the containing set and contained in the next set and ends up being a set element of Q. So the Natural number 1 is also a Whole number, and an Integer and a Rational number.

Let's look at an element from each subset starting with a Natural number and then picking one that is strictly in the containing subset. We will show that each of these is a rational number by rewriting it in the ratio form ($\frac{a}{b}, \quad b \neq 0$) in the definition above.

- 12 is a Natural number and it can be written as $\frac{12}{1}$ so it's a Rational number. We write $12 \in$ Q.
- 0 is the new number that makes the Natural numbers a proper subset of the Whole numbers. We can write it as $\frac{0}{19}$ to show one of many ways it can be written in as a rational number so $0 \in$ Q.
- – 27 is an Integer that is neither a Whole number nor a Natural number. It, too, is a Rational number. The simplest ratio representation is $-\frac{27}{1}$ but notice that $\frac{81}{-3}$ would be another perfectly adequate representation to make this point so $-27 \in$ Q.
- Mixed numbers are strictly Rational numbers (meaning that they are Rational numbers that are not elements of the proper subsets that are included in the Rational numbers. We can rewrite them as improper fractions (those with the numerator that is larger than the denominator) to illustrate that they, too, fit the form $\frac{a}{b}, \quad b \neq 0$.

Let's look at $5\frac{1}{3}$ and $-3\frac{1}{4}$.

Formally: $5\frac{1}{3} = 5 + \frac{1}{3} = \frac{15}{3} + \frac{1}{3} = \frac{16}{3}$

The short cut is to say 5(3) + 1 to get the 16 in the numerator and to reuse the denominator.

This representation of the number has an integer in the numerator: a = 16 and an integer in the denominator: b = 3. So $5\frac{1}{3}$ is a Rational number.

Formally: $-3\frac{1}{4} = -1(3 + \frac{1}{4}) = -1(\frac{12}{4} + \frac{1}{4}) = -1(\frac{13}{4}) = -\frac{13}{4}$

The short cut is to say 3(4) + 1 = 13, and to reuse the negative sign and the denominator with this new numerator. NOTE that the short cut is NOT −3(4) + 1 = −11…you actually "hold" the negative sign out front of the fraction while you're taking the short cut. Our numerator is − 13 and our denominator is 4. Could we also say that our numerator is 13 and our denominator is −4?

With respect to decimal representation, this format is perfect for the strictly Rational numbers since each one has a fractional part. There are two types of decimals that show up in the Rational numbers:

- terminating and
- infinitely repeating.

An example of a strictly rational number that is a terminating decimal is

$$\frac{3}{4} = .75$$

An example of a strictly rational number that is an infinitely repeating decimal is

$$\frac{1}{11} = .090909\ldots = .\overline{09}$$

We use the superscripted bar to indicate that this pair of number is repeated forever. The repeating part need not be 2 digits. One-seventh has a repeat of length 6.

So now, we've got one huge set of numbers, the Rational numbers, Q. Are there any other numbers? Well, yes, there are lots of other numbers that are not in the Rational numbers. We will look at some and acknowledge, but not study, some of the others.

Here are some types of numbers that might be familiar to you that are not in any of the sets we've discussed so far.

$\sqrt{17}$, π, $\sqrt{-1}$ = i, $2+3i$.

The first two are Irrational numbers and the last two are Complex numbers. We'll study the irrationals and just note that all Real numbers are contained, properly, in the set of Complex numbers. Once we've studied the Irrationals, we will be "done" but there are many, many more types of numbers that are beyond the scope of our class. Complex numbers are often used in electrical engineering and medical imaging – they're important to our everyday life but we won't spend time on them here.

In fact, if you put dots for all the Rational numbers on a blank sheet of paper, there would still be itsy bitsy gaps between one rational number and the next rational number. Hard to see, but definitely there. And these gaps are filled by the Irrational numbers.

Note that the prefix "ir" means "beside" and it's true: beside each rational number is an irrational number. This "beside-ness" is so pronounced that it is given a special name: density. We say that the Irrational numbers are dense in the Real numbers. Rational numbers are also dense in the Real numbers. This means that if you pick any two numbers of any type, there is both an Irrational and a Rational number between them. And, practically speaking, this means that there are no longer any gaps between numbers and a solid line is an accurate representation of the Real numbers.

Let's focus on the Irrationals. There are actually more Irrational numbers than Rational numbers but few people are comfortable with them because they're so hard to work with. The proof that there's more Irrational numbers than Rational numbers is part of a senior level course for Math majors called "Real Analysis". We'll just take it as a fact.

First let's have some discussion about the nature of the set of Irrational numbers.

An irrational number can be represented as a non-terminating and non-repeating decimal. The decimal representation of an Irrational number generally displays a few numbers and concludes with ellipsis…which is an interesting problem in itself. Let's look at a short decimal representation of pi, arguably the most famous Irrational number.

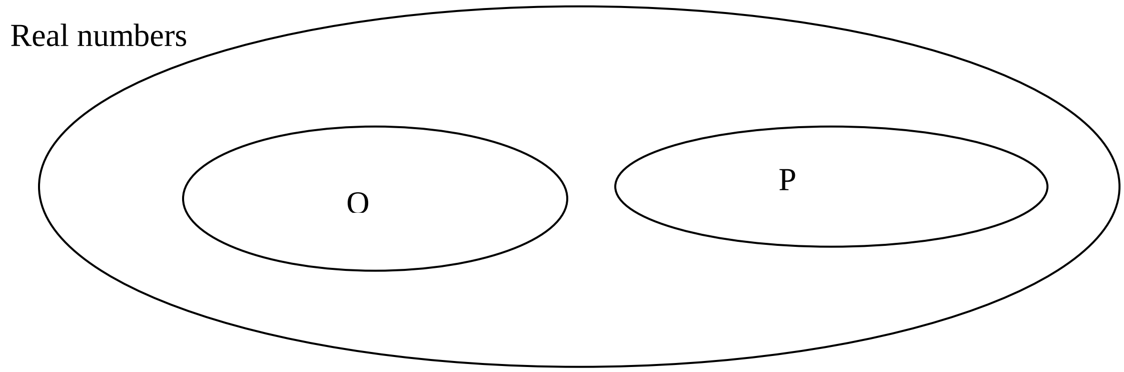
$\pi = 3.14...$

Now we generally read "…" as "and so on in this pattern"…Unfortunately the "pattern" is non-repeating so there's nowhere to put a superscripted bar and saying "and so on in this pattern of NO pattern" generally makes a student's brain hurt just a bit. But we do write Irrationals with an ellipsis meaning "and so on in no pattern".

Now let's look at the set relationship with the Rational numbers. Each Rational number can be written as a decimal that either terminates or has an infinite repeating pattern. Irrational numbers do not fit this pattern. So Irrational numbers are put in their own set, separate from Rational numbers. We've already used "I" for integers so we will use "P" for the set of Irrational numbers. The technical way to discuss the fact that Rational numbers and Irrational numbers are in separate sets is to note that the sets are "disjoint". If you pick a single number from the number line, it is either Rational or Irrational (not both, not neither, it's one or the other).

If you take both the sets and put the numbers in size order with a point for each number: small numbers on the left moving to the right for each larger number, you will create the Real number line. To say this in set builder notation: $R = Q \cup P$.

A Venn diagram of this looks like:



So, which numbers are irrational? Here's a short list of some of the Irrational numbers:

- Pi (π) and the Euler number, e
- the square root of any prime number
- the square root of any composite number that is not a perfect square
- decimals that are non-terminating and non-repeating
- all multiples of the elements of the above list

Examples of some Irrational numbers:

$\frac{\pi}{2}$ and 3e are irrational.

$\sqrt{17}$ is the square root of a prime number.

$\sqrt{6}$ is the square root of a composite number that is not a perfect square.

3.01001000100001… is a non-terminating, non-repeating decimal
(it has an understandable pattern, but no place to put a superscripted bar).

.112123123412345… is another nice Irrational number

Let's look at some computation with irrational numbers:

When you add, subtract, multiply, or divide two Rational terminating decimals, you go to the last place on the right of the decimals and start working your way to the left, borrowing and carrying until you finish up in the digit place of the largest number.

Since there is NO last place with an Irrational number you cannot start the borrowing and carrying process to add, subtract multiply or divide. Generally you just leave the problem as stated and declare it "done", unless there are some special conditionals that come with the problem.

For example, you might have a multiple of one Irrational number being combined with another multiple of the SAME Irrational number:

$3\pi + 5\pi = 8\pi$

Here, the irrational number is just carried along as though it were and "x" in an algebra problem. These terms are "like terms" really.

Similarly: $\frac{\sqrt{17}}{3} + \frac{\sqrt{17}}{4} = \frac{4}{4} \cdot \frac{\sqrt{17}}{3} + \frac{3}{3} \cdot \frac{\sqrt{17}}{4} = \frac{7\sqrt{17}}{12}$

Neither of these examples are like:

$\sqrt{11} + \pi$

This problem is stated and finished with this presentation because neither term is "like".

Now sometimes you really do need to come up with a number answer that is "close enough" to the answer. In that case you use nearby rational numbers to come up with an approximate answer. In this class, every approximation will come with a request for a certain number of decimal places in the work and the answer. Let's do this addition to 3 decimal places: $\sqrt{11} + \pi \approx 3.317 + 3.142 = 6.459$. So the strictly Rational number 6.459 is near the actual exact answer which is an Irrational number.

Not every arithmetic computation with two Irrational numbers results in an Irrational number. For example $\sqrt{15} + -\sqrt{15} = 0$, a Rational number.

Arithmetic can move you from the Irrational numbers to the Rational numbers but not the other way.

So we now have two disjoint sets that union up to make the Real Numbers.