Section 2.3

Exponents and Radicals

Exponents allow us to

- give instructions in an efficient way,
- write large numbers in few symbols,
- give a concise way to write some irrational numbers and combine the ones that can be combined easily in arithmetic computations, and
- work some problems in an efficient way.

An exponent for this class can be any Rational number. The exponent is superscripted and in smaller type, the number that is being worked on is on the line of type, in the regular sized fort, and is called the base.

For example: 47^{-3} 47 is the base and -3 is the exponent.

The following summary of exponents and their translation to another number is for your reference.

<u>Exponent Reference</u>

1. A **Natural number exponent** means to multiply the base times itself as many times as the number that is one less than the exponent.

For example: 3^3 = 3 times itself two more times for 3(3)(3) = 27.
You have a factor string that is 3 long with an exponent of 3.

Now, there's no real efficiency in writing 3^3 for 27, but how about 5^7 for 78,125?

2. An exponent of **0** means that the number affected by the exponent is 1.

For example: $11^0 = 1$

Note that a number that is NOT under the direct influence of an exponent is NOT affected by it.

Examples: $2x^0$ = 2 here the x is affected by the 0 exponent and the 2 is not.

$(2x)^0 = 1$ note that parentheses give the 0 the power to change BOTH the 2 and the x to 1

-3^2 note that the base is considered to be 3, the negative part of the number is actually -1 times the base. So this is $-1 \cdot 3^0$ and only the 3 is acted upon. The answer is -1. If the base is intended to be strictly an Integer, you need to use parentheses to let the reader know. $(-3)^0 = 1 \neq -3^0$

3. An exponent of -1 means change the number to it's reciprocal.

Examples: $12^{-1} = \frac{1}{12}$

$(\frac{1}{4})^{-1} = 4$

$-5^{-1} = -\frac{1}{5}$ here, the exponent is working only on the 5

4. An exponent that is **strictly an Integer** gives TWO instructions. Each strict Integer is a Natural number multiplied by -1. The Natural number gives the multiplication instruction explained above (in 1) while the -1 still means to turn the base into it's reciprocal.

Examples: $4^{-3} = 4^{-1 \cdot 3} = (\frac{1}{4})^3 = \frac{1}{64}$

$-9^{-2} = -1 \cdot 9^{-1 \cdot 2} = -1(\frac{1}{9})^2 = -\frac{1}{81}$

$(-2)^{-4} = (-\frac{1}{2})^4 = \frac{1}{16}$

$-2^{-2} = -4$ here, the exponent is working only on the 2

5. An exponent that is strictly a **Rational number and a unit fraction** (the numerator is 1) tells us to find the root of the number. A root is a base from a previous exponential instruction.

An exponent of $\frac{1}{2}$ means find the number that was squared to give us our current base.

Example: $25^{\frac{1}{2}} = 5$ because 5^2 gave us our current base of 25

Notice that the numerator for this is always 1 and the denominator tells us the previous exponential instruction.

Examples: $81^{\frac{1}{4}} = 3$ because 3^4 gave us the current base of 81

$32^{\frac{1}{5}} = 2$ because 2^5 gave us the current base of 32

Now what if the denominator is NOT one and you don't have a unit fraction? Then there are two instructions to do. First factor the fraction to a Natural number times the unit fraction, then follow the instructions doing the whichever makes the problem easiest first.

Examples: $25^{\frac{3}{2}}$ note that $\frac{3}{2} = \frac{3}{1} \cdot \frac{1}{2} = 3 \cdot \frac{1}{2}$ so the instructions are cube the base and take the square root of the base. Now cubing 25 leads to a big, big number so let's take the square root first, then cube that answer.

$$25^{\frac{3}{2}} = 25^{\frac{1}{2} \cdot 3} = 5^3 = 125$$

$$16^{\frac{3}{4}} = 16^{\frac{1}{4} \cdot 3} = 2^3 = 8$$

Once in a while, we fall back on traditional notation for root-finding:

$\sqrt[u]{b} = b^{\frac{1}{u}}$ u means the denominator of the unit fraction.

And

$\sqrt[u]{b^e} = b^{\frac{e}{u}}$ e gives you your non-one denominator

Examples: $\sqrt{25^3} = 25^{\frac{3}{2}} = 125$ note that if “u” is not showing, it’s 2, a square root

$\sqrt[4]{16^3} = 16^{\frac{3}{4}} = 8$

One problem with the traditional notation is the students do not really “see” that they have a choice about which instruction to do first…they almost always do the natural number exponent before the unit fraction – and this makes for big numbers and hard work unnecessarily.

One nice thing about traditional notion is that there’s no ambiguity about what the instruction is.

Now, you know that lots of Irrational numbers involve square roots. In fact, any base to a unit fraction that doesn’t have a nice answer is probably an Irrational number!

Example: Let’s look at $30^{\frac{1}{3}}$.
What number times itself in a factor string 3 long gives you 30?
$30 = 2 \cdot 3 \cdot 5$. This isn’t 30 = x^3 for some nice single rational x.
No nice Rational or Natural numbers do the trick so $30^{\frac{1}{3}}$ is Irrational and we often use traditional notation for this situation and write $\sqrt[3]{30}$.

Rules for Computation with Exponents

Computing with numbers that have exponents is easily accomplished when you’ve learned the Rules for Computation with Exponents. There are 5 of these:

- Multiplication/Addition Rule
- Division/Subtraction Rule
- Powered Number Rule
- Powered Product Rule
- Powered Ratio Rule

These are all short cuts to make your work easier and more efficient.

Multiplication/Addition Rule:

If you have two numbers, each with an exponent, and they have the SAME base AND you're multiplying them, you may simply add the exponents to do the work efficiently.

Examples: $2^4(2^{-2}) = 2^{4+-2} = 2^2 = 4$

Just this once, let's do it the long way to show that it's really true:

$2^4(2^{-2}) = 16(\frac{1}{4}) = 4$

$57^{102}(57^{-3}) = 57^{99}$

You really wouldn't want to deal with the actual numbers for this problem – they're HUGE.

$5^{\frac{1}{2}}(5^{\frac{3}{2}}) = 5^{\frac{1}{2}+\frac{3}{2}} = 5^{\frac{4}{2}} = 5^2 = 25$

Note that if you have different bases or you're doing something besides multiply, well, you're going to have to use the actual numbers. No short cut rules apply to those situations.

Examples:

$3^2(2^3) = 9(8) = 72$

These are multiplied but are not the same base, so you have to convert to the Natural numbers and do the work.

$5^{\frac{1}{2}} + 5^{\frac{1}{3}} = \sqrt{5} + \sqrt[3]{5}$

These are NOT multiplied and they are each Irrational and they are not "like". They can be converted to traditional notation, but that's as far as it can go.

Division/Subtraction Rule:

If you have two numbers, each with an exponent, and they have the SAME base AND they're divided, you may simply subtract the exponents, in this order:

Numerator exponent – Denominator exponent

to work the problem efficiently.

Examples: $\frac{19^4}{19^2} = 19^{4-2} = 19^2$

$$\frac{5^{-4}}{5^{-5}} = 5^{-4--5} = 5^{-4+5} = 5$$

$$\frac{3^{\frac{1}{2}}}{3^{\frac{3}{4}}} = 3^{\frac{1}{2}-\frac{3}{4}} = 3^{\frac{2}{4}-\frac{3}{4}} = 3^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{3}}$$

the answer is still in improper form

$$\frac{1}{\sqrt[4]{3}} = \frac{1}{3^{\frac{1}{4}}} \cdot \frac{3^{\frac{3}{4}}}{3^{\frac{3}{4}}} = \frac{3^{\frac{3}{4}}}{3^{\frac{1}{4}+\frac{3}{4}}} = \frac{3^{\frac{3}{4}}}{3}$$

Note that using modern notation makes this relatively easy.

Powered Number Rule:

Suppose you have a number that can be written with in exponential form. Suppose further that you want to multiply this number times itself several times. You may do this very efficiently with exponents.

Example: $8^2 = (2^3)^2 = 2^3 \cdot 2^3 = 2^{3+3=2(3)} = 2^6$

Multiplying a number that is in exponential form times itself results in adding that original exponent to itself – which is the same as multiplying the original exponent times the new exponent.

Example: $(19^5)^3 = 19^5 \cdot 19^5 \cdot 19^5 = 19^{15}$ which is a 20 digit number

$(.5)^{14} = (\frac{1}{2})^{14} = (2^{-1})^{14} = 2^{-14}$ which is a VERY tiny number

Now this process works for fractional exponents as well. In traditional notation, this sort of problem seems very difficult, but with this rule, simplifying is really just multiplying fractions.

Example: $(\sqrt[3]{2})^6 = (2^{\frac{1}{3}})^6 = 2^{\frac{1}{3}\cdot\frac{6}{1}} = 2^2 = 4$

$(\sqrt[3]{5^2})^{12} = (5^{\frac{2}{3}})^{12} = 5^{\frac{2}{3}\cdot 12} = 5^8$

This doesn't always result in a Natural number; it the resulting number is irrational, just leave it with the fractional exponent or change it to traditional notation.

Example: $(\sqrt[3]{7^2})^{\frac{1}{5}} = (7^{\frac{2}{3}})^{\frac{1}{5}} = 7^{\frac{2}{3}\cdot\frac{1}{5}} = 7^{\frac{2}{15}} = \sqrt[15]{7^2}$

Every piece of this work has Irrational numbers being manipulated to a number that, at least, doesn't have that fifth root to be done.

Powered Product Rule

Suppose you have a product and you have exponential instructions to do "something" with this product, you may separate the factors of the product and apply the instructions to each factor individually.

Example: $14^2 = (2 \cdot 7)^2 = 2^2 \cdot 7^2$

This can be very handy in simplifying fractions: $\frac{14^2}{4} = \frac{2^2 \cdot 7^2}{2^2} = 49$

Notice that by factoring the 14 and applying the Powered Product rule, the problem is efficiently solved by cancellation.

Example $\sqrt{24} = (24)^{\frac{1}{2}} = (4)^{\frac{1}{2}}(6)^{\frac{1}{2}} = 2(6)^{\frac{1}{2}} = 2\sqrt{6}$

This type of problem generally comes with the instruction "simplify". It is not proper to leave a factor with an exponent that can be reduced by actually doing the work that the exponent suggests.

This rule also works "in reverse". If you have different numbers, each to the same power, you may combine them to one composite number to that power.

Example: $4^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = (40)^{\frac{1}{3}}$

Powered Ratio Rule:

If you have a fraction ("ratio") that has an exponent affecting both the numerator and the denominator, you may apply that exponent to the numerator and denominator separately.

Example: $(\frac{3}{2})^2 = \frac{3^2}{2^2} = \frac{9}{4}$

$$(\frac{5}{16})^{\frac{1}{2}} = \frac{5^{\frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{\sqrt{5}}{4}$$

Now it is not proper to have a fractional exponent in the denominator of a fraction. The only kind of number that is acceptable is an Integer. So there is some work involved to make each fraction proper. Note that any kind of number will be fine in the numerator. You may have encountered this before and used a process called "rationalizing the denominator".

Example: $(\frac{1}{3})^{\frac{1}{2}} = \frac{1^{\frac{1}{2}}}{3^{\frac{1}{2}}} = \frac{1}{3^{\frac{1}{2}}} \cdot \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}}} = \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}+\frac{1}{2}}} = \frac{3^{\frac{1}{2}}}{3^1}$

Getting that Integer "3" in the denominator is what makes the ratio proper.

The key to this process is to look at the fractional exponent in the denominator and figure out what fraction must be added to it to get an exponent of 1 on that base. Using this fraction on the same base to get what we'll call the rationalizer, multiply the original fraction by "1" (rationalizer/rationalizer) and simplify.

Example: $(\frac{2}{5})^{\frac{2}{3}} = \frac{2^{\frac{2}{3}}}{5^{\frac{2}{3}}}$

That exponent $\frac{2}{3}$ is what's causing this ratio to be improper.

What number plus $\frac{2}{3}$ = 1? $\frac{1}{3}$ Now, take one third and use it as an exponent for the base 5. Form a fraction that is equal to 1 and multiply it times the original fraction. Then simplify the denominator.

$$\frac{2^{\frac{2}{3}}}{5^{\frac{2}{3}}} \cdot \frac{5^{\frac{1}{3}}}{5^{\frac{1}{3}}} = \frac{2^{\frac{2}{3}} \cdot 5^{\frac{1}{3}}}{5^{\frac{2}{3}+\frac{1}{3}}} = \frac{2^{\frac{2}{3}} \cdot 5^{\frac{1}{3}}}{5^{1}}$$

Now, there is one more simplifying step:

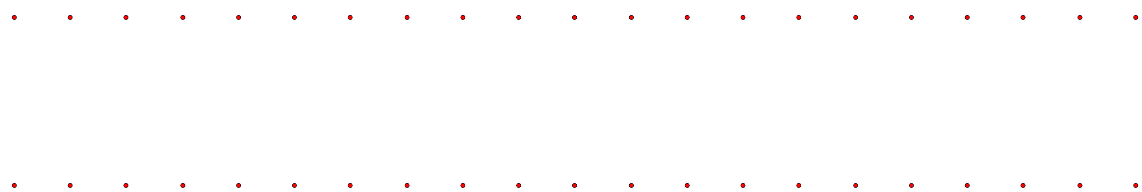
In the numerator the base 2 number has an exponent that is not a unit fraction. The 2 in the numerator of the exponent means "square" the base. If we do that, then we may apply the Powered Product rule to make a single Irrational numerator (remember, any kind of number is fine for a numerator).

$$\frac{2^{\frac{2}{3}} \cdot 5^{\frac{1}{3}}}{5} = \frac{(2^2)^{\frac{1}{3}} \cdot 5^{\frac{1}{3}}}{5} = \frac{(4 \cdot 5)^{\frac{1}{3}}}{5} = \frac{\sqrt[3]{20}}{5}$$

Exploration: Square roots of the whole numbers:

How do you figure out what the square root of 7 is without a calculator?

Well, let's explore the number line and a home-made number line of the square roots:



Pick the top line to be the number line for the whole numbers; place a whole numbers on each point, 0 – 20. The second line will be our square root line. Draw a line down from each whole number to the dot below. Don't write in any numbers yet.

Now, write in the square roots of the perfect square whole numbers on the bottom line. 0 and 1 will have 0 and 1 below them. 4 will have a 2 below it; 9 will have a 3 and 16 will have a 4 below it.

You will notice that there are dots in between the whole numbers on the bottom line. In fact, the top line has the numbers 0 – 20 and the bottom line only has 0 – 4 with unlabeled dots in between and at the end. This is ok. We are now going to name those dots.

Notice that you have natural numbers 2 and 3 in between 1 and 4 on the top line and two dots on the bottom line. Those two dots represent the $\sqrt{2}$ and the $\sqrt{3}$ respectively; these are irrational numbers. Label those.

Now go along and fill in the square roots on the second line, noting that for each natural number that is not a perfect square, you have a dot that is a square root below and each of these dots is an irrational number. For example, $\sqrt{5}$ is on the dot below 5 and $\sqrt{6}$ is on the dot below 6 and so on. You may simplify the square roots that do simplify, but write the simplification below the un-simplified square root.

You should end up with $\sqrt{20}$ on the last dot on the right of the bottom row. And you may want to write that this is $2\sqrt{5}$ below.
Now back to our original question. Notice that $\sqrt{7}$ is between 2 and 3 on the bottom line. Notice, too, that there are 4 irrational square roots between 2 and 3 making 5 steps between 2 and 3. Now count over from 2 to $\sqrt{7}$ …there are 3 steps to there.

We will approximate $\sqrt{7}$ as $2\frac{3}{5}$ where 2 is the closest natural number on the left and the numerator of the fraction is the number of steps from 2 to $\sqrt{7}$ and the denominator is the number of steps between 2 and 3.

Now let's check our estimate.

If you put $\sqrt{7}$ in your calculator you get: 2.645751311…
and if you put $2\frac{3}{5}$ in there you get: 2.6

Not bad for an estimate.

Exercise:

Now approximate the $\sqrt{33}$ on your own or in your group using the method shown above. Extend the dots so that you have 36 above and matching dots below.

Did you get $5\frac{8}{11}$? How does this check out?

Did you notice that you can subtract 25 from 36 to get the denominator of the fraction quickly?

Discuss how to find the value of the square root of a natural number.

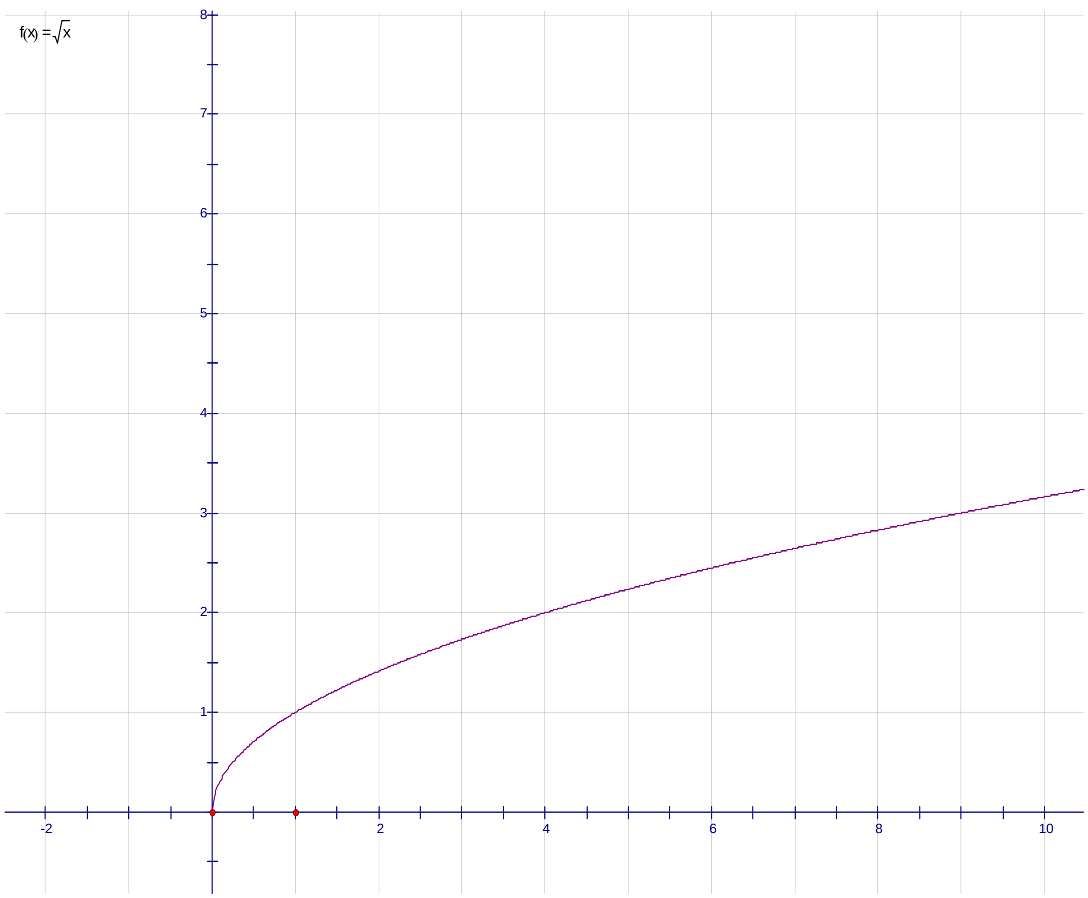
Exercise:

Find $\sqrt{17}$ and $\sqrt{23}$ using the method shown in class in groups.

Compare your answers to the answers that the other groups got.

One caveat:

This way of estimating is NOT good for $\sqrt{2}$ and $\sqrt{3}$. You'd expect to get $1\frac{1}{3}$ and $1\frac{2}{3}$ and actually they are $1\frac{4}{10}$ and $1\frac{7}{10}$. To see why, look at the following graph of the square root function. The initial curving on the function makes our proportional process be slightly off.



Once you move past 3, this method works well.