Section 5.1

Functions, Domains and Ranges

In this chapter, we'll look at relations and functions. Functions are a very important idea in mathematics, and you'll see function notation used in many different types of application problems. For example, we can use a function to model the cost of purchasing multiple copies of a book through a website (including shipping costs of $25), $C(x) = 32.00x + 25$.

A relation is any set of ordered pairs. A function is a set of ordered pairs that has an additional requirement: each value of *x* can have one and only one value for *y*.

A function sets up a dependent relationship between quantities. That is, the value of the function depends on the number that is substituted into the function in place of the variable.

The notation for a function is $f(x)$ and is read "*f* of *x*". For the function $f(x) = 5x + 1$, the right hand side of the equation tells us to multiply 5 by the value of *x* and then add 1. The value of this quantity depends on what number we choose for *x*.

We can use function notation when we wish to evaluate a function a various values for *x*. For example, if we want to find the value of $f(x) = 5x + 1$ when *x* is 0, 1 or 2, we can write:

$$f(0) = 5 \cdot 0 + 1 = 1$$
$$f(1) = 5 \cdot 1 + 1 = 6$$
$$f(2) = 5 \cdot 2 + 1 = 11$$

In each instance, we replaced *x* with the desired number, and then we used the order of operations rule to evaluate.

While we will usually use the letters *f* and *g* to denote functions, we could really use any letter. If we wanted to write down a function for the cost of purchasing 10 copies of a book, for example, we might choose to call that function *C*, since it is a cost function. Similarly, we will typically use the variable *x*, but we can use any variable we'd like. In a function that depends on how much time has passed, we might use variable *t* to denote time.

Definition of a Function

A function is a rule that assigns to each element in a set *A*, called the domain, exactly one element in a set *B*, called the range.

In this definition, we have an independent variable, that is, the value that we choose from set *A*, and a dependent variable, an element in set *B* that will depend on what value we choose from *A*.

Example 1: Given the set of ordered pairs, $\{(2,3),(3,-1),(5,-1),(3,8),(-2,3)\}$, determine if the relation is a function.

Solution:

To answer this question, we will first state the set of all of the *x* coordinates from the ordered pairs: $A = \{-2,2,3,5\}$. The definition of a function requires that each element of this set produce one *and only one* value for *y*. In this case, -2, 2 and 5 each produce only one *y* value, but 3 is paired with both -1 and 8. This violates the definition of a function. So we can conclude that the set is not a function.

Note that the fact that some of the *y* values are repeated is not relevant to the definition of a function.

Example 2: Suppose $f(x) = 3x - 5$ and $A = \{-2,-1,0,1,2\}$. Find the value of *f* for each element in the domain. Then state the range.

Solution:

We are given five elements in the domain. We'll substitute each one of these into the function and evaluate. Then we can state the range using the five values that we found for the function.

$$f(x) = 3x - 5$$
$$f(-2) = 3(-2) - 5 = -6 - 5 = -11$$
$$f(-1) = 3(-1) - 5 = -3 - 5 = -8$$
$$f(0) = 3(0) - 5 = 0 - 5 = -5$$
$$f(1) = 3(1) - 5 = 3 - 5 = -2$$
$$f(2) = 3(2) - 5 = 6 - 5 = 1$$

The range of this function is $\{-11,-8,-5,-2,1\}$.

Interval Notation

We'll need to be able to use interval notation when we write the domain of a function. Interval notation is a very concise way of describing a set of real numbers.

We'll use parentheses or brackets when writing an interval. When we use a bracket, we're indicating that the set includes that number. When we use a parenthese, we're indicating that is does not.

For example, the interval $[2,6)$ means "all real numbers between 2 and 6, including 2 but not including 6. This interval corresponds to the inequality $2 \le x < 6$.
We'll need to be able to write an inequality such as $x > 3$ using interval notation. In this case, we understand that the interval will not include 3, but it will include all of the real numbers bigger than 3. Since there is no "biggest" number, we'll use the infinity symbol, ∞.

The interval corresponding to $x > 3$ is $(3, \infty)$.

Here is a summary of interval notation that we will use:

(-3, 5)	all x such that $-3 < x < 5$
[-3, 5]	all x such that $-3 \le x \le 5$
[-3, 5)	all x such that $-3 \le x < 5$
$[-3, \infty)$	all x such that $x \ge 3$
$(-\infty, 5)$	all x such that $x < 5$
$(-\infty, \infty)$	all real numbers

Example 3: Write using interval notation: $x < 7$

Solution:

The interval does not include 7, so we'll use a parenthese. We want all values less than 7, so the interval is $(-\infty, 7)$.

Example 4: Write using interval notation: $-2 < x \le 4$.

Solution:

The interval includes 4 but does not include -2. We want all values between the two numbers. The interval is $(-2, 4]$.

Domain of a Function

Most often, you will not be given a set as the domain of a function. Determining the domain will be one of your tasks. If the domain is not specified, then the domain will be the set of all real numbers for which the function is defined. Next, we'll look at some situations where the domain of a function is restricted. We can have restrictions on the domain if our function is a fraction with variables in the denominator, if our function contains a square root, or if it has both features. There are other situations where the domain is restricte, but they are beyond the scope of this course.

Example 5: State the domain of the function: $f(x) = 3x^2 + 4$.

Solution:

There are no restrictions on the values of *x* that can be used with this function. The domain is $(-\infty, \infty)$.

Example 6: State the domain of the function: $f(x) = \dfrac{3}{x-2}$.

Solution:

Recall that division by zero is not defined. Since we have variable in the denominator of this function, we have to make sure that the denominator is never zero. We will have a restriction:

$$x - 2 \neq 0$$
$$x \neq 2$$

The domain will be all real numbers, except 2. We'll write this using interval notation as $(-\infty, 2) \cup (2, \infty)$.

Example 7: State the domain of the function: $f(x) = \sqrt{x+5}$.

Solution:

We can only find the square root of a number if it is non-negative. In this case, we can't allow $x + 5$ to be negative. To find the domain, we will solve the inequality $x + 5 \geq 0$:

$$x + 5 \geq 0$$
$$x \geq -5$$

The domain is $[-5, \infty)$.

Example 8: State the domain of the function $f(x) = \dfrac{5}{\sqrt{2-x}}$.

Solution:

Here, we have a square root in the denominator. In this case, $2-x$ must be strictly greater that zero. We can't let it equal zero, as that would give us division by zero. So we'll solve the inequality $2-x>0$:

$$2-x>0$$
$$-x>-2$$
$$x<2$$

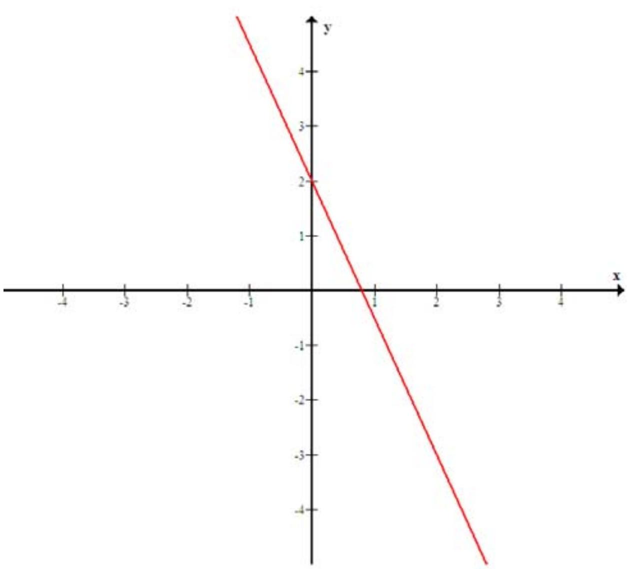
In the last step, we had to reverse the direction of the inequality, since we divided by a negative number. Now we can state the domain using interval notation: $(-\infty,2)$.

Functions and Graphs

A function is a set of ordered pairs. That means that we can graph any function in the coordinate plane. The *y* value in each ordered pair is the value of *f* at *x*, so we can write $y=f(x)$.

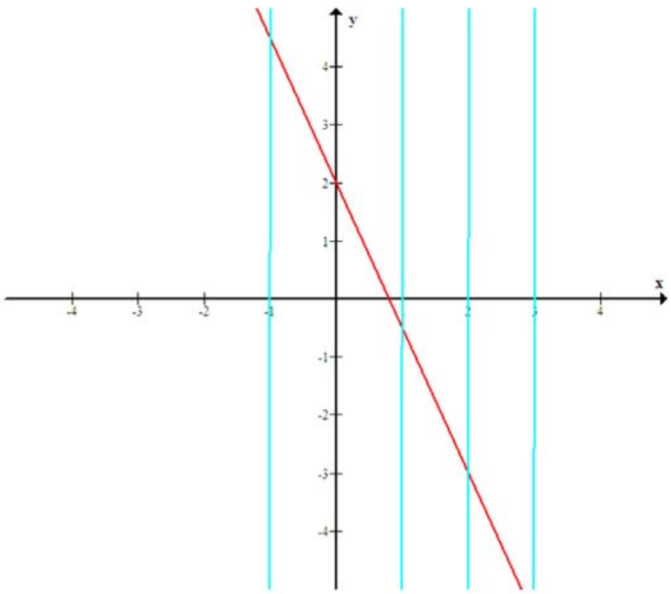
We can also look at a graph and easily determine if it is the graph of a function. We can use the **vertical line test** to accomplish this. The vertical line test states that the relation graphed is a function so long as no vertical line can intersect the graph in more than one point.

Example 9: Determine if the graph given is the graph of a function:



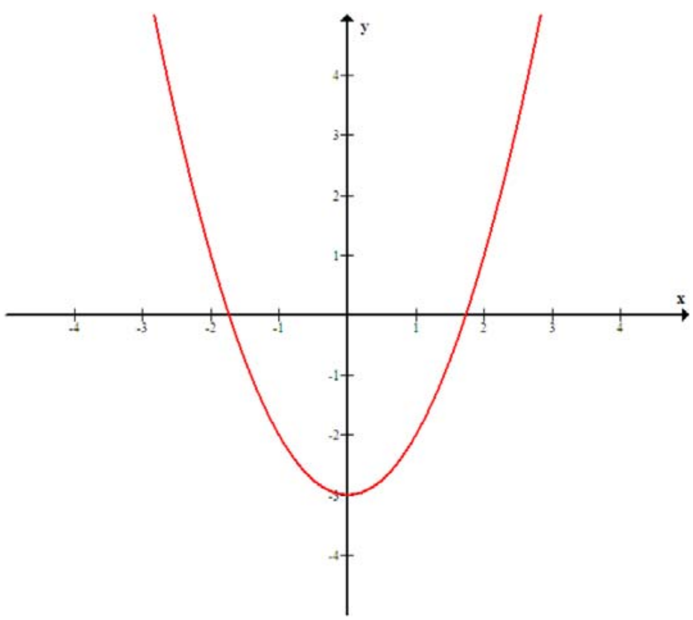
Solution:

We can draw several vertical lines to test this graph:



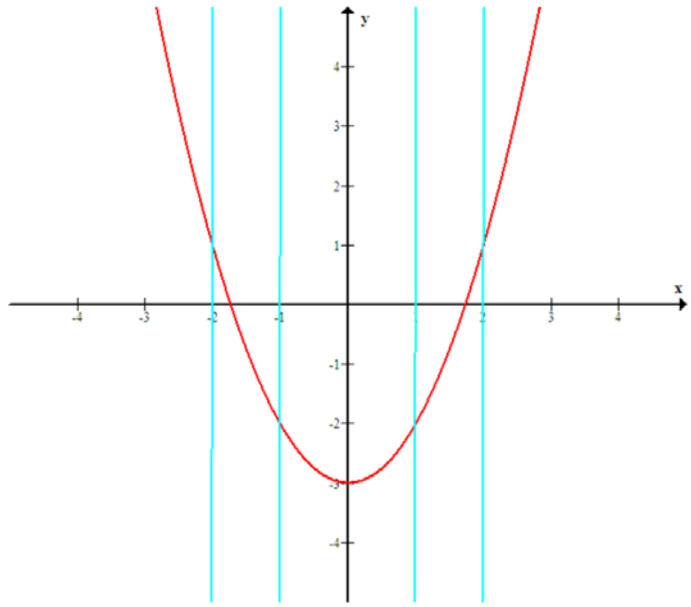
From these sample vertical lines, we can see that no vertical line will ever intersect the graph in more than one point. The equation graphed here is a function.

Example 10: Determine if the graph given is the graph of a function:



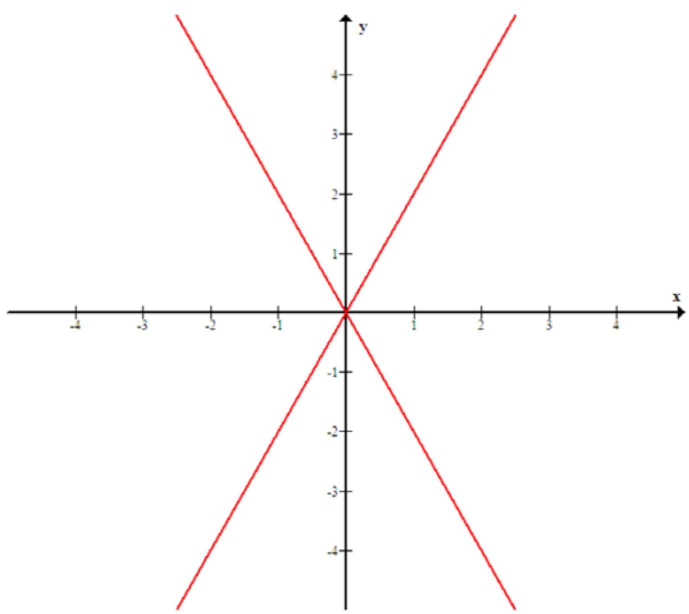
Solution:

We can draw several vertical lines to test this graph:



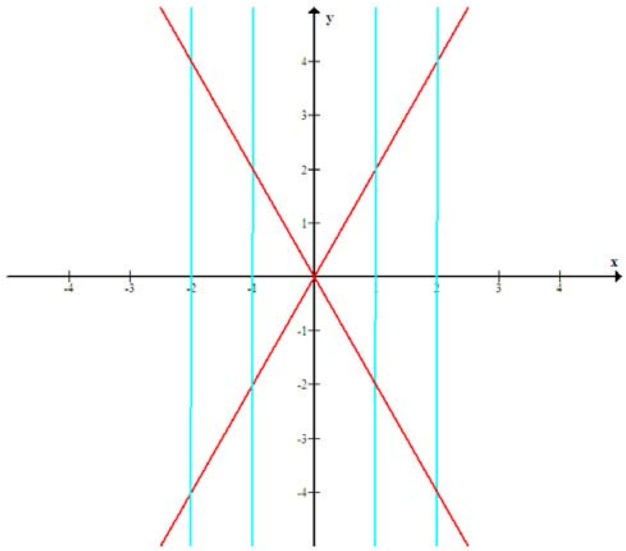
From these sample vertical lines, we can see that no vertical line will ever intersect the graph in more than one point. The equation graphed here is a function.

Example 11: Determine if the graph given is the graph of a function:

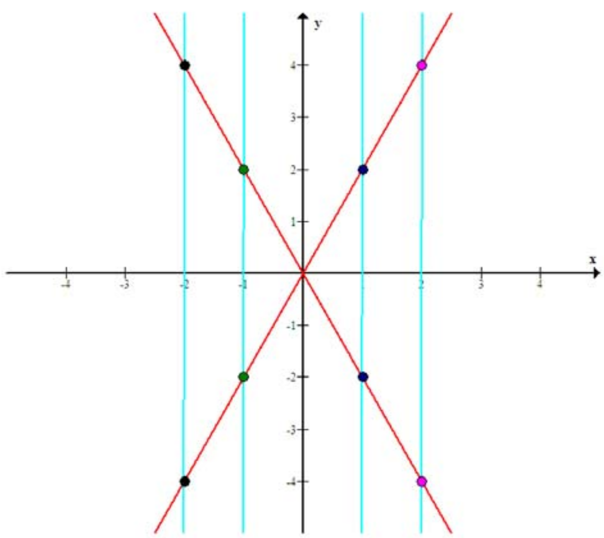


Solution:

We can draw several vertical lines to test this graph:



We can see that each of these vertical lines intersects the graph in two points as shown here:

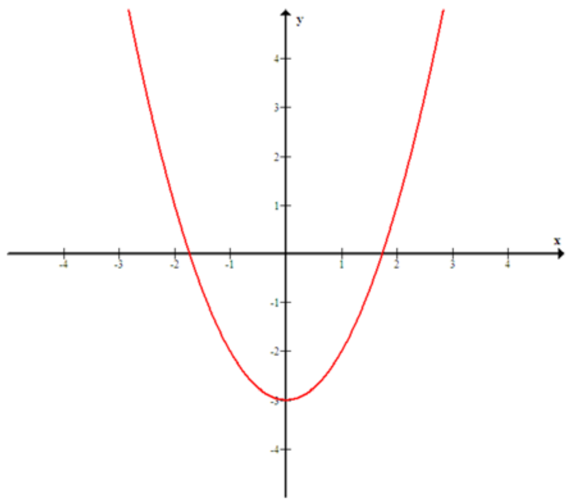


We can conclude that the equation graphed here is not a function.

Range of a Function

Just as we can state the domain of a function, we can state the range of a function. The range of a function is the set of all possible *y* values. We'll usually do this by looking at a graph of the function. Once the function is graphed, we'll look to see if there are any top points, bottom points, holes or other gaps that exclude values from the range. Then we'll use interval notation to state the range.

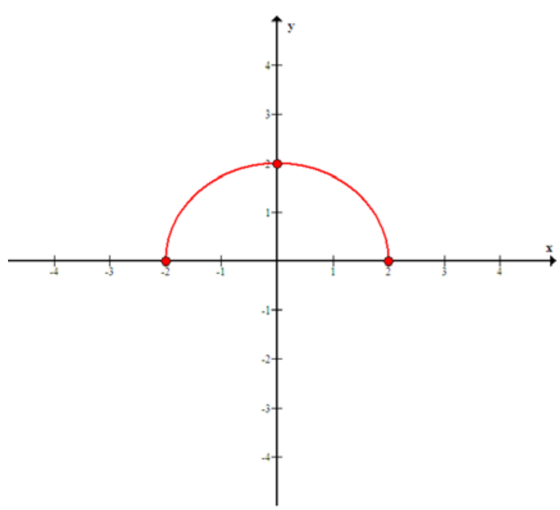
Example 12: State the range of the function that is graphed here:



Solution:

There is a definite bottom point to this graph, (0, -3). There are no points on the graph below this point. However, the graph includes all *y* values above -3. We can conclude that the range of the function is $[-3,\infty)$.

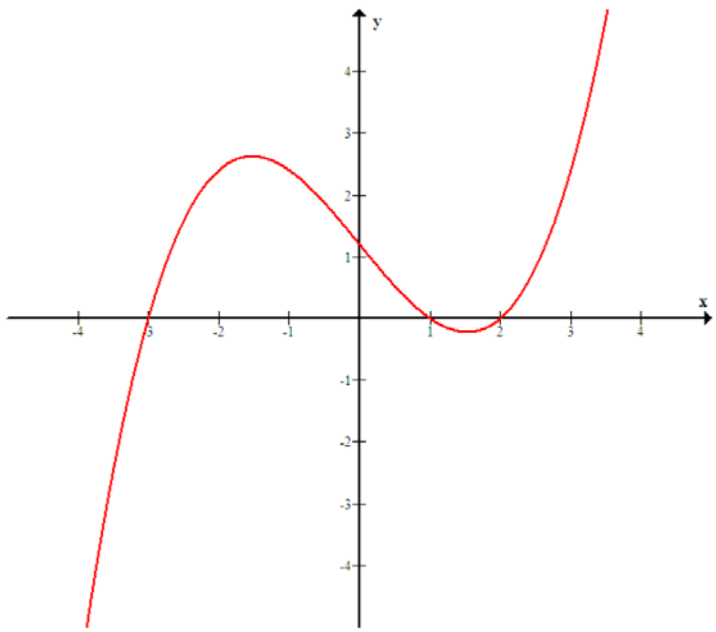
Example 13: State the range of the function that is graphed here:



Solution:

There is a definite top point to this graph, and we can see that the graph never goes below the *x* axis. The top point is the point (0, 2). The bottom points are the points (-2, 0) and (2, 0). The graph contains all *y* values between 0 and 2, so the range is $[0,2]$.

Example 14: State the range of the function that is graphed here:



Solution:

There is no top point and no bottom point to this graph. This graph contains every possible value for *y*. The range is $(-\infty,\infty)$.