Section 5.3

Factoring

Now we want to focus our attention on quadratic functions. A quadratic function is a function of the form $f(x) = ax^2 + bc + c$. These equations will have a term that contains x^2. Before we begin our study of these functions, we'll review some topics from algebra.

Multiplying Binomials

A binomial is a polynomial that contains two terms. We often need to multiply two of these together. We'll distribute each term in the first binomial through the second binomial. Many students have learned an acronym to help them remember how to do this: FOIL, which stands for First term times First term; Outside term times Outside term; Inside term times Inside term ; Last term times Last term. After we do the multiplying, we'll need to combine any like terms.

Example 1: Multiply $(x-7)(x+3)$.

Solution:

Multiply the first term in each binomial together: $x \cdot x = x^2$
Multiply the outside term in each binomial together: $x \cdot 3 = 3x$
Multiply the inside term in each binomial together: $-7 \cdot x = -7x$
Multiply the last term in each binomial together: $-7 \cdot 3 = -21$

Now we can add these four terms together: $x^2 + 3x - 7x - 21$

We can combine like terms to arrive at the final answer: $x^2 - 4x - 21$

Example 2: Multiply $(3x^2 + 2)(4x^2 - 5)$.

Solution:

Multiply the first term in each binomial together: $3x^2 \cdot 4x^2 = 12x^4$
Multiply the outside term in each binomial together: $3x^2(-5) = -15x^2$
Multiply the inside term in each binomial together: $2 \cdot 4x^2 = 8x^2$
Multiply the last term in each binomial together: $2(-5) = -10$

Now we can add these four terms together: $12x^4 - 15x^2 + 8x^2 - 10$

We can combine like terms to arrive at the final answer: $12x^4 - 7x^2 - 10$

In both of these examples, the answer is a trinomial. That is, it's a polynomial that contains three terms. The exponents for all polynomials are natural numbers, so you won't see any fractions or negative numbers as exponents as we work through these problems.

Next we want to be able to factor a quadratic expression. You can think of this as taking the answer in these past two examples and working backwards to find the problem. There are several different patterns for factoring, and we'll look at all of them.

Factoring Trinomials of the form $x^2 + bx + c$

When given an expression of this form, we can break it into two binomials that can be multiplied together to get the given expression. We'll do this by listing the pairs of factors of *c* and finding a pair that we can multiply together to get *c* and add together to get *b*. We can then write two binomials that can be multiplied together to get the original expression.

Example 3: Factor: $x^2 + 7x + 12$

Solution:

When we factor this expression, we'll need to find two binomials of the form

$(x + \_\_\_\ \ )(x + \_\_\_\ \ )$. Our task is to fill in the blanks. We'll look for a pair of factors of 12 that add up to 7.

The factors of 12 are:

1 and 12
-1 and -12
2 and 6
-2 and -6
3 and 4
-3 and -4

The pair of factors we need is 3 and 4, since it's the only pair of factors that adds up to 7.

Now we can write our answer: $(x+3)(x+4)$.

To check this, we can simply multiply the two binomials together.

$(x+3)(x+4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$

Example 4: Factor: $x^2 - 4x - 21$

Solution:

The factors of -21 are

1 and -21
-1 and 21
3 and -7
-3 and 7

The pair of factors that adds up to -4 is 3 and -7. We can write our answer: $(x+3)(x-7)$.

Example 5: Factor: $x^2 - 5x + 6$

Solution:

The factors of 6 are

1 and 6
-1 and -6
2 and 3
-2 and -3

The pair of factors that adds up to -5 is -2 and -3. We can write our answer: $(x-2)(x-3)$.

Example 6: Factor: $x^2 + 10x + 25$.

Solution:

The factors of 25 are

1 and 25
-1 and -25
5 and 5
-5 and -5

The pair of factors that adds up to 10 is 5 and 5. We can write our answer: $(x+5)(x+5) = (x+5)^2$

Factoring the Difference of Two Squares

Sometimes the expression will be in the form $(ax)^2 - b^2$. We call this the difference of two squares because each term is a perfect square and the two terms are subtracted. These expressions can be factored as follows: $(ax)^2 - b^2 = (ax+b)(ax-b)$.

Example 7: Factor $x^2 - 9$.

Solution:

This expression is in the form $(ax)^2 - b^2$ since we have $(x)^2 - 3^2$. Using the factoring pattern, we can write our answer: $(x+3)(x-3)$.

Example 8: Factor $4x^2 - 25$.

Solution:

This expression is in the form $(ax)^2 - b^2$ since we have $(2x)^2 - 5^2$. Using the factoring pattern, we can write our answer: $(2x+5)(2x-5)$.

Note that an expression of the form $(ax)^2 + b^2$ cannot be factored over the real numbers. We refer to an expression of this sort as "prime."

Example 9: Factor $x^2 + 1$

Solution:

This is the sum of two squares, not the difference of two squares. This expression is prime.

Factoring by Grouping

Sometimes the expression we need to factor will contain four terms. In this case, we can try to use factoring by grouping. With this method, we group together the first two terms and the last two terms and look for common factors in each pair of terms. We may be able to find a common binomial factor, which we can then factor from the expression.

Example 10: Factor: $x^2 + 3x + xy + 3y$

Solution:

We'll group the first two terms together and the last two terms together. The first two terms have a common factor of *x*, and the last two terms have a common factor of *y*. We'll start by factoring out these common factors:

$$x^2 + 3x + xy + 3y$$
$$x(x+3) + y(x+3)$$

Now we can see a common binomial factor, $x + 3$. We can factor it from the two terms in our expression:

$$(x+3)(x+y)$$

We can check this by multiplying the two binomials together:

$$(x+3)(x+y) = x^2 + xy + 3x + 3y$$

Rearrange the terms to get the original expression: $x^2 + 3x + xy + 3y$

Example 11: Factor: $xy + 5y - 3x - 15$

Solution:

The first two terms have a common factor of *y* and the last two terms have a common factor of -3. We'll start by factoring out these common factors:

$$xy + 5y - 3x - 15 = y(x+5) - 3(x+5)$$

Now we can see a common binomial factor, $x + 5$. We can factor it from the two terms in our expression. The final answer is

$$(x+5)(y-3).$$

Factoring Trinomials of the form $ax^2 + bx + c,\ a \neq 1$

We'll look at three different methods for factoring trinomials of this form.

First, trial and error. When the leading coefficient is not 1, we have many more possibilities for the factors of the trinomial. Using this method, we take a guess and then multiply our binomials together to see if our guess is correct. If so, we're done. If not, we take another guess. With practice, you may be able to do much of the work mentally.

Example 12: Factor: $6x^2 - 5x - 4$

Solution:

First, we'll look at the factors of $6x^2$. One pair of factors is $2x$ and $3x$. The other pair is x and $6x$. We'll work with $2x$ and $3x$ first. If we don't find the right answer, we'll try the other pair.

The factors of -4 are

-1 and 4
1 and -4
2 and -2

Possible factorings are listed:

$(2x-1)(3x+4)$
$(2x+4)(3x-1)$
$(2x+1)(3x-4)$
$(2x-4)(3x+1)$
$(2x+2)(3x-2)$
$(2x-2)(3x+2)$

Multiply these pairs of binomials together until you find the pair of factors that works.

$(2x-1)(3x+4) = 6x^2 + 8x - 3x - 4 = 6x^2 + 5x - 4 \quad \textit{doesn't work}$
$(2x+4)(3x-1) = 6x^2 - 2x + 12x - 4 = 6x^2 + 10x - 4 \quad \textit{doesn't work}$
$(2x+1)(3x-4) = 6x^2 - 8x + 3x - 4 = 6x^2 - 5x - 4 \quad \textit{works}!!$
$(2x-4)(3x+1)$
$(2x+2)(3x-2)$
$(2x-2)(3x+2)$

Once we've found the factors that work, we can stop. Correct answer is $(2x+1)(3x-4)$.

Example 13: Factor $3x^2 - 10x - 8$.

Solution:

We'll start by looking at factors of $3x^2$. The only possibly factors are 3x and x, so we know that they will be the first terms in the two binomials. Next we'll look at the factors of -8:

-1 and 8
1 and -8
-2 and 4
2 and -4

Possible factorings are listed:

$(3x-1)(x+8)$
$(3x+8)(x-1)$
$(3x+1)(x-8)$
$(3x-8)(x+1)$
$(3x-2)(x+4)$
$(3x+4)(x-2)$
$(3x+2)(x-4)$
$(3x-4)(x+2)$

We'll multiply until we find the pair of factors that works.

$(3x-1)(x+8)=3x^2+24x-x-8=3x^2+23x-8 \quad \textit{doesn't work}$
$(3x+8)(x-1)=3x^2-3x+8x-8=3x^2+5x-8 \quad \textit{doesn't work}$
$(3x+1)(x-8)=3x^2-24x+x-8=3x^2-23x-8 \quad \textit{doesn't work}$
$(3x-8)(x+1)=3x^2+3x-8x-8=3x^2-5x-8 \quad \textit{doesn't work}$
$(3x-2)(x+4)=3x^2+12x-2x-8=3x^2+10x-8 \quad \textit{doesn't work}$
$(3x+4)(x-2)=3x^2-6x+4x-8=3x^2-2x-8 \quad \textit{doesn't work}$
$(3x+2)(x-4)=3x^2-12x+2x-8=3x^2-10x-8 \quad \textit{works}!!$
$(3x-4)(x+2)$

Since we've found the pair of factors that works, we can stop. The correct answer is $(3x+2)(x-4)$.

We can also use factoring by grouping. There are several steps to this method

1. Find *ac*.
2. List all of the pairs of factors of *ac*.
3. From your list, find the pair of factors of *ac* that add up to *b*.
4. Rewrite the *bx* term as the sum of two terms with the factors from step 3 as coefficients.
5. Factor the new expression using factoring by grouping.

Example 14: Factor: $5x^2 + 13x - 6$

Solution:

First, we'll find *ac*. In this problem it's $5(-6) = -30$. Next, we'll list all of the pairs of factors of -30:

1 and -30
-1 and 30
2 and -15
-2 and 15
3 and -10
-3and 10
5 and -6
-5 and 6

We need a pair from this list that adds up to *b*, which in this case is 13. That pair is -2 and 15.

We'll rewrite 13*x* as -2*x* + 15*x*. Now we can factor by grouping:

$$5x^2 + 13x - 6 = 5x^2 - 2x + 15x - 6$$
$$= x(5x - 2) + 3(5x - 2)$$
$$= (5x - 2)(x + 3)$$

Correct answer is $(5x - 2)(x + 3)$.

Example 15: Factor $12x^2 - 23x + 10$

Solution:

First, we'll find *ac*. In this case, *ac* = 120. Well write down all of the pairs of factors of 120. Keep in mind that the two factors can both be positive or can both be negative to give us a value of 120. Since *b* = -23, we know that the two factors must both be negative.

-1 and -120
-2 and -60
-3 and -40
-4 and -30
-5 and -24
-6 and -20
-8 and -15
-10 and -12

We need a pair of factors that adds up to -23. From this list, we see that the only pair that meets this criterion is -8 and -15.

Next we'll rewrite -23x as -8x – 15x. Now we can factor by grouping:

$$12x^2 - 23x + 10 = 12x^2 - 8x - 15x + 10$$
$$= 4x(3x-2) - 5(3x-2)$$
$$= (3x-2)(4x-5)$$

Correct answer is $(3x-2)(4x-5)$.

Here's a third method for factoring these trinomials.

Example 16: Factor $6x^2 - 7x - 20$

Solution:

For this problem, we'll start by writing down the first term in each set of parentheses – we know these won't work, but this is just our starting point. We'll choose 6x because it's the leading coefficient in the problem:

$$(6x \qquad)(6x \qquad).$$

Now find *ac*. In our case, that's (6)(-20) = -120. We need to find a pair of factors of -120 that adds up to our value for *b*, which is -7. We'll list all of the pairs of factors of -120.

1 and -120
-1 and 120
2 and -60
-2 and 60
3 and -40
-3 and 40
4 and -30
-4 and 30
5 and -24
-5 and 24
6 and -20
-6 and 20
8 and -15
-8 and 15
10 and -12
-10 and 12

We need factors that add up to -7, so 8 and -15 will do the trick.

We can now rewrite our factors:

$(6x+8)(6x-15)$

Now get rid of any common factors in the two factors. We can factor out 2 in the first set of parentheses and 3 in the second.

Final answer is $(3x+4)(2x-5)$.

You can check this by multiplying the two binomials together.

Example 17: Factor $8x^2-2x-3$

Solution:

We'll start by writing down the two binomials, each with 8*x* as the first term:

$(8x\quad)(8x\quad)$.

Our value for *ac* is $8(-3)=-24$. We need to find a pair of factors of -24 that adds up to our value for *b*, which is -2. Here is a list of all of the pairs of factors of -24

1 and -24
-1 and 24
2 and -12
-2 and 12
3 and -8
-3 and 8
4 and -6
-4 and 6

The only pair in the list that adds up to -2 is 4 and -6. We'll write these into our binomials.

$(8x+4)(8x-6)$

Now we can factor out any common factors and discard them. We can factor 4 from the first set of parentheses and 2 from the second set.

$(2x+1)(4x-3)$

The final answer is $(2x+1)(4x-3)$.

Factoring the Greatest Common Factor

Sometimes, the terms of a given polynomial have a common factor. For example, if we look at the polynomial, $2x^2 + 8x$, we can see that each term has a factor of $2x$. Before we consider other types of factoring, we start by factoring out the greatest common factor (GCF). Note that we call it the *greatest* common factor, because it is the largest factor that divides both terms. Both 2 and x are common factors, but $2x$ is the greatest common factor.

Example 18: Factor: $2x^2 + 8x$

Solution:

Since we've identified the GCF as $2x$, our answer is $2x(x+4)$. This cannot be factored any further.

Example 19: Factor: $5x^2 + 10xy + 15yz$

Solution:

The GCF is 5. We can start by factoring 5 out of each term:

$$5\left(x^2 + 2xy + 3yz\right)$$

The trinomial inside the parentheses cannot be factored any further.

Example 20: Factor $3x^2 - 27$

Solution:

The GCF is 3. We can start by factoring 3 out of each term:

$$3(x^2 - 9)$$

Since $x^2 - 9$ is the difference of two squares, we can continue factoring:

$$3(x^2 - 9) = 3(x+3)(x-3)$$

The factored form of this polynomial is $3(x+3)(x-3)$.

Example 21: Factor: $x^4 - 4x^3 + 3x^2$

Solution:

The GCF is x^2. We can start by factoring x^2 out of each term:

$x^2\left(x^2 - 4x + 3\right)$

Since $x^2 - 4x + 3$ can be factored, we must continue:

$x^2(x-3)(x-1)$

The factored form of this polynomial is $x^2(x-3)(x-1)$.

Example 22: Factor: $3x^3 + 6x^2 - 72x$

Solution:

The GCF is 3*x*. We can start by factoring 3*x* out of each term:

$3x\left(x^2 + 2x - 24\right)$

Since $x^2 + 2x - 24$ can be factored, we must continue:

$3x(x+6)(x-4)$

The factored form of this polynomial is $3x(x+6)(x-4)$.