Section 5.6

Solving Quadratic Equations Using the Quadratic Formula

The last method for finding solutions to a quadratic equation is called the quadratic formula. This method will work for any quadratic equation, although it will often be easier to take square roots or factor. For some equations, those methods will not work and your only choice for a method will be the quadratic formula.

The Quadratic Formula

The solutions to a quadratic equation, $ax^2 + bx + c = 0,\ a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that if $b^2 - 4ac$ is negative, the problem has no real solutions.

Example 1: Solve using the quadratic formula: $x^2 - 2x - 8 = 0$.

Solution:

In this problem, $a = 1$, $b = -2$ and $c = -8$. We'll substitute these values into the quadratic formula and then simplify.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 32}}{2}$$

$$x = \frac{2 \pm \sqrt{36}}{2}$$

$$x = \frac{2 \pm 6}{2}$$

$$x = \frac{2+6}{2} \ or \ x = \frac{2-6}{2}$$

$$x = \frac{8}{2} \ or \ x = \frac{-4}{2}$$

$$x = 4 \ or \ x = -2$$

The two solutions to the equation are $x = 4$ or $x = -2$.

Since we have two rational answers to this equation, we could have solved it by factoring.

Example 2: Solve using the quadratic formula: $x^2 - 6x = -3$

Solution:

First, we'll rewrite the problem so that the quadratic equation equals 0 by adding 3 to both sides of the equation:

$$x^2 - 6x + 3 = 0$$

Now we can see that a = 1, b = -6 and c = 3. We'll substitute these values into the quadratic formula and then simplify:

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{6 \pm \sqrt{36 - 12}}{2}$$
$$x = \frac{6 \pm \sqrt{24}}{2}$$

We can simplify $\sqrt{24}$, so we'll do that next.

$$x = \frac{6 \pm \sqrt{4 \cdot 6}}{2}$$
$$x = \frac{6 \pm 2\sqrt{6}}{4}$$

We can factor 2 out of the numerator and reduce the fraction:

$$x = \frac{2\left(3 \pm \sqrt{6}\right)}{4}$$
$$x = \frac{3 \pm \sqrt{6}}{2}$$
$$x = \frac{3 + \sqrt{6}}{2} \; or \; \frac{3 - \sqrt{6}}{2}$$

The two answers to the problem are $x = \frac{3 + \sqrt{6}}{2} \; and \; \frac{3 - \sqrt{6}}{2}$.

Example 3: Solve using the quadratic formula: $4x^2 - 12x + 9 = 0$

Solution:

In this example, a = 4, b = -12 and c = 9. We'll substitute these values into the quadratic formula and then simplify:

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$
$$x = \frac{12 \pm \sqrt{144 - 144}}{8}$$
$$x = \frac{12 \pm \sqrt{0}}{8}$$
$$x = \frac{12}{8} = \frac{3}{2}$$

In this example, the value under the square root sign was 0, so we have only one solution. The answer to the problem is $x = \frac{3}{2}$.

Example 4: Solve using the quadratic formula: $2x^2 + \frac{4}{3}x = \frac{5}{3}$

Solution:

We'll first need to write the problem in the appropriate form. We'll also clear the problem of fractions before using the quadratic formula. This will make the work easier to simplify:

$$2x^2 + \frac{4}{3}x = \frac{5}{3}$$
$$2x^2 + \frac{4}{3}x - \frac{5}{3} = 0$$

The Least Common Denominator for this problem is 3, so we'll multiply both sides of the equation by 3 to eliminate the fractions:

$$3\left(2x^2 + \frac{4}{3}x - \frac{5}{3}\right) = 3(0)$$
$$6x^2 + 4x - 5 = 0$$

Now we can identify the values for *a*, *b* and *c* and use the quadratic formula. In this case, *a* = 6, *b* = 4 and *c* = -5.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(6)(-5)}}{2(6)}$$
$$x = \frac{-4 \pm \sqrt{16 + 120}}{12}$$
$$x = \frac{-4 \pm \sqrt{136}}{12}$$

We can simplify $\sqrt{136}$ so we'll do that next.

$$x = \frac{-4 \pm \sqrt{4 \cdot 34}}{12}$$
$$x = \frac{-4 \pm 2\sqrt{34}}{12}$$
$$x = \frac{2\left(-2 \pm \sqrt{34}\right)}{12}$$
$$x = \frac{-2 \pm \sqrt{34}}{6}$$

The two answer are $x = \frac{-2 + \sqrt{34}}{6}$ and $x = \frac{-2 - \sqrt{34}}{6}$.

Example 5: Solve using the quadratic formula: $x^2 - 4x + 5 = 0$

Solution:

In this example, a = 1, b = -4 and c = 5. We'll substitute these values into the quadratic formula and simplify:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$
$$x = \frac{4 \pm \sqrt{-4}}{2}$$

Since we have a negative number under the square root sign, we know that the problem has no real solutions.

You may have worked with complex numbers in another course. This problem has imaginary answers, but we will not work with imaginary answers in this course.

Choosing a Method

We have now covered three methods for solving quadratic equations. Now, we'll work on selecting the best method to use to solve some equations.

Example 6: Solve: $25x^2 - 9 = 0$

Solution:

The left hand side of the equation is the difference of two squares. We can either factor this problem or isolate the x^2 and take the square root of both sides:

Method 1: Factoring

$$25x^2 - 9 = 0$$
$$(5x+3)(5x-3) = 0$$
$$5x+3 = 0 \; or \; 5x-3 = 0$$
$$x = \frac{-3}{5} \; or \; x = \frac{3}{5}$$

Method 2: Taking the square root of both sides of the equation

$$25x^2 - 9 = 0$$
$$25x^2 = 9$$
$$x^2 = \frac{9}{25}$$
$$x = \pm\frac{3}{5}$$

Using either method, we have the same two answers, $x = \frac{-3}{5} \; or \; x = \frac{3}{5}$.

Example 7: Solve $x^2 + 8x + 15 = 0$

Solution:

This problem factors easily. We have two factors of 15, 5 and 3, that add up to 8. We can quickly factor this problem.

$$x^2 + 8x + 15 = 0$$
$$(x+3)(x+5) = 0$$
$$x+3 = 0 \; or \; x+5 = 0$$
$$x = -3 \; or \; x = -5$$

The two solutions to the problem are x = -3 and x = -5.

Example 8: Solve $4x^2 + 20x = -23$

Solution:

We'll start by rewriting the problem:

$$4x^2 + 20x + 23 = 0$$

This problem cannot be solved by taking the square root of both sides, and trying to factor it will take some time. The quickest way to solve this problem will be to use the quadratic formula.

We'll identify the values for *a*, *b*, and *c* and then substitute them into the quadratic formula.

$a = 4$, $b = 20$ and $c = 23$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(4)(23)}}{2(4)}$$

$$x = \frac{-20 \pm \sqrt{400 - 368}}{8}$$

$$x = \frac{-20 \pm \sqrt{32}}{8}$$

We can simplify $\sqrt{32}$, so we'll do that next.

$$x = \frac{-20 \pm \sqrt{16 \cdot 2}}{8}$$

$$x = \frac{-20 \pm 4\sqrt{2}}{8}$$

$$x = \frac{4\left(-5 \pm \sqrt{2}\right)}{8}$$

$$x = \frac{-5 \pm \sqrt{2}}{2}$$

The two solutions to the problem are $x = \frac{-5 + \sqrt{2}}{2}$ and $x = \frac{-5 - \sqrt{2}}{2}$