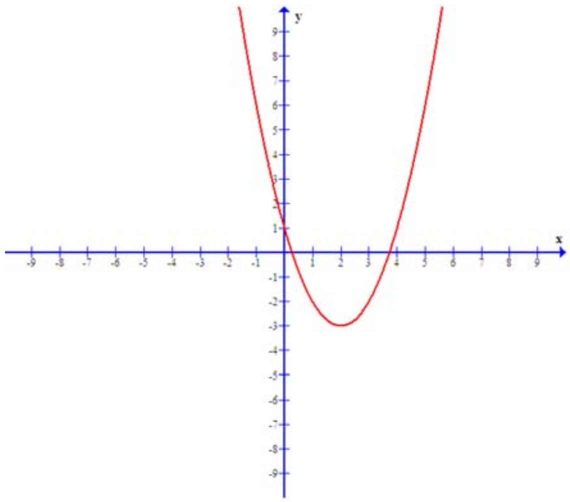
Section 5.8

Graphing Quadratic Functions

We also want to be able to graph quadratic functions. A quadratic function is a function of the form $f(x) = ax^2 + bx + c$. The graph of a quadratic function is a parabola. The general shape of the graph (when a > 0) will look like this:



Graphing Quadratic Functions

For each problem, we will gather some information about the graph of the parabola and then we'll use that information to sketch the graph.

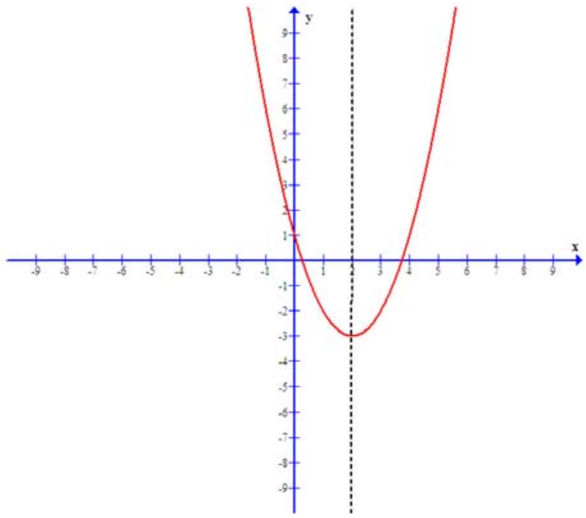
Here is a list of the information we'll find as we work through these problems:

1. direction the parabola opens (up or down)
2. *x* and *y* coordinates of the vertex
3. equation of the axis of symmetry
4. maximum or minimum value of the function
5. *x* and *y* coordinates of the *y* intercept
6. *x* and *y* coordinates of any zeros of the function

We can find the direction the parabola opens by looking at the leading coefficient of the function. If the leading coefficient is positive, the graph will open upwards, like the one that is pictured above. If the leading coefficient is negative, the graph will open downwards.

We can find the *x* coordinate of the vertex using the formula $x = \frac{-b}{2a}$. To find the *y* coordinate of the vertex, we'll find $f\left(\frac{-b}{2a}\right)$.

The graph of a quadratic function is symmetric about a vertical line that passes through the vertex. This line is called the axis of symmetry. The equation of the axis of symmetry will be the equation $x = \frac{-b}{2a}$. The graph below shows a parabola with its axis of symmetry drawn as a dotted line:



If the parabola opens upwards, it has a minimum value (bottom point). If it opens downwards, it has a maximum value (top point). The maximum or minimum value is the *y* coordinate of the vertex.

We can find the *y* intercept of the graph by computing $f(0)$.

We can find the *x* intercept(s) of the graph by setting the function equal to 0 and solving the quadratic equation for *x*. The graphs of some quadratic functions do not have any *x* intercepts.

Once we have found all of this information, we can use it to graph the function.

Example 1: Graph $f(x) = x^2 - 4x - 5$

Solution:

Since the leading coefficient is positive, the graph will open upwards. We know that the function has a minimum value.

Next, we'll find the coordinates of the vertex:

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

To find the *y* coordinate, we'll compute $f(2)$:

$f(2) = 2^2 - 4(2) - 5 = 4 - 8 - 5 = -9$

The vertex is $(2, -9)$

The axis of symmetry is $x = 2$; the minimum value is -9.

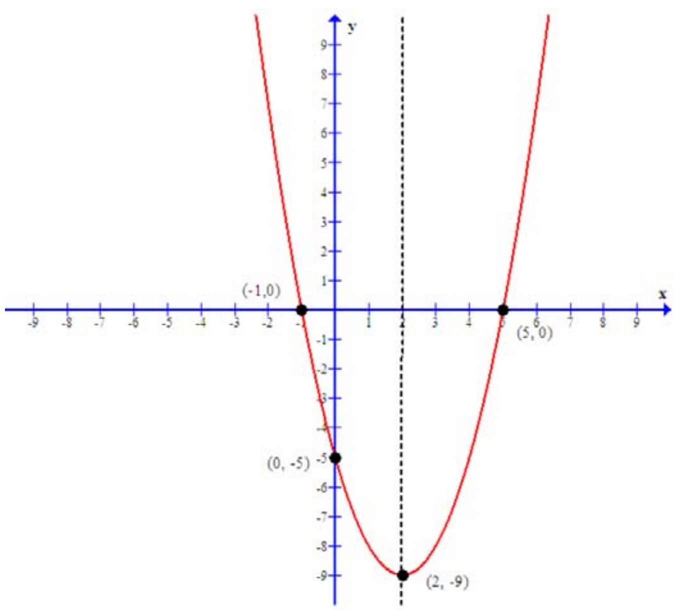
We can find the *y* intercept by computing $f(0) = 0^2 - 4(0) - 5 = -5$. The *y* intercept is (0, -5).

We can set the function equal to zero and solve to find any *x* intercepts:

$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0$$
$$x - 5 = 0 \ or \ x + 1 = 0$$
$$x = 5 \ or \ x = -1$$

The coordinates of the *x* intercepts are (5, 0) and (-1, 0).

Now we can use all of this information to graph the parabola:



Example 2: Graph $f(x) = -x^2 - 6x - 3$

Solution:

The graph opens downward, since the leading coefficient is negative. The function has a maximum value.

The x coordinate of the vertex is $x = \frac{-(-6)}{2(-1)} = \frac{6}{-2} = -3$.
The y coordinate of the vertex is $f(-3) = -(-3)^2 - 6(-3) + 2 = -9 + 18 - 3 = 6$.
The vertex is the point (-3, 6).

The axis of symmetry is $x = -3$; the maximum value of the function is 6.

We can compute $f(0)$ to find the y intercept: $f(0) = -0^2 - 6(0) - 3 = -3$. The y intercept is the point (0, -3).

We can set the function equal to 0 and solve for x to find the x intercepts.

$$-x^2 - 6x - 3 = 0$$
$$x^2 + 6x + 3 = 0$$

The left hand side does not factor, so we'll use the quadratic formula to find the x intercepts.

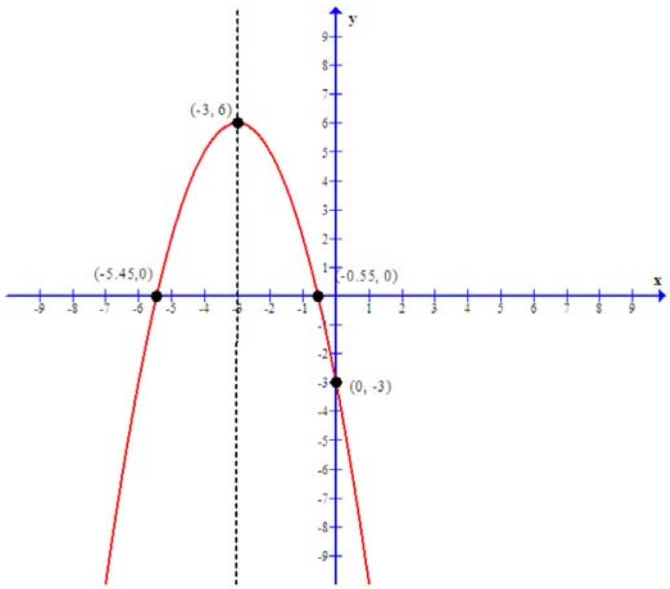
$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{-6 \pm \sqrt{36 - 12}}{2}$$
$$x = \frac{-6 \pm \sqrt{24}}{2}$$
$$x = \frac{-6 \pm \sqrt{4 \cdot 6}}{2}$$
$$x = \frac{-6 \pm 2\sqrt{6}}{2}$$
$$x = \frac{2\left(-3 \pm \sqrt{6}\right)}{2}$$
$$x = -3 \pm \sqrt{6}$$

To assist in graphing, we'll use a calculator to write the x intercepts in decimal form:

$x = -3 + \sqrt{6} \approx -0.55$, $x = -3 - \sqrt{6} \approx -5.45$

The x intercepts are (-0.55, 0) and (-5.45, 0).

Now we can use all of this information to graph the function:



Example 3: Graph $f(x) = 2x^2 + 4x + 5$

Solution:

The graph will open upward since the function has a positive leading coefficient. The function has a minimum value.

The x coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-4}{2(2)} = \frac{-4}{4} = -1$.

We'll find the y coordinate by computing $f(-1)$:

$$f(-1) = 2(-1)^2 + 4(-1) + 5 = 2 \cdot 1 - 4 + 5 = 3$$

The vertex is (-1, 3).

The axis of symmetry is $x = -1$; the minimum value is 3.

We'll compute $f(0)$ to find the y intercept: $f(0) = 2(0)^2 + 4(0) + 5 = 5$. The y intercept is (0, 5).

We'll set the function equal to zero and solve for x to find the x intercepts:

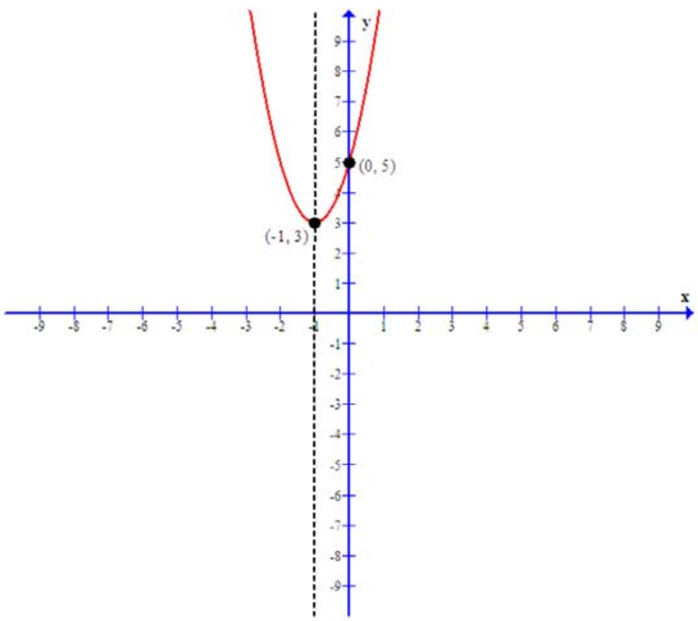
$$2x^2 + 4x + 5 = 0$$

We'll use the quadratic formula to solve this equation.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(5)}}{2(2)}$$
$$x = \frac{-4 \pm \sqrt{16 - 20}}{4}$$
$$x = \frac{-4 \pm \sqrt{-4}}{4}$$

Since we have a negative number under the square root sign, we know that there are no real solutions to this quadratic equation. There are no x intercepts.

We'll use the information we've found to help us graph the function.



Domain and Range of a Quadratic Function

We should also be able to state the domain and range of any quadratic function. The domain of all quadratic functions is $(-\infty, \infty)$.

For a quadratic function with a minimum value, the range will be an interval of the form $[m, \infty)$ where m is the minimum value of the function.

For a quadratic function with a maximum value, the range will be an interval of the form $(-\infty, M]$ where M is the maximum value of the function.

Example 4: Find the domain and range of each of the three functions graphed in examples 1 – 3.

Solution:

For example 1, the domain is $(-\infty, \infty)$. The range is $[-9, \infty)$ since the minimum value is -9.
For example 2, the domain is $(-\infty, \infty)$. The range is $(-\infty, 6]$ since the maximum value is 6.
For example 3, the domain is $(-\infty, \infty)$. The range is $[3, \infty)$ since the minimum value is 3.

Applications

We can solve some application problems using quadratic functions. Often these involve finding a maximum or a minimum value of a function. We'll need to find the vertex of the function to help us solve these problems.

Example 5: A rocket is launched from the top of a building. Its path from launch until it hits the ground will be in the shape of a parabola. The function that gives the rocket's height in feet with respect to time in seconds is $h(t) = -16t^2 + 64t + 125$. When will the rocket reach its maximum height? Find the maximum height that the rocket will reach.

Solution:

We're asked to determine when the rocket reaches its maximum height, and to find the maximum height that the rocket will reach. The graph of the path is a parabola. The leading coefficient of the function is negative, so the graph will open downward. The vertex will be the point (t , $h(t)$). The time at which it reaches its maximum height will be given by the first coordinate of the vertex. The maximum value will be the second coordinate of the vertex.

We'll first find t:

$$t = \frac{-b}{2a}$$
$$t = \frac{-64}{2(-16)} = \frac{-64}{-32} = 2$$

To find the maximum value, we'll compute $h(2)$.

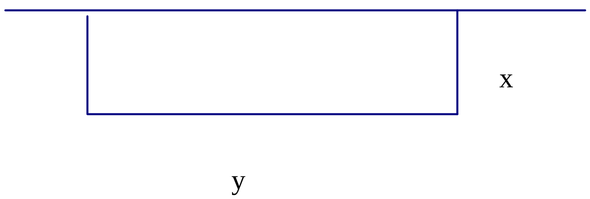
$$h(2) = -16(2)^2 + 64(2) + 125 = 189$$

The rocket reaches its maximum height of 189 feet 2 seconds after launch.

Example 6: A farmer wants to fence in a rectangular region using one side of a building as one side and 300 feet of fencing for the other three sides. The side along the building will not be fenced. Find the dimensions of the fenced-in region that will give the greatest area.

Solution:

We'll start by drawing a picture of the situation:



We'll define some variables: let x = the width of the region and let y = the length of the region.

The area of the region is $A = xy$.

Three sides will be fenced, and we have 300 feet of fencing to use, so we can write $2x + y = 300$. We'll solve this equation for y and then substitute into the area formula:

$$2x + y = 300$$
$$y = 300 - 2x$$
$$A(x) = x(300 - 2x)$$
$$A(x) = 300x - 2x^2$$

The graph of this function will open downward, so we know it will have a maximum value. The maximum value will occur at the vertex. In this problem, we're looking for the dimension (x value) that will generate the maximum area. We need to find the x coordinate of the vertex:

$$x = \frac{-b}{2a}$$
$$x = \frac{-300}{2(-2)} = \frac{-300}{-4} = 75$$

To find the other dimension, we'll use the statement $y = 300 - 2x$ with $x = 75$.

$$y = 300 - 2x$$
$$y = 300 - 2(75) = 300 - 150 = 150$$

The dimensions of the region with maximum area are 75 feet by 150 feet.