Math 2303
Section 2.8
The Properties of Real Number Operations

Formal Definitions
Binary Operation – Given a set S, a binary operation ⊙ is a function that takes every pair of elements a and b in S and defines a unique "value" to a⊙b.

a + b

Example: The usual operation of addition of natural numbers is a binary operation on the set of natural numbers, since for every natural numbers a and b, a+b is a clearly defined natural number.

Example: The usual operation of squaring a number is NOT a binary operation on the set of integers, since it only is performed on ONE number at a time, not two.

$-a$ a^2

Mathematical System
A mathematical system is a set of objects (numbers in this case) together with one or more binary operations defined on the set.

Properties of a Mathematical Systems
Let S be a set with a binary operation called "⊙" defined on it. Then the system consisting of S with the operation ⊙ is called closed if a⊙b is an element of S for every elements a and b in S.

1. operation defined with all combos of elements in set

Example: The set of Natural Numbers with the operation of addition is closed.

2. answer in set

The set of Natural Numbers with the operation of subtraction is not closed, why?

2 - 3 = -1 not Natural #
NO CLOSED

A binary operation ⊙ on a set S is called commutative if for every elements a and b in S, $a \odot b = b \odot a$

3 + 2 = 2 + 3

A binary operation ⊙ on a set S is called associative if for every elements a, b and c in S, $a \odot (b \odot c) = (a \odot b) \odot c$.

(3 + 4) + 2 = 7 + 2 = 9
3 + (4 + 2) = 3 + 6 = 9

Given a set S with a binary operation called "⊙" defined on it. The set S is said to have an identity for the operation ⊙ if there exists a an element, we'll call it "I" for the moment, such that I⊙a=a⊙I=a for all a in S.

I = 0
I + a = a + I = a

Given a set S with a binary operation called "⊙" defined on it, and suppose I is the identity for the operation ⊙. Then, the element a in S is said to have an inverse with respect to ⊙ if there exists an element b in S such that a⊙b = b⊙a = I.

-2 + 2 + 0
3 + -3 = 0
$\frac{3}{4} \times \frac{4}{3} = 1$
$2 \times \frac{1}{2} = 1$

Given a set S with two operations defined on it, call them “+” and “*”. Then S is said to have the distributive property if for all a, b and c in S,
$a*(b+c)=a*b+a*c$ and $(a+b)*c=a*c+b*c$.

The formal definitions of mathematical systems and operations look fairly intimidating to the novice mathematician, but the “translation” into common language should look very familiar to most college students.

Properties of the Real Numbers with the operations of addition and multiplication:

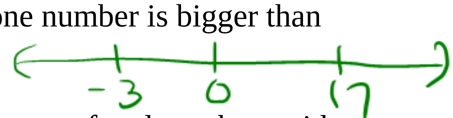
Property	Operation	Example
Associative	Addition	2+(3+5)=(2+3)+5
Associative	Multiplicatio	$3\times(6\times 2)=(3\times 6)\times 2$
Commutative	Addition	3 + 6 = 6 + 3
Commutative	Multiplication	8*7 = 7*8
Distributive	Mult. Over Addition	2(7+5) = 2*7+2*5
Identity for Addition is 0		0 + - 2 = -2
Identity for Multiplication is 1		1*3=3 3*1=3
Additive Inverses	Negative	$\sqrt{2}+-\sqrt{2}=0$
Multiplicative Inverses (except for 0)		$\frac{3}{82}*\frac{82}{3}=1$
Closure	Addition	real + real = real
Closure	Multiplication	real * real = real

Closure (1) Can perform operation with all combination
(2) amount in set

There are two more properties of the real number system that we use all the time, but probably don’t appreciate as we should.

$a \times b = 0$

1. The Zero Product Property: the only way you can multiply two real numbers together and have an answer of 0 is if one of the two numbers is 0.
2. Linear Order: there is an order to the real number system. We can place all real numbers on our number line and we can always determine if one number is bigger than another by which one is farther right on the number line.



These properties make the real number system consisting of the set of real numbers with the operations of addition and multiplication into an ordered field. This is incredibly important to modern mathematics, and the basis for most of the algebra we do every day. We will explore properties of abstract mathematical systems and the field of real numbers in more detail in chapter 7 of this textbook.

Meanwhile, let's see how these properties are used in basic arithmetic.

Examples: For the following arithmetic problems state the property that is involved at each step of the process.

$2(3+4)+(6-(-3))$

$= 2*3+2*4+(6-(-3))$ distribute

$= 6+8+(6-(-3))$ mult fact

$= 6+8+(6+3)$ inverse fact $-(-3) = +3$

$= 6+(8+6)+3$ associative

$= (6+14)+3$ added

$= 20+3 = 23$ add

$8h-3h+(2h-3)-4$

$= 8h-3h+2h+(-3-4)$ associative

$= (8h-3h+2h)+(-3-4)$ associative

$= (8h-3h+2h)+(-7)$ added

$= (8-3+2)h+(-7)$ distributive

$= (8+2-3)h+(-7)$ commutative

$= 7h+(-7)$ add

$= 7h-7$ subtraction is adding a negative

$= 7(h-1)$ distribute

$5x+3x = 8x$

$5x+3x = (5+3)x = 8x$

$\frac{1}{2}(8-4)+2$

$= \frac{1}{2}*8-\frac{1}{2}*4+2$ distribute

$= 4-2+2$ multiplied

$= 4+(-2+2)$ associative

$= 4+0$ −2 and 2 are inverse

$= 4$ 0 is the identity for addition

$2x + 2y = 2(x+y)$

A few extra questions to test our knowledge:

Consider the set of even integers which consists of all integers (positive and negative) whose units digit is even (0,2,4,6,8). Is this set closed under addition? Under multiplication?

yes

yes

$2x \cdot 2y = 4xy$

Is subtraction a commutative operation?

$3 - 7 = -4$

$7 - 3 = 4$

Is division an associative operation?

$(12 \div 4) \div 3 = 3 \div 3 = 1$

$12 \div (4 \div 3) \quad 12 \div \frac{4}{3} = 12 \times \frac{3}{4} = 9$

What is the additive inverse of $\frac{3}{5}$?

$-3/5$

What is the multiplicative inverse of $\frac{3}{5}$? $= 5/3$

What is the additive inverse of -55? $= +55$

What is the multiplicative inverse of -55? $= -\frac{1}{55}$

What is the additive inverse of $\sqrt{7}$? $= -\sqrt{7}$

What is the multiplicative inverse of $\sqrt{7}$? $\frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{7}$

Is 0 an identity for multiplication?

$0 \times 3 = 3$ NOT

Is 1 an identity for addition?

$1 + 4 = 4$

NOT

$1 \times 4 = 4$