MATH 4332

Homework 7
due Monday, March 24

Recall that the Bernoulli numbers B_n satisfy

(a) $B_0 = 1$,

(b) $B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$, $n > 1$,

and that the Bernoulli polynomials satisfy

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \ n \geq 0.$$

Q1.

(a) Show that $B_2 = \frac{1}{6}, B_3 = 0$ and $B_4 = -\frac{1}{30}$.

(b) Show that $B_2(x) = x^2 - x + \frac{1}{6}$.

(c) Show that $B_n(0) = B_n(1) = B_n$, $n \neq 1$. (Note $B_1(0) = -B_1(1) = B_1$!)

Q2. Show that for $n \geq 1$, $B_n'(x) = nB_{n-1}(x)$. Hence, using the results of Q1, find $B_3(x)$ and $B_4(x)$.

Q3. Show that for all $z \in \mathbb{C}$, $\lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$. (You may assume that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$. Some hints: Let $\varepsilon > 0$. Fix z and pick N so that $\sum_{n=N}^{\infty} \frac{|z|^n}{n!} < \varepsilon/2$. Using the binomial expansion of $(1 + \frac{z}{n})^n$, show that for all sufficiently large $n > N$, $|(1 + \frac{z}{n})^n - \sum_{j=0}^{n} \frac{z^j}{j!}| < \varepsilon$. This will require writing $(1 + \frac{z}{n})^n - \sum_{j=0}^{n} \frac{z^j}{j!}$ as a sum of terms from $j = 2$ to $N - 1$ and a sum of terms from N to n. The first sum can be made small by taking n large.)

Q4. Let (p_n) denote the sequence of prime numbers ≥ 1 written in ascending order: $2 = p_2 < p_3 < \ldots$. Let $x > 1$. Show that

(a) $\Pi_{n=1}^{\infty}(1 - \frac{1}{p_n^x})$ converges and, in particular, is non-zero. (You may assume the convergence test given in lectures).

(b) $\Pi_{n=1}^{\infty}(1 - \frac{1}{p_n^x})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^x}$ (the Riemann zeta function).

Deduce that $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is divergent. (For part (b), you will need to consider $\sum_{n \in X_N} \frac{1}{n^x}$, where $X_N \subset \mathbb{N}$ consists of all integers not divisible by prime numbers $> p_N$. That is, numbers in X_N have prime factorization $p_1^{a_1} \ldots p_N^{a_N}$.)