

MATH 4332

Homework 8

due Wednesday, April 16

Q1. Let $\| \cdot \|$ denote the Euclidean norm on \mathbb{R}^n and (x, y) ($= x \cdot y$) denote the standard inner (or dot) product of vectors $x, y \in \mathbb{R}^n$. Define $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\phi(x) = \|x\|^2$.

(a) Show that ϕ is differentiable on \mathbb{R}^n and find $D\phi_x, x \in \mathbb{R}^n$.

(b) Show that $D\phi_x(h) = 0$ iff $h \perp x$ (that is, x is perpendicular to h).

(c) Let u be a unit vector in \mathbb{R}^n . Given x , find u so that $D\phi_x(u)$ is (a) maximal, (b) minimal.

Q2. Let $x \in \mathbb{R}^m$. Define $g : \mathbb{R} \rightarrow \mathbb{R}^m$ by $g(t) = tx$. What is $g'(t) = Dg_t(1)$?

Q3. Let $Q : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable and suppose that $Q(tx) = t^d Q(x)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^n$ (d is assumed to be a strictly positive integer).

(a) Show that $Q(0) = 0$.

(b) $DQ_x(x) = dQ(x)$, for all $x \in \mathbb{R}^n$ (Euler's theorem). (Hint: Apply the chain rule to $Q \circ g : \mathbb{R} \rightarrow \mathbb{R}^m$ where $g(t) = tx$ — see also Q2.)

Q4. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^1 and $f(0) = 0$. Show that $f(x) = \int_0^1 Df_{tx}(x) dt$ and deduce that we can write

$$f(x) = \sum_{i=1}^n x_i g_i(x),$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt$. (Hint: $f(x) = \int_0^1 \frac{d}{dt} f(tx) dt$.) Can you find a formula if, in addition $\frac{\partial f}{\partial x_i}(0) = 0, 1 \leq i \leq n$?

Q5. Find an example of a C^1 map $f : \mathbb{R} \rightarrow \mathbb{R}^2$ such that

$$f(1) - f(0) \neq f'(t), \text{ for all } t \in [0, 1].$$

(Failure of the Mean Value theorem as an *equality* for maps into $\mathbb{R}^n, n > 1$. Note that if the result were true then it would apply to the components and so for $j = 1, 2, f_j(1) - f_j(0) = f'_j(t)$ at the *same* point $t \in (0, 1)$. So it is enough to find two functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ where the equality occurs at *different* points.)