MATH 4332 SPRING 2008

Abel & Dirichlet's tests

ABEL'S TEST
Let $I \subset \mathbb{R}$. Given sequences (a_n), (u_n) of functions defined on I then the series $\sum_{n=1}^{\infty} a_n(x)u_n(x)$ is uniformly convergent on I if

(1) The series $\sum_{n=1}^{\infty} a_n(x)$ is uniformly convergent on I.
(2) $\exists K \geq 0$ such that $0 \leq u_n(x) \leq K$, for all $x \in I$, $n \geq 1$.
(3) $(u_n(x))$ is decreasing for all $x \in I$.

In particular, if $a_n, u_n \in C^0(I)$, $n \geq 1$, then $U(x) = \sum_{n=1}^{\infty} a_n(x)u_n(x)$ is continuous on I.

DIRICHLET'S TEST
Let $I \subset \mathbb{R}$. Given sequences (u_n), (v_n) of functions defined on I then the series $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ is uniformly convergent on I if

(1) $\exists K \geq 0$ such that $|u_1(x) + \ldots + u_n(x)| \leq K$ for all $x \in I$ and $n \geq 1$.
(2) $(v_n(x))$ is decreasing for all $x \in I$.
(3) (v_n) is uniformly convergent to the zero function on I.

In particular, if $u_n, v_n \in C^0(I)$, $n \geq 1$, then $U(x) = \sum_{n=1}^{\infty} u_n(x)v_n(x)$ is continuous on I.

Note that (2,3) imply that $v_n(x) \geq 0$ for all $x \in I$, $n \geq 0$.

REMARK Both tests work when the sequences of functions are constant functions. So these tests apply to infinite series of numbers as well as functions.

The series $\sum_{m=1}^{n} \sin mx$

Provided x is not an integer multiple of 2π, we prove that

$$\sum_{m=1}^{n} \sin mx = \frac{1}{\sin(\frac{x}{2})} \left(\sin(\frac{nx}{2}) \sin(\frac{(n+1)x}{2}) \right).$$

We use the following trigonometric identities

$$1 - \cos A = 2\sin^2(A/2), \tag{1}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}, \tag{2}$$

$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}. \tag{3}$$

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Observe that $\sin A = \mathrm{Im}(e^{\imath A})$. Hence

$$\sum_{m=1}^{n} \sin mx = \mathrm{Im}(\sum_{m=1}^{n} e^{\imath mx}).$$

The geometric series $X + X^2 + \ldots + X^n$ has sum $\frac{X(1-X^n)}{1-X}$ and so, setting $X = e^{\imath mx}$, we have

$$\sum_{m=1}^{n} e^{\imath mx} = \frac{e^{\imath x}(1-e^{\imath nx})}{1-e^{\imath x}}.$$

Multiply numerator and denominator by $(1-e^{-\imath x})$ so as to make the denominator real. We have

$$\begin{aligned}\sum_{m=1}^{n} e^{\imath mx} &= \frac{(e^{\imath x}-1)(1-e^{\imath nx})}{2-e^{\imath x}-e^{-\imath x}}, \\ &= \frac{e^{\imath nx}+e^{\imath x}-e^{\imath(n+1)x}-1}{2-2\cos x}\end{aligned}$$

Now take the imaginary part to get

$$\mathrm{Im}\left(\sum_{m=1}^{n} e^{\imath mx}\right) = \frac{\sin nx + \sin x - \sin(n+1)x}{2-2\cos x}.$$

Using formula (1) we have

$$2 - 2\cos x = 4\sin^2\frac{x}{2}.$$

Using formula (2) we have

$$\sin nx + \sin x = 2\sin\frac{(n+1)x}{2}\cos\frac{(n-1)x}{2}.$$

Since $\sin(n+1)x = 2\sin\frac{(n+1)x}{2}\cos\frac{(n+1)x}{2}$ it follows that

$$\sin nx + \sin x - \sin(n+1)x = 2\sin\frac{(n+1)x}{2}(\cos\frac{(n-1)x}{2} - \cos\frac{(n+1)x}{2}).$$

Now apply (3) to get

$$\cos\frac{(n-1)x}{2} - \cos\frac{(n+1)x}{2} = 2\sin\frac{nx}{2}\sin\frac{x}{2}.$$

Hence

$$\sin nx + \sin x - \sin(n+1)x = 4\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}\sin\frac{x}{2}.$$

Dividing by $2 - 2\cos x = 4\sin^2\frac{x}{2}$, we get

$$\frac{\sin nx + \sin x - \sin(n+1)x}{2-2\cos x} = \frac{1}{\sin(\frac{x}{2})}\left(\sin(\frac{nx}{2})\sin(\frac{(n+1)x}{2})\right).$$

and so

$$\sum_{m=1}^{n} \sin mx = \frac{1}{\sin(\frac{x}{2})}\left(\sin(\frac{nx}{2})\sin(\frac{(n+1)x}{2})\right).$$