MATH 4332 SPRING 2008

Landau's example of a nowhere differentiable function

Define

$$\begin{aligned} u_1(x) &= x,\ 0 \le x \le \frac{1}{2}, \\ &= 1 - x,\ \frac{1}{2} \le x \le 1. \end{aligned}$$

Extend u_1 to $\mathbb{R}$ as a 1-periodic function. That is, if $x \in [n, n+1]$, $n \in \mathbb{Z}$, then $u_1(x) = u_1(x - n)$ (note $x - n \in [0,1]$). For all $m \in \mathbb{Z}$, $x \in \mathbb{R}$ we have

$$u_1(x+m) = u_1(x).$$

For $n > 1$, define

$$u_n(x) = \frac{1}{10^{n-1}} u_1(10^{n-1}x),\ x \in \mathbb{R}.$$

The function u_n is $\frac{1}{10^{n-1}}$-periodic. Indeed, for all $m \in \mathbb{Z}$, $x \in \mathbb{R}$,

$$u_n(x + \frac{m}{10^{n-1}}) = u_n(x).$$

To see this, observe that

$$\begin{aligned} u_n(x + \frac{m}{10^{n-1}}) &= \frac{1}{10^{n-1}} u_1(10^{n-1}(x + \frac{m}{10^{n-1}})) \\ &= \frac{1}{10^{n-1}} u_1(10^{n-1}x + m) \\ &= \frac{1}{10^{n-1}} u_1(10^{n-1}x) \\ &= u_n(x). \end{aligned}$$

Let $\mathbb{Z}^{\frac{1}{2}} = \{\frac{m}{2} \mid m \in \mathbb{Z}\}$. So $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots \in \mathbb{Z}^{\frac{1}{2}}$. Observe that the set of points where u_1 is not differentiable is precisely $\mathbb{Z}^{\frac{1}{2}}$. Elsewhere the derivative of u_1 is ± 1. The set of points where u_n is not differentiable is $\frac{1}{10^{n-1}}\mathbb{Z}^{\frac{1}{2}} = \{\frac{m}{2\times 10^{n-1}} \mid m \in \mathbb{Z}\}$. Elsewhere the derivative of u_n is ± 1.

Define $U : \mathbb{R} \to \mathbb{R}$ by

$$U(x) = \sum_{n=1}^{\infty} u_n(x).$$

Since $|u_n(x)| \le \frac{1}{10^{n-1}}$, it follows by the M-test that $\sum_{n=1}^{\infty} u_n$ is uniformly convergent on $\mathbb{R}$ and that U is continuous. We claim that U is nowhere differentiable. We prove the nowhere differentiability of U by showing that for each $x_0 \in \mathbb{R}$, there exists a sequence (x_N) converging to x_0 such that the limit as $N \to \infty$ of $\frac{U(x_N) - U(x_0)}{x_N - x_0}$ does

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not exist (if U is differentiable at x_0 then $\lim_{N\to\infty} \frac{U(x_N)-U(x_0)}{x_N-x_0} = U'(x_0)$ if $x_N \to x_0$.)

Let $x_0 \in \mathbb{R}$ and $N \geq 1$. Then $x_0 \in [\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}}]$ where $m \in \mathbb{Z}^{\frac{1}{2}}$. Observe that the length of the interval $[\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}}]$ is $\frac{1}{2}\frac{1}{10^{N-1}}$. Certainly either $x_0 - \frac{m}{10^{N-1}} > 10^{-N}$ or $\frac{m+\frac{1}{2}}{10^{N-1}} - x_0 > 10^{-N}$ (as $\frac{1}{2}\frac{1}{10^{N-1}} > \frac{2}{10^N}$). Define $x_N = x_0 \pm \frac{1}{10^N}$ so that $x_N \in [\frac{m}{10^{N-1}}, \frac{m+\frac{1}{2}}{10^{N-1}}]$. This completes the construction of the sequence (x_N).

For $n \leq N$, we have

$$\frac{u_n(x_N) - u_n(x_0)}{x_N - x_0} = \pm 1.$$

(The set of points where u_n is not differentiable is a proper subset of the set of points where u_N is not differentiable if $n < N$.) On the other hand if $n > N$ then $u_n(x_N) = u_n(x_0 \pm \frac{1}{10^N}) = u_n(x_0)$ by the $\frac{1}{10^{n-1}}$ periodicity of u_n ($\frac{1}{10^N}$ is an integer multiple of $\frac{1}{10^{n-1}}$ if $n > N$). Hence if $n > N$,

$$\frac{u_n(x_N) - u_n(x_0)}{x_N - x_0} = 0.$$

We have

$$\begin{aligned} \frac{U(x_N) - U(x_0)}{x_N - x_0} &= \sum_{n=1}^{\infty} \frac{u_n(x_N) - u_n(x_0)}{x_N - x_0}, \\ &= \sum_{n=1}^{N} \frac{u_n(x_N) - u_n(x_0)}{x_N - x_0}, \\ &= \sum_{n=1}^{N} \pm 1, \\ &= Q_N, \end{aligned}$$

where Q_N must be an even integer if N is even and an odd integer if N is odd. Hence the limit of $\frac{U(x_N)-U(x_0)}{x_N-x_0}$ as $N \to \infty$ does not exist and so U cannot be differentiable at x_0.