MATH 4332 SPRING 2008

Notes on the Euler-Maclaurin formula

Bernoulli numbers and Bernoulli polynomials

The Bernoulli numbers $B_0, B_1, \ldots$ are defined by

(a) $B_0 = 1$,

(b) $B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$, $n > 1$,

It follows from (b) that for $n > 1$, $B_n = B_0 + \binom{n}{1} B_1 + \ldots + nB_{n-1} + B_n$ and so

$$B_{n-1} = -\frac{1}{n}\left(B_0 + \binom{n}{1} B_1 + \ldots + \binom{n}{n-2} B_{n-2}\right).$$

We can use this formula to recursively compute the Bernoulli numbers. We find

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}$$

It can be shown that $B_{2n+1} = 0$, $n \geq 1$. One proof uses the identity $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ together with the fact that $\frac{x}{e^x-1} + \frac{x}{2}$ is even (we proved this in lectures). Sometimes, B_1 is taken to be zero — it works better for us to take $B_1 = -\frac{1}{2}$. Appearances to the contrary, $|B_n| \to \infty$ as $n \to \infty$.

The Bernoulli polynomials $B_n(x)$ are defined for $n \geq 0$ by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

Computing, we find that

$$B_0(x) = 1,\ B_1(x) = 1B_1 + 1B_0 x = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6}.$$

LEMMA A
For $n \neq 1$,

$$B_n(0) = B_n(1) = B_n.$$

Note that $B_1(0) = -\frac{1}{2} \neq B_1(1) = \frac{1}{2}$.

LEMMA B

$$B_n'(x) = nB_{n-1}(x),\ n \geq 1.$$

(1) Proofs of Lemma A and B are set as exercises on Homework 7.
(2) We can use Lemma A and B to compute $B_n(x)$ recursively since Lemmas A, B

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imply that $B_n(x) = n\int_0^x B_{n-1}(t)\,dt + B_n$.

The 1-periodic functions $\widetilde{B}_n$

Let $\widetilde{B}_n$ denote the 1-periodic extension of B_n restricted to $[0,1]$ to $\mathbb{R}$.

That is, if $x \in \mathbb{R}$, choose $p \in \mathbb{Z}$ such that $x - p \in [0,1]$ and define

$$\widetilde{B}_n(x) = B_n(x-p).$$

Since $B_n(0) = B_n(1)$ when $n \neq 1$, it is immediate that $\widetilde{B}_n$ is uniquely determined provided $n \neq 1$. When $n = 1$, we need to be careful as $B_1(0) \neq B_1(1)$. What we do is take $\widetilde{B}_1(x) = B_1(x-p)$ if $x \notin \mathbb{Z}$ and define $\widetilde{B}_1(x) = 0$ if x is an integer. The resulting function will then have a jump discontinuity at integer points.

For all $x \in \mathbb{R}$, $p \in \mathbb{Z}$, $n \geq 0$ we have

$$\widetilde{B}_n(x+p) = \widetilde{B}_n(x).$$

(That is, the functions $\widetilde{B}_n$ are all *1-periodic.*)

If $x \in (j, j+1)$, then

$$\widetilde{B}_1(x) = x - j - \frac{1}{2}.$$

LEMMA C

For all $A \geq 1$,

$$\left|\int_1^A \widetilde{B}_1(x)\,dx\right| \leq \frac{1}{8}.$$

Proof. We have $\int_j^{j+1} \widetilde{B}_1(x)\,dx = 0$, for all integers $j \geq 1$. Examination of the graph of $\widetilde{B}_1(x)$ shows that if $j + y \in [j, j+1]$ we maximize $|\int_j^{j+y} \widetilde{B}_1(x)\,dx|$ when $y = \frac{1}{2}$. Computing, we see easily that $\int_j^{j+\frac{1}{2}} \widetilde{B}_1(x)\,dx = \frac{1}{8}$. The result follows since we can write $A = j + y$, $y \in [0,1)$. $\square$

More generally, we also have

LEMMA D

$\int_j^{j+1} \widetilde{B}_n(x)\,dx = 0$, $n \geq 1$, $j \geq 0$.

$|\int_1^A \widetilde{B}_n(x)\,dx| \leq |B_n|$, all $A \geq 1$ if n is even.

Proof. We have

$$\int_j^{j+1} \widetilde{B}_n(x)\,dx = \int_j^{j+1} \frac{d}{dx}\frac{\widetilde{B}_{n+1}(x)}{n+1}\,dx = (\widetilde{B}_{n+1}(j+1) - \widetilde{B}_{n+1}(j))/(n+1) = 0,$$

where the last statement follows by periodicity.

The second statement follows from the first using periodicity and the result $|B_n(x)| \leq |B_n|$ if n even and $x \in [0,1]$ (we do not give a proof of this but it is not hard to verify directly for $n = 2, 4, 6$). □

The Euler-Maclaurin formula

THEOREM
Let n, r be positive integers with $n > 0$ and let $f : [1, \infty) \to \mathbb{R}$ be at least C^{2r+1}. Then

$$\begin{aligned}\int_1^n f(x)\,dx &= \sum_{k=1}^n f(k) - \left(\frac{f(1)+f(n)}{2}\right) \\ &\quad - \sum_{j=1}^r \frac{B_{2j}}{(2j)!}[f^{(2j-1)}(n) - f^{(2j-1)}(1)] \\ &\quad + \frac{1}{(2r)!}\int_1^n \widetilde{B}_{2r}(x) f^{(2r)}(x)\,dx.\end{aligned}$$

Moreover,

$$\frac{1}{(2r)!}\int_1^n \widetilde{B}_{2r}(x) f^{(2r)}(x)\,dx = -\frac{1}{(2r+1)!}\int_1^n \widetilde{B}_{2r+1}(x) f^{(2r+1)}(x)\,dx$$

(1) The interest in this result depends on being able to show that the remainder or error term $\frac{1}{(2r)!}\int_1^n \widetilde{B}_{2r}(x) f^{(2r)}(x)\,dx$ is small for *small* values of n. This is typically the case provided that f is reasonably well behaved — in particular if f is analytic.
(2) If $r = 0$, then the second term is zero and the last term is $-\int_1^n \widetilde{B}_1(x) f^{(1)}(x)\,dx$.
(3) The *proof* of the Euler-Maclaurin formula is rather easy, quite formal and somewhat similar to proofs of Taylor's theorem with integral remainder. Matters get more interesting when one starts to estimate.

Proof of Euler-Maclaurin formula: We proceed by induction on r. Suppose we have established the result for $0 < r \leq R$. We prove for $R+1$. We have

$$\begin{aligned}
\frac{1}{(2R)!}\int_1^n \widetilde{B}_{2R}(x)f^{(2R)}(x)\,dx &= \frac{1}{(2R)!}\int_1^n \frac{1}{2R+1}\widetilde{B}'_{2R+1}(x)f^{(2R)}(x)\,dx \text{ (Lemma B)} \\
&= [\frac{1}{(2R+1)!}\widetilde{B}_{2R+1}(x)f^{(2R)}(x)]_{x=1}^{x=n} \\
&\quad -\frac{1}{(2R+1)!}\int_1^n \widetilde{B}_{2R+1}(x)f^{(2R+1)}(x)\,dx \ (\int \text{ by parts}), \\
&= -\frac{1}{(2R+1)!}\int_1^n \widetilde{B}_{2R+1}(x)f^{(2R+1)}(x)\,dx \ (B_{2R+1}=0,\ R>0) \\
&= -\frac{1}{(2R+2)!}\int_1^n \widetilde{B}'_{2R+2}(x)f^{(2R+1)}(x)\,dx \\
&= -[\frac{1}{(2R+2)!}\widetilde{B}_{2R+2}(x)f^{(2R+1)}(x)]_{x=1}^{x=n} \\
&\quad +\frac{1}{(2R+2)!}\int_1^n \widetilde{B}_{2R+2}(x)f^{(2R+2)}(x)\,dx \\
&= -\frac{B_{2R+2}}{(2R+2)!}[f^{(2R+1)}(n) - f^{(2R+1)}(1)] \\
&\quad +\frac{1}{(2R+2)!}\int_1^n \widetilde{B}_{2R+2}(x)f^{(2R+2)}(x)\,dx.
\end{aligned}$$

This proves the Euler-Maclaurin formula for $r = R+1$. It remains to prove the case $r = 0$. We have

$$\int_1^n f(x)\,dx = \sum_{k=1}^{n-1}\int_k^{k+1} f(x)\,dx.$$

Now

$$\begin{aligned}
\int_k^{k+1} f(x)\,dx &= \int_k^{k+1} f(x)\frac{d}{dx}(x-k-\frac{1}{2})\,dx \\
&= [f(x)(x-k-\frac{1}{2})]_{x=k}^{x=k+1} - \int_k^{k+1}(x-k-\frac{1}{2})f'(x)\,dx, \\
&= \frac{f(k+1)+f(k)}{2} - \int_k^{k+1}\widetilde{B}_1(x)f'(x)\,dx.
\end{aligned}$$

(Note that we can replace $x - k - \frac{1}{2}$ by $\widetilde{B}_1(x)$ without changing the value of the integral — the functions agree except at the points $k, k+1$.) Next we substitute in

$\int_1^n f(x)\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx$ to obtain

$$\begin{aligned}\int_1^n f(x)\,dx &= \frac{f(1)+f(n)+2(f(2)+\ldots+f(n-1))}{2} \\ &\quad - \int_1^n \widetilde{B}_1(x) f'(x)\,dx, \\ &= \sum_{k=1}^{n} f(k) - \left(\frac{f(1)+f(n)}{2}\right) - \int_1^n \widetilde{B}_1(x) f'(x)\,dx\end{aligned}$$

This is the Euler-Maclaurin formula in case $r = 0$.

Finally we show that

$$\frac{1}{(2r)!}\int_1^n \widetilde{B}_{2r}(x) f^{(2r)}(x)\,dx = -\frac{1}{(2r+1)!}\int_1^n \widetilde{B}_{2r+1}(x) f^{(2r+1)}(x)\,dx$$

This uses integration by parts and $\widetilde{B}'_{2r+1}(x) = (2r+1)\widetilde{B}_{2r}(x)$ (except when x is an integer).

$$\begin{aligned}\int_1^n \widetilde{B}_{2r}(x) f^{(2r)}(x)\,dx &= \frac{1}{2r+1}\int_1^n \widetilde{B}'_{2r+1}(x) f^{(2r)}(x)\,dx \\ &= \frac{\widetilde{B}_{2r+1}(x) f^{(2r)}(x)}{2r+1}\Big|_{x=1}^{x=n} - \frac{1}{2r+1}\int_1^n \widetilde{B}_{2r+1}(x) f^{(2r+1)}(x)\,dx \\ &= 0 - \frac{1}{(2r+1)!}\int_1^n \widetilde{B}_{2r+1}(x) f^{(2r+1)}(x)\,dx,\end{aligned}$$

where the last statement uses $\widetilde{B}_{2r+1}(1) = \widetilde{B}_{2r+1}(n) = B_{2r+1} = 0$. □

Stirling's Formula

THEOREM (Stirling's formula, version 1)

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-\delta_n},$$

where $\lim_{n\to\infty} \delta_n = 0$.

Proof. We give a proof based on the Euler-Maclaurin formula. There is an alternative proof in Rudin. The proof we give has the advantage that it extends to give precise information about the error term e^{δ_n} — see below.

We apply the Euler-Maclaurin formula with $f(x) = \log x$ and $r = 0$. We have

$$\int_1^n \log x\,dx = n\log n - n + 1, \quad \sum_{k=1}^{n} \log k = \log n!.$$

Taking $r = 0$ and substituting in the Euler-Maclaurin formula we obtain

$$n \log n - n + 1 = \log n! - \frac{\log n}{2} - \int_1^n \frac{\widetilde{B}_1(x)}{x}\, dx.$$

Hence

$$\begin{aligned}
\log n! &= (n + \frac{1}{2}) \log n - n + 1 + \int_1^n \frac{\widetilde{B}_1(x)}{x}\, dx, \\
&= \log(n^{n+\frac{1}{2}} e^{-n}) + 1 + \int_1^\infty \frac{\widetilde{B}_1(x)}{x}\, dx - \delta_n, \text{ where} \\
\delta_n &= \int_{n+1}^\infty \frac{\widetilde{B}_1(x)}{x}\, dx
\end{aligned}$$

Set $C = 1 + \int_1^\infty \frac{\widetilde{B}_1(x)}{x}\, dx$ so that $\log n! = \log(n^{n+\frac{1}{2}} e^{-n}) + C - \delta_n$. Exponentiate to obtain

$$n! = e^C n^{n+\frac{1}{2}} e^{-n} e^{-\delta_n}.$$

It remains to prove that (a) $e^{-\delta_n} \to 0$ as $n \to \infty$ and (b) $e^C = \sqrt{2\pi}$.

(a) By Lemma C and the 1-periodicity of $\widetilde{B}_1$, we have

$$|\int_{n+1}^A \widetilde{B}_1(x)\, dx| = |\int_1^{A-n} \widetilde{B}_1(x)\, dx| \le 1/8, \text{ for all } A \ge n+1.$$

Set $F(x) = \int_{n+1}^x \widetilde{B}_1(t)\, dt$. Integrating by parts, we have

$$\begin{aligned}
\int_{n+1}^A \frac{\widetilde{B}_1(x)}{x}\, dx &= F(x)/x|_{x=n+1}^A + \int_{n+1}^A F(x)/x^2\, dx, \\
&= F(A)/A + \int_{n+1}^A F(x)/x^2\, dx
\end{aligned}$$

Hence for $A \ge n+1$ we have

$$\begin{aligned}
|\int_{n+1}^A \frac{\widetilde{B}_1(x)}{x}\, dx| &\le \frac{1}{8A} + \int_{n+1}^A |F(x)|/x^2\, dx \\
&\le \frac{1}{8A} + \frac{1}{8} \int_{n+1}^A \frac{1}{x^2}\, dx \\
&= \frac{1}{8A} + \frac{1}{8} [-\frac{1}{x}]_{x=n+1}^A, \\
&= \frac{1}{8(n+1)}.
\end{aligned}$$

Since this estimate holds for all $A \ge n+1$, we have shown that $|\delta_n| \le \frac{1}{8(n+1)}$. Hence $\lim_{n\to\infty} \delta_n = 0$, proving (a).

(b) We recall Wallis' formula: $\lim_{n\to\infty} \frac{4^{2n}(n!)^4}{[(2n)!]^2(2n+1)} = \frac{\pi}{2}$. Taking square roots and the reciprocal, we have

$$\lim_{n\to\infty} \frac{(2n)!\sqrt{2n+1}}{4^n(n!)^2} = \sqrt{\frac{2}{\pi}}.$$

Substituting our expressions for $n!$ and $(2n)!$ in $\frac{(2n)!\sqrt{2n+1}}{4^n(n!)^2}$, we have

$$\begin{aligned}\frac{(2n)!\sqrt{2n+1}}{4^n(n!)^2} &= \frac{e^C(2n)^{2n+\frac{1}{2}}e^{-2n}e^{-\delta_{2n}}\sqrt{2n+1}}{2^{2n}[e^C n^{n+\frac{1}{2}}e^{-n}e^{-\delta_n}]^2} \\ &= e^{-C}\frac{\sqrt{2}}{\sqrt{n}}\sqrt{2n+1}e^{-\delta_{2n}-2\delta_n}.\end{aligned}$$

Since $\lim_{n\to\infty}\delta_n = 0$, we have

$$\sqrt{\frac{2}{\pi}} = \lim_{n\to\infty} e^{-C}\frac{\sqrt{2}}{\sqrt{n}}\sqrt{2n+1}e^{-\delta_{2n}-2\delta_n} = 2e^{-C}.$$

Hence $e^C = \sqrt{2\pi}$. □

If we apply the Euler-Maclaurin formula with $r > 0$, we can obtain more precise estimates of $n!$. For example, we have

THEOREM (Stirling's formula, version 2)

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n} \le n! \le \sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}},\ n \ge 1$$

Proof. It follows from the Euler-Maclaurin formula with $f(x) = \log x$ that

$$\log\left(\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}\right) = \sum_{j=1}^{r}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{n^{2j-1}} - 1\right] + \frac{1}{2r}\int_1^n \frac{\widetilde{B}_{2r}(x)}{x^{2r}}\,dx.$$

Let $n \to \infty$ and we get (using Stirling's formula)

$$\log(\sqrt{2\pi}) = -\sum_{j=1}^{r}\frac{B_{2j}}{2j(2j-1)} + \frac{1}{2r}\int_1^\infty \frac{\widetilde{B}_{2r}(x)}{x^{2r}}\,dx$$

Subtract this from our expression for $\log\left(\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}\right)$ to get

$$\log\left(\frac{n!}{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}}\right) = \sum_{j=1}^{r}\frac{B_{2j}}{2j(2j-1)}\frac{1}{n^{2j-1}} - \frac{1}{2r}\int_n^\infty \frac{\widetilde{B}_{2r}(x)}{x^{2r}}\,dx.$$

Take $r = 1$. The right hand side equals $\frac{1}{12n} - \frac{1}{2}\int_n^\infty \frac{\widetilde{B}_2(x)}{x^2}\,dx$. We have

$$\begin{aligned}
\frac{1}{2}\int_n^\infty \frac{\widetilde{B}_2(x)}{x^2}\,dx &= \frac{1}{2}\int_n^\infty \frac{1}{x^2}\frac{d}{dx}(\frac{\widetilde{B}_3(x)}{3})\,dx \\
&= -\frac{1}{6}\int_n^\infty \widetilde{B}_3(x)\frac{d}{dx}(\frac{1}{x^2})\,dx \\
&= \frac{1}{3}\int_n^\infty \frac{\widetilde{B}_3(x)}{x^3}\,dx
\end{aligned}$$

Using the 1-periodicity of $\widetilde{B}_3$ we can show that $0 \le \int_n^\infty \frac{\widetilde{B}_3(x)}{x^3}\,dx \le 1/12n$ and so

$$0 \le \log\left(\frac{n!}{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}}\right) \le \frac{1}{12n},\ n \ge 1.$$

Exponentiating, we get the result. □

Computing Euler's constant

Take $f(x) = \frac{1}{x}$ and $r = 2$ in the Euler-Maclaurin formula. We have $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$, $f^{(4)}(x) = -\frac{24}{x^5}$, and $\int_1^n \frac{dx}{x} = \log n$. Substituting, we get

$$\begin{aligned}
\log n &= \sum_{k=1}^n \frac{1}{k} - \frac{1+\frac{1}{n}}{2} - \frac{\frac{1}{6}}{2}\left[(-\frac{1}{n^2}) - (-\frac{1}{1^2})\right] \\
&\quad -\frac{(-\frac{1}{30})}{24}\left[(\frac{-6}{n^4} - (\frac{-6}{1^4})\right] \\
&\quad +\frac{1}{4!}\int_1^n \widetilde{B}_4(x)\frac{4!}{x^5}\,dx.
\end{aligned}$$

After some simplifying, this gives

$$\log n = \sum_{k=1}^n \frac{1}{k} - \frac{1}{2} - \frac{1}{12} + \frac{1}{120} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \int_1^n \frac{\widetilde{B}_4(x)}{x^5}\,dx,$$

and so

$$\sum_{k=1}^n \frac{1}{k} - \log n = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \int_1^n \frac{\widetilde{B}_4(x)}{x^5}\,dx,$$

Letting $n \to \infty$, we get

$$\gamma = \frac{1}{2} + \frac{1}{12} - \frac{1}{120} - \int_1^\infty \frac{\widetilde{B}_4(x)}{x^5}\,dx.$$

Since $\int_1^n = \int_1^\infty - \int_n^\infty$, this gives us

$$\sum_{k=1}^n \frac{1}{k} - \log n = \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \int_n^\infty \frac{\widetilde{B}_4(x)}{x^5}\,dx.$$

and so we obtain an asymptotic formula for Euler's constant:

$$\gamma = \sum_{k=1}^{n} \frac{1}{k} - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} - \int_n^\infty \frac{\widetilde{B}_4(x)}{x^5}\,dx.$$

If we take n=10, we find that

$$\begin{aligned} \log 10 &= 2.302585092994 \\ 1/20 &= 0.05 \\ 1/1200 &= 0.0008\dot{3} \\ 1/1200000 &= 0.00000083\dot{3} \end{aligned}$$

From this, we get

$$\sum_{k=1}^{10} \frac{1}{k} - \log 10 - \frac{1}{20} + \frac{1}{1200} - \frac{1}{12000000} = 0.577215660974\ldots$$

In fact the true value of γ is $\gamma = 0.577215664901\ldots$ so our estimate is accurate to 8 decimal places. We can verify this by estimating the error term $\int_{10}^\infty \frac{\widetilde{B}_4(x)}{x^5}\,dx$.

We start by recalling from lectures a lemma on improper integrals.

LEMMA E
Suppose $f(x) = g(x)h(x)$, where g is continuous and h is C^1. Suppose that (a) $h(x)$ is decreasing and converges to zero as $x \to \infty$, (b) $|\int_a^A g(x)\,dx| \le M$ for all $A \ge a$. Then $\int_a^\infty f(x)\,dx$ exists and

$$|\int_a^\infty f(x)\,dx| \le Mh(a).$$

Proof. We assume the existence (see lecture notes) and just verify the estimate. Let $A \ge a$. Set $G(x) = \int_a^x g(t)\,dt$. Integrating by parts we have

$$\int_a^A f(x)\,dx = G(x)h(x)|_{x=a}^{x=A} - \int_a^A G(x)h'(x)\,dx = G(A)h(A) - \int_a^A G(x)h'(x)\,dx.$$

We have to estimate both terms in this equation. Since $h'(x) \le 0$, we have

$$|G(A)h(A)| \le M(-h(A)).$$

Again using $h'(x) \le 0$, we have

$$|-\int_a^A G(x)h'(x)\,dx| = \int_a^A G(x)(-h'(x))\,dx| \le M\int_a^A -h'(x)\,dx = M(h(a) - h(A)).$$

Therefore

$$|\int_a^A f(x)\,dx| \le M(-h(A)) + M(h(a) - h(A)) = Mh(a).$$

Letting $A \to \infty$ the result follows. $\square$

We start with an easy crude estimate.

$$\begin{aligned}
|\int_{10}^{\infty}\frac{\widetilde{B}_4(x)}{x^5}\,dx| &\leq \int_{10}^{\infty}\frac{|B_4|}{x^5} \quad \text{(Lemma D)}\\
&= \frac{1}{30}\int_{10}^{\infty}\frac{1}{x^5}\,dx\\
&= \frac{1}{30}\frac{x^{-4}}{-4}|_{10}^{\infty}\\
&= \frac{1}{120}\times 10^{-4}\\
&\leq 10^{-6}
\end{aligned}$$

Not bad, but we can do better (again using Lemma D). Using the second form of the error term in the Euler-Maclaurin formula we find

$$\int_{10}^{\infty}\frac{\widetilde{B}_4(x)}{x^5}\,dx = \int_{10}^{\infty}\frac{\widetilde{B}_5(x)}{x^6}\,dx.$$

It follows from Lemmas D, E that

$$|\int_{10}^{\infty}\frac{\widetilde{B}_5(x)}{x^6}\,dx| \leq 10^{-6}\times\sup_{A\geq 1}|\int_1^A \widetilde{B}_5(x)\,dx|.$$

But

$$\begin{aligned}
|\int_1^A \widetilde{B}_5(x)\,dx| &= |\int_1^A \frac{d}{dx}\frac{\widetilde{B}_6(x)}{6}\,dx|\\
&= |\frac{\widetilde{B}_6(A)-B_6}{6}|\\
&\leq \frac{2B_6}{6}, \quad \text{(Lemma D)}\\
&= \frac{1}{3\times 42} \quad (B_6 = \frac{1}{42}).
\end{aligned}$$

Hence $|\int_{10}^{\infty}\frac{\widetilde{B}_4(x)}{x^5}\,dx| \leq \frac{1}{126}\times 10^{-6} < 10^{-8}$.

Estimating $\sum_{k=1}^{\infty}\frac{1}{k^2}$

This time we apply Euler-Maclaurin to $f(x) = 1/x^2$ and take $r = 1$. We have $f'(x) = -\frac{2}{x^3}$, $f''(x) = \frac{6}{x^4}$, and $\int_1^n \frac{dx}{x^2} = 1 - \frac{1}{n}$. Substituting, we get

$$1 - \frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k^2} - \frac{1+\frac{1}{n^2}}{2} - B_2\left[(-\frac{2}{n^3}) - (-\frac{2}{1^3})\right] + 3\int_1^n \frac{\widetilde{B}_2(x)}{x^4}\,dx.$$

Taking $B_2 = 1/6$, we get

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{11}{6} - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{3n^3} - 3\int_1^n \frac{\widetilde{B}_2(x)}{x^4}\,dx.$$

Letting $n \to \infty$, this gives

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{11}{6} - 3\int_1^\infty \frac{\widetilde{B}_2(x)}{x^4}\,dx.$$

Writing $\int_1^n = \int_1^\infty - \int_n^\infty$ and substituting, we get an asymptotic formula for $\sum_{k=1}^\infty \frac{1}{k^2}$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - 3\int_n^\infty \frac{\widetilde{B}_2(x)}{x^4}\,dx.$$

As you can check using the known value $\sum_{k=1}^\infty \frac{1}{k^2} = \pi^2/6$, this gives an estimate accurate to 4 decimal places if we take $n = 10$. We can do much better if we take $r = 2 \ldots$