DYNAMICAL EQUIVALENCE OF NETWORKS OF COUPLED DYNAMICAL SYSTEMS: SYMMETRIC INPUTS

N AGARWAL AND M FIELD

Abstract. We show that two networks of coupled dynamical systems are dynamically equivalent if and only if they are output equivalent. We also obtain necessary and sufficient conditions for two dynamically equivalent networks to be input equivalent. These results were previously described in the companion paper 'Dynamical equivalence of networks of coupled dynamical systems' but only proved there for the case of asymmetric inputs. In this paper, we allow for symmetric inputs. We also provide a number of examples to illustrate the main results in the case when there are both symmetric and asymmetric inputs.

1. Introduction

In this work we provide the proofs of two general results on equivalence of networks of coupled dynamical systems that were stated in the companion paper [1]. In what follows we assume some familiarity with the notational conventions of [1] and, in particular, with the definitions of dynamical equivalence and input and output equivalence. (We review the definitions for symmetric inputs in sections 2 and 3. See also [1, 2, 4] for general background and results on coupled dynamical systems.) We prove two basic results. In section 2, we show that networks $\mathcal{M}$ and $\mathcal{N}$ are dynamically equivalent if and only if they are output equivalent. In particular, if $\mathcal{M}$ and $\mathcal{N}$ both have n identical cells, then we have output equivalence of $\mathcal{M}$ and $\mathcal{N}$ if and only if we can order the cells of $\mathcal{M}$ and $\mathcal{N}$ so that the adjacency matrices of $\mathcal{M}$ and $\mathcal{N}$ span the same linear subspace of $M(n, n; \mathbb{Q})$. In section 3 we give necessary and sufficient conditions for the input equivalence of two dynamically equivalent networks. We recall from [1] that dynamically equivalent networks with asymmetric inputs are always input equivalent. This is not the case when there are symmetric inputs and we

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provide, in sections 3 and 4, simple examples of dynamically equivalent networks $\mathcal{M}$, $\mathcal{N}$ that are not input equivalent (for the example in section 4, $\mathcal{M}$ is not input dominated by $\mathcal{N}$ and $\mathcal{N}$ is not input dominated by $\mathcal{M}$ — see [1, §3.1] and section 3 for the formal definition of input domination). As a corollary of our proofs, we obtain algorithms for moving from one network architecture to an input or output equivalent architecture so that each system in the second architecture is expressed in terms of cells from the first architecture and conversely. We illustrate these algorithms, as well as instances of the input and output equivalence theorems, in section 4. Finally, we remark that although all the results are stated for the case of identical cell networks, the extension to networks with more than one class of cell is routine.

2. Output Equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be coupled n identical cell networks. Denote the cells of $\mathcal{N}$ by $D_1, \cdots, D_n$ (this fixes an ordering of the cells). Suppose cells in $\mathcal{N}$ have s inputs and q input types and let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in \mathcal{M}_{s_i}(n;\mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be a connection matrix for $\mathcal{N}$ and recall from [1, §3.1] that for $j \in \mathbf{n}$, $\mathfrak{n}^j = (\mathfrak{n}^j_1, \cdots, \mathfrak{n}^j_q)$, $\mathfrak{n}^j_i = (\mathfrak{n}^j_{i1}, \cdots, \mathfrak{n}^j_{is_i})$, $i \in \mathbf{q}$, where $\mathfrak{n}^j_{i\ell} \in \mathbf{n}$ is specified by the requirement that there is an output from $C_{\mathfrak{n}^j_{i\ell}}$ to the ℓth input of type i of C_j. In this section we always assume that $\mathfrak{n}$ is the default connection matrix [1, §2.2] and so the vectors $\mathfrak{n}^j_i$ are uniquely determined by the condition $\mathfrak{n}^j_{i\ell} \leq \mathfrak{n}^j_{i\ell'}$ if $\ell \leq \ell'$. We suppose cells in $\mathcal{M}$ have r inputs and p input types with r_i inputs of type i, for $i \in \mathbf{p}$. Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in \mathcal{M}_{r_i}(n;\mathbb{Z}^+),\ i \in \mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Denote the associated (default) connection matrix of $\mathcal{M}$ by $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ where the components of $\mathfrak{m}$ are defined exactly as for $\mathfrak{n}$.

We now formalize the concepts of output dominance and output equivalence for networks with symmetric inputs.

Let $G_{\mathcal{N}} = \prod_{i=0}^q S_{s_i}$, where S_{s_i} denotes the symmetric group on s_i symbols and we have taken $s_0 = 1$ (so that $S_{s_0} = S_1$ is the trivial group consisting of the identity). We define $G_{\mathcal{M}} = \prod_{i=0}^q S_{r_i}$, where $r_0 = 1$.

We take the natural action of $G_{\mathcal{N}}$ on $\bar{\mathbf{s}}$ (we regard $\mathbf{s}$ as identified with $\{\mathbf{s}_1, \cdots, \mathbf{s}_q\}$). Let $\mathbf{A}(r, s)$ denote the set of all maps $\gamma : \{1, \cdots, r\} = \mathbf{r} \to \{0, 1, \cdots, s\} = \bar{\mathbf{s}}$. We have natural left and right actions of $G_{\mathcal{N}}$ and

$G_{\mathcal{M}}$ on $\mathbf{A}(r,s)$ defined by

$$\begin{aligned} \gamma &\mapsto \sigma\gamma, \quad && \gamma \in \mathbf{A}(r,s),\ \sigma \in G_{\mathcal{N}}, \\ \gamma &\mapsto \gamma\beta, \quad && \gamma \in \mathbf{A}(r,s),\ \beta \in G_{\mathcal{M}}. \end{aligned}$$

A map $C : \mathbf{A}(r,s) \to \mathbb{Q}$ will be $G_{\mathcal{N}}$-*invariant* if $C(\gamma) = C(\sigma\gamma)$ for all $\sigma \in G_{\mathcal{N}}$.

Let M be a smooth manifold. We write points $\mathbf{X} \in M \times \prod_{i=1}^{p} M^{r_i}$ in the form $\mathbf{X} = (\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_p)$, where $\mathbf{X}_i = (x_1^i, \cdots, x_{r_i}^i)$, $i \in \mathbf{p}$. We often write x_0 rather than $\mathbf{X}_0$ as the variable belongs to a single factor rather than a product of factors. We use similar notation for points in $M \times \prod_{i=1}^{q} M^{s_i}$. Given $j \in \mathbf{n}$, $i \in \mathbf{p}$, we let $\mathbf{X}_{\mathfrak{m}_i^j} \in M^{r_i}$ be the variables defined by the connection vector $\mathfrak{m}^j$. We similarly define $\mathbf{X}_{\mathfrak{n}_i^j} \in M^{s_i}$ for $i \in \mathbf{q}$.

Let $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$ be a family of $G_{\mathcal{M}}$-invariant vector fields on the smooth manifold M. For $\gamma \in \mathbf{A}(r,s)$, define $f_\gamma : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$f_\gamma(x_0; x_1, \cdots, x_s) = f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}),$$

where $(x_0; x_1, \cdots, x_s) \in M \times \prod_{i=1}^{q} M^{s_i}$.

Definition 2.1. (Notation and assumptions as above.) Suppose that $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$ is $G_{\mathcal{M}}$-invariant, $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$, and and $C : \mathbf{A}(r,s) \to \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. We say that f is $(C, \mathfrak{m}, \mathfrak{n})$-*output dominated by* g, written $f <^{O}_{(C,\mathfrak{m},\mathfrak{n})} g$, if

(1) $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.
(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

Remark 2.2. Since C is $G_{\mathcal{N}}$-invariant, $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$ is automatically $G_{\mathcal{N}}$-invariant, even if f is not $G_{\mathcal{M}}$-invariant. We use this remark below to obtain a useful simplification of the formula $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.

Lemma 2.3. *(Notation and assumptions as above.) If f is $G_{\mathcal{M}}$-invariant, then $f_\gamma = f_{\gamma\beta}$ for all $\beta \in G_{\mathcal{M}}$.*

Proof. The model f is $G_{\mathcal{M}}$-invariant and so we have $f(x_0; x_1, \cdots, x_r) = f(x_0; x_{\beta(1)}, \cdots, x_{\beta(r)})$ for all $\beta \in G_{\mathcal{M}}$. Hence, if $\beta \in G_{\mathcal{M}}$, $\gamma \in \mathbf{A}(r,s)$, we have

$$\begin{aligned} f_\gamma(x_0; x_1, \cdots, x_s) &= f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}), \\ &= f(x_0; x_{\gamma\beta(1)}, \cdots, x_{\gamma\beta(r)}), \\ &= f_{\gamma\beta}(x_0; x_1, \cdots, x_s). \end{aligned}$$

Therefore $f_\gamma = f_{\gamma\beta}$. □

Let $\tilde{\mathbf{A}}(r,s) = \mathbf{A}(r,s)/G_{\mathcal{M}}$ denote the orbit space of $\mathbf{A}(r,s)$ under the right action by $G_{\mathcal{M}}$. Since the actions of $G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ on $\mathbf{A}(r,s)$ commute, the $G_{\mathcal{N}}$-action on $\mathbf{A}(r,s)$ induces a (left) $G_{\mathcal{N}}$-action on $\tilde{\mathbf{A}}(r,s)$. Although a $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ will not generally induce a map on $\tilde{\mathbf{A}}(r,s)$, we do have a trivial converse.

Lemma 2.4. *(Notation and assumptions as above.) If $\tilde{C} : \tilde{\mathbf{A}}(r,s) \rightarrow \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant, then $\tilde{C}$ lifts to a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map*

$$\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}.$$

We regard the orbit space $\mathbf{A}(r,s)/G_{\mathcal{M}}$ as the set of group orbits for the $G_{\mathcal{M}}$-action on $\mathbf{A}(r,s)$. It is convenient to fix a subset $R = \{\gamma \in \mathbf{A}(r,s)\}$ such that the $\{G_{\mathcal{M}}\gamma \mid \gamma \in R\}$ partitions $\mathbf{A}(r,s)$. That is, $\cup_{\gamma \in R} G_{\mathcal{M}}\gamma = \mathbf{A}(r,s)$ and $G_{\mathcal{M}}\gamma \cap G_{\mathcal{M}}\nu \neq \emptyset$ iff $\gamma = \nu$.

Lemma 2.5. *(Notation as above.) Suppose that f is $G_{\mathcal{M}}$-invariant and $C : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. Then there exists a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map $\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ such that*

$$\sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma = \sum_{\gamma \in R} \hat{C}(\gamma) f_\gamma.$$

Proof. We have

$$\sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma = \sum_{\gamma \in R} \left(\sum_{\tau \in G_{\mathcal{M}}\gamma} C(\tau) f_\tau \right).$$

By lemma 2.3, $f_\tau = f_\nu$ for all $\tau, \nu \in G_{\mathcal{M}}\gamma$. Letting $[\gamma] \in \tilde{\mathbf{A}}(r,s)$ denote the coset defined by γ, we define $\tilde{C}([\gamma]) = \sum_{\tau \in G_{\mathcal{M}}\gamma} C(\tau)$, $\gamma \in R$. This defines a $G_{\mathcal{N}}$-invariant map $\tilde{C} : \tilde{\mathbf{A}}(r,s) \rightarrow \mathbb{Q}$. Let $\hat{C} : \mathbf{A}(r,s) \rightarrow \mathbb{Q}$ be the $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant lift given by lemma 2.4. □

Example 2.6. Let the single input type networks $\mathcal{M}$ and $\mathcal{N}$ have respective adjacency matrices $M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. We have $M_1 = I + N_1$. If $\mathcal{F} \in \mathcal{M}$ has model f and we define

$$g(x_0; x_1, x_2) = f(x_0; x_0, x_1, x_2), \tag{2.1}$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with identical dynamics to $\mathcal{F}$. In this case, $G_{\mathcal{N}} = \langle \sigma \rangle = S_2$, where $\sigma(x_1, x_2) = (x_2, x_1)$. Obviously, $g(x_0; \sigma(x_1, x_2)) = f(x_0; x_0, x_2, x_1) = f(x_0; x_0, x_1, x_2)$ and so g is $G_{\mathcal{N}}$-invariant. Following definition 2.1, we may also define g by

$$g(x_0; x_1, x_2) = \frac{1}{2}(f(x_0; x_0, x_1, x_2) + f(x_0; x_0, x_2, x_1)).$$

Since f is $G_{\mathcal{M}}$-invariant, this expression for g is equal to that of (2.1).

Definition 2.7. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks such that

(a) $n(\mathcal{M}) = n(\mathcal{N}) = n$.
(b) Cells in $\mathcal{M}$ have p input types, cells in $\mathcal{N}$ have q input types.
(c) If we fix an ordering of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$.

We write $\mathcal{M} \prec_O \mathcal{N}$ and say $\mathcal{M}$ *is output dominated by* $\mathcal{N}$, if there exist an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, and a $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r,s) \to \mathbb{Q}$, such that for every $\mathcal{F} \in \mathcal{M}$, there exists $\mathcal{G} \in \mathcal{N}$ for which $f_{\mathcal{F}} <^O_{(C,\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. (Recall $\mathcal{F}$ is modelled by $f_{\mathcal{F}}$, and $\mathcal{G}$ is modelled by $g_{\mathcal{G}}$.) If $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{N} \prec_O \mathcal{M}$, we say $\mathcal{N}$ and $\mathcal{M}$ are output equivalent and write $\mathcal{M} \sim_O \mathcal{N}$.

Before we give the main result of this section, we state and prove a useful result about output domination (an analogous result holds for input domination). We continue with our assumptions on $\mathcal{M}$ and $\mathcal{N}$ and assume that we have fixed an ordering of the cells in $\mathcal{N}$. Given an ordering of the cells in $\mathcal{M}$, denote the associated set of adjacency matrices by $M_0, M_1, \cdots, M_p$. For $j \in \mathbf{p}$, Let $\mathcal{M}_j$ denote the n-cell network with 1 input type and adjacency matrices $\{M_0, M_j\}$. Denote the connection matrix associated to $\{M_0, M_j\}$ by $\mathfrak{m}_j$.

Lemma 2.8. *(Notation and assumptions as above). The following conditions are equivalent.*

(1) $\mathcal{M} \prec_O \mathcal{N}$.
(2) *There exists an ordering of the cells in* $\mathcal{M}$ *such that* $\mathcal{M}_j \prec_O \mathcal{N}$, *for all* $j \in \mathbf{p}$.

Proof. Suppose first that $\mathcal{M} \prec_O \mathcal{N}$. By definition of output domination, we have an associated ordering of the cells of $\mathcal{M}$, connection matrix $\mathfrak{m}$ and $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r,s) \to \mathbb{Q}$. If $\mathcal{F} \in \mathcal{M}$ has model f, there exists $\mathcal{G} \in \mathcal{N}$ with model g such that

(1) $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.
(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}^j_1}, \cdots, \mathbf{X}_{\mathfrak{n}^j_q}) = f(x_j; \mathbf{X}_{\mathfrak{m}^j_1}, \cdots, \mathbf{X}_{\mathfrak{m}^j_p})$.

Now suppose that f depends only on the variables $(x_0, \mathbf{X}_j) \in V \times V^{p_j}$. Then the associated system can be identified with a system in $\mathcal{M}_j$. The input matching condition (2) implies trivially that we have the correct input matching for the connection matrix $\mathfrak{m}_j$ of $\mathcal{M}_j$. Hence $\mathcal{M}_j \prec_O \mathcal{N}$. Conversely, suppose that there exists an ordering of the cells in $\mathcal{M}$ such that $\mathcal{M}_j \prec_O \mathcal{N}$, for all $j \in \mathbf{p}$. For each $j \in \mathbf{p}$, there exists a $G_{\mathcal{N}}$-invariant map $C_j : \mathbf{A}(r_j, s) \to \mathbb{Q}$ such that if f^j is the model for $\mathcal{F}_j \in \mathcal{M}_j$, there exists $\mathcal{G}_j \in \mathcal{N}$ with model g^j such that

$$g^j = \sum_{\gamma \in \mathbf{A}(r_j,s)} C_j(\gamma) f^j_\gamma.$$

and the input matching conditions hold (with $\mathfrak{m}$ replaced by $\mathfrak{m}_j$). Now suppose $\mathcal{F} \in \mathcal{M}$ has model f. We define g by

$$g = \sum_{\gamma_1 \in \mathbf{A}(r_1,s)} \cdots \sum_{\gamma_p \in \mathbf{A}(r_p,s)} C_1(\gamma_1) \cdots C_p(\gamma_p) f_{\gamma_1 \cdots \gamma_p}.$$

where we define $f_{\gamma_1 \cdots \gamma_p}$ by making the natural identification between $\prod_{j=1}^p \mathbf{A}(r_j, s)$ and $\mathbf{A}(r, s)$ (that is, using the identification of $\mathbf{r}$ and $\{\mathbf{r}_1, \cdots, \mathbf{r}_p\}$). It is straightforward to verify that g does define a system $\mathcal{G} \in \mathcal{N}$ which satisfies the input matching conditions (2). □

Theorem 2.9. *(Notation as above.)* $\mathcal{M} \sim_O \mathcal{N}$ *iff* $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ *iff* $\mathcal{M} \sim \mathcal{N}$.

In order to prove theorem 2.9 it suffices to show that

(A) $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N}) \Longrightarrow \mathcal{M} \prec_O \mathcal{N}$.
(B) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathcal{M} \prec \mathcal{N}$.
(C) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.
(D) $\mathcal{M} \prec \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.

Statement (B) is trivial. We prove (C,D) by reducing to the case of linear vector fields. Most of the work involves the proof of (A) and we start with the proof of (A) and conclude with the proofs of (C,D).

We break the proof of (A) into a number of lemmas. These lemmas also give an algorithm for computing an explicit output equivalence or dominance. Throughout we assume that $\mathcal{M}$, $\mathcal{N}$ are identical cell networks and follow our established notational conventions. In particular, we assume given orderings of the cells of $\mathcal{M}$, $\mathcal{N}$ and associated adjacency and connection matrices and the inclusion $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.

Lemma 2.10. *If* $p = q$, *and* $M_i = N_i$, $i \notin \{a, b\}$, $N_a = M_b$, $N_b = M_a$ *then* $\mathcal{M} \prec_O \mathcal{N}$.

Proof. If $a = b$, there is nothing to prove. Suppose without loss of generality that $a < b$. We have $r_i = s_i, i \in \mathbf{p} \setminus \{a, b\}$, $r_a = s_b, r_b = s_a$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^p M^{r_i} \rightarrow TM$. Define $g : M \times \prod_{i=1}^p M^{s_i} \rightarrow TM$ by

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_p) \\ = f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_p).$$

It is easy to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. □

Remark 2.11. As a consequence of lemma 2.10, we see that if the adjacency matrices of $\mathcal{M}$ are a permutation of those of $\mathcal{N}$, then $\mathcal{M} \sim_O \mathcal{N}$.

Lemma 2.12. *Let $p = 2$, and $M_1 = \sum_{i\in A} \alpha_i N_i$, $M_2 = \sum_{j\in B} \epsilon_j N_j$, where $A, B \subset \overline{\mathbf{q}}$, and $\alpha_i, \epsilon_j \in \mathbb{Z}^+$, $i \in A, j \in B$. Then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Suppose that $A = \{a_1, \cdots, a_u\}, B = \{b_1, \cdots, b_w\} \subset \overline{\mathbf{q}}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{2} M^{r_i} \to TM$. Define $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$g(x_0; \mathbf{X}_1, .., \mathbf{X}_k) = f(x_0; \overline{\mathbf{X}_{a_1}^{\alpha_1}, \cdots, \mathbf{X}_{a_u}^{\alpha_u}}, \overline{\mathbf{X}_{b_1}^{\epsilon_1}, \cdots, \mathbf{X}_{b_w}^{\epsilon_w}}),$$

where $\mathbf{X}_i \in M^{s_i}$ (variables corresponding to inputs of type i, $i \in \mathbf{q}$) and $\mathbf{X}_i^{\alpha}$ denotes $\mathbf{X}_i$ repeated α times. It is straightforward to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. □

Lemma 2.13. *Let $p = 1$ and suppose $M_1 = N_1 - N_2$ then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Set $r_1 = r$, $s_2 = \tilde{s}$ so that $s_1 = r + \tilde{s}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^r \to TM$. Set $\mathbf{Z} = (\mathbf{X}_3, \cdots, \mathbf{X}_q) \in \prod_{i=3}^{q} M^{s_i}$ (the variables represented by $\mathbf{Z}$ play no role in what follows). Define $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$\begin{aligned} g(x_0; &\overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) \\ &= \sum_{i=0}^{r} (-1)^i \sum_{\mathcal{C}_i} f(x_0; \overline{y_1^{a_1}, \cdots, y_{\tilde{s}}^{a_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-i}}}), \end{aligned}$$

where $\mathcal{C}_i$ is the set of all $(\tilde{s} + r - i)$-tuples $(a_1, \cdots, a_{\tilde{s}}, j_1, \cdots, j_{r-i})$ satisfying

$$a_1 + \cdots + a_{\tilde{s}} = i, a_p \in \mathbb{Z}^+, 1 \le j_1 < \cdots < j_{r-i} \le r + \tilde{s}.$$

Let $x_{r+i} = y_i$, $i = 1, \cdots, \tilde{s}$. It suffices to show that

$$g(x_0; \overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) = f(x_0; x_1, \cdots, x_r).$$

For any $t \in \mathbf{r}$, $b_1, \cdots, b_{\tilde{s}} \in \mathbb{Z}^+$ such that $\sum_{i=1}^{\tilde{s}} b_i = t$, we find the coefficient of $f(x_0; \overline{y_1^{b_1}, \cdots, y_{\tilde{s}}^{b_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-t}}})$ where $j_v \in \mathbf{r}$, $v \in \mathbf{r} - \mathbf{t}$. Let $(b_1, \cdots, b_{\tilde{s}}, j_1, \cdots, j_{r-t}) \in \mathcal{C}_t$ and m denote the number of b_i that are greater than equal to 1. Now $f(x_0; \overline{y_1^{b_1}, \cdots, y_{\tilde{s}}^{b_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-t}}})$ appears in the sum for g when $t - m \le i \le t$ and has coefficient $(-1)^i \binom{m}{t-i}$. Hence, the net coefficient is $\sum_{i=t-m}^{t} (-1)^i \binom{m}{t-i}$ which is zero unless $m = 0$ ($t = 0$), in which case the coefficient is 1 and we get $f(x_0; x_1, \cdots, x_r)$. Hence g defines the required system $\mathcal{G} \in \mathcal{N}$. □

Lemma 2.14. *If $p = 1$ and $M_1 = \frac{1}{m} N_1$, then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Just as in the proof of lemma 2.13, the variables $\mathbf{X}_j \in M^{s_j}$ play no role if $j > 1$ and so it is no loss of generality to take $p = q = 1$. The computations do not use the internal variable which we also omit. Since $p = q = 1$ and there is no internal variable, all functions will be symmetric and we omit the overline signifying symmetry. Since the case when $m = 1$ is trivial we assume $m \geq 2$. Set $r_1 = r, s_1 = s$ and note that $s = mr$. Let $\mathcal{J}$ denote the set of all tuples $\boldsymbol{j} = (j_1, \cdots, j_u)$ of positive integers such that $j_1 \geq j_2 \geq \cdots \geq j_u \geq 1$ and $\sum_{i=1}^u j_i = r$. We define a total order on $\mathcal{J}$ by

$$\boldsymbol{j} = (j_1, \cdots, j_u) > \boldsymbol{j}' = (j'_1, \cdots, j'_{u'}),$$

if $\exists k \in \mathbf{u}$ such that

$$j_i = j'_i,\ i < k, \text{ and } j_k > j'_k.$$

Note that $\boldsymbol{j} > \boldsymbol{j}'$ does not imply $u \lesseqgtr u'$. The unique maximal and minimal elements of $\mathcal{J}$ are (r) and $(1, 1, \cdots, 1)$ respectively.

Suppose $f : M \times M^r \to TM$ models $\mathcal{F} \in \mathcal{M}$. Define $g : M \times M^{rm} \to TM$ by

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}),$$

where $c_{\boldsymbol{j}} \in \mathbb{Q}$ are constants to be determined. For fixed $\boldsymbol{j} \in \mathcal{J}$, define

$$g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}) = \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}).$$

Thus

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}).$$

We remark that each $g_{\boldsymbol{j}}$ is symmetric in $(x_1, \cdots, x_{rm})$. Hence g is symmetric in $(x_1, \cdots, x_{rm})$.

Given $\boldsymbol{j} = (j_1, \cdots, j_u) \in \mathcal{J}$, define $\mathcal{J}(\boldsymbol{j}) \subset \mathcal{J}$ to consist of all $\boldsymbol{\ell} = (\ell_1, \cdots, \ell_{u'}) \geq \boldsymbol{j}$ such that each ℓ_t can be written as a sum $\sum_{i \in I_t} j_i$, $I_t \subset \mathbf{u}$.

Suppose we are given $y_1, \cdots, y_r$ and $\boldsymbol{j} \in \mathcal{J}$. Suppose $x_1, \cdots, x_p = y_1, \cdots, x_{(r-1)m+1}, \cdots, x_{rm} = y_r$. Then there exist strictly positive integers $A_{\boldsymbol{\ell}}^{\boldsymbol{j}}$ such that

$$g_{\boldsymbol{j}}(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r),$$

where

$$f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) = \sum f(y_{i_1}^{\ell_1}, \cdots, y_{i_{u'}}^{\ell_{u'}}),$$

and the sum is taken all distinct u'-tuples $(i_1, \cdots, i_{u'})$ of integers in $\mathbf{r}$. Each $f_{\boldsymbol{\ell}}$ is symmetric in $y_1, \cdots, y_r$. We have

$$g(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \left(\sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) \right).$$

We choose the coefficients $c_{\boldsymbol{j}}$ so that $g(y_1^m, \cdots, y_r^m) = f(y_1, \cdots, y_r)$. The term $f(y_1, \cdots, y_r)$ only occurs once in the sum we have for g (when $\boldsymbol{j}$ is the minimal element $(1, 1, 1, \cdots, 1)$ of $\mathcal{J}$). Hence $c_{(1,\cdots,1)}$ is uniquely determined. Our choice of order on $\mathcal{J}$ orders the the rows of the matrix of the linear system and our construction implies that the matrix is in upper triangular form. □

Lemma 2.15. *If $p = 1$, then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Since $M_1 \in \mathbf{A}(\mathcal{N})$, we may write $M_1 = \sum_{i \in A} \lambda_i N_i - \sum_{i \in B} \lambda_i N_i$, where A, B are disjoint subsets of $\overline{\mathbf{q}}$ and for $i \in A \cup B$, $\lambda_i = \frac{a_i}{b_i}$, where $a_i, b_i \in \mathbb{Z}^+$, and $(a_i, b_i) = 1$.

Let $\lambda = \mathrm{lcm}\{b_i \mid i \in A \cup B\}$ and define $\alpha_i = \lambda \lambda_i \in \mathbb{Z}^+, i \in A \cup B$. If we define $P = \sum_{i \in A} \alpha_i N_i$, $Q = \sum_{i \in B} \alpha_i N_i$, then

$$M_1 = \frac{1}{\lambda}(P - Q)$$

Let $\mathcal{N}_1$ be the network with adjacency matrices $\{I, P, Q\}$, and $\mathcal{N}_2$ be the network with adjacency matrices $\{I, R = P - Q\}$. Note that

(1) If $Q = 0$, $\mathcal{N}_1 = \mathcal{N}_2$.
(2) If $\lambda = 1$, $\mathcal{N}_2 = \mathcal{M}$.
(3) If $Q = 0$ and $\lambda = 1$, $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{M}$.

We claim that

$$\mathcal{M} \prec_O \mathcal{N}_2 \prec_O \mathcal{N}_1 \prec_O \mathcal{N}.$$

Assuming the claim, the transitivity of $\prec_O$ gives $\mathcal{M} \prec_O \mathcal{N}$. The claim follows since lemma 2.12 implies $\mathcal{N}_1 \prec_O \mathcal{N}$, lemma 2.13 implies $\mathcal{N}_2 \prec_O \mathcal{N}_1$ and lemma 2.14 implies $\mathcal{M} \prec_O \mathcal{N}_2$. □

Lemma 2.16. *If $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$, then $\mathcal{M} \prec_O \mathcal{N}$ (statement (A) is true).*

Proof. By lemma 2.8, it suffices to show that $\mathcal{M}_j \prec_O \mathcal{N}$ for all $j \in \mathbf{p}$. By lemma 2.10, we may assume $j = 1$. The result follows from lemma 2.15. □

Lemma 2.17. *If $\mathcal{M} \prec_O \mathcal{N}$ then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ (statement (C) is true).*

Proof. Suppose $\mathcal{M} \prec_O \mathcal{N}$. The method we use is based on the linear equivalence ideas described in [3]. Specifically, we prove that $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$ by restricting to the case where phase spaces equal $\mathbb{R}$ and vector fields are linear. (Notice that output domination preserves linearity of vector fields.)

Let $\mathcal{F} \in \mathcal{M}$ have (linear) model $f : \mathbb{R} \times \prod_{i=1}^{p} \mathbb{R}^{r_i} \to \mathbb{R}$. Then there exists a system $\mathcal{G} \in \mathcal{N}$ with linear model $g : \mathbb{R} \times \prod_{i=1}^{q} \mathbb{R}^{s_i} \to \mathbb{R}$ such that for each $j \in \mathbf{n}$ we have

$$g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j}), \tag{2.2}$$

where $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{n}_{i1}^j}, \cdots, x_{\mathfrak{n}_{is_i}^j})$, $i \in \mathbf{q}$, and $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{m}_{i1}^j}, \cdots, x_{\mathfrak{m}_{ir_i}^j})$, $i \in \mathbf{p}$. Let $k \in \mathbf{p}$ and take

$$f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_p) = \sum_{i=1}^{r_k} x_{ki}$$

where $\mathbf{X}_v = (x_{v1}, \cdots, x_{vr_v})$, $v \in \mathbf{p}$. The corresponding g given by output domination is linear and so, noting the symmetry of inputs, we may write

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = c_{k0}x_0 + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{i\ell},$$

where $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$, $i \in \mathbf{q}$, and the $c_{\alpha\beta}$ are constants. From (2.2) we get

$$c_{k0}x_j + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{\mathfrak{n}_{i\ell}^j} = \sum_{i=1}^{r_k} x_{\mathfrak{m}_{ki}^j},\ j \in \mathbf{n}.$$

Putting these equations in matrix form, we get

$$\sum_{i=0}^{q} c_{ki} N_i = M_k.$$

Hence for each $k \in \mathbf{q}$, we have shown that $M_k \in \mathbf{A}(\mathcal{N})$ and so $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$. $\square$

Lemma 2.18. *If* $\mathcal{M} \prec \mathcal{N}$ *then* $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ *(statement (D) is true).*

Proof. (Sketch) Working within the class of C^1-vector fields with phase space $\mathbb{R}$, it follows that if $\mathcal{F}$ has linear model f, then there exists $\mathcal{G} \in \mathcal{N}$ with C^1-model g such that $\mathcal{G}$ has identical dynamics to $\mathcal{F}$. The statement remains true if we replace g by the derivative of g at $0 \in \mathbb{R} \times \mathbb{R}^q$ and then the method of proof of lemma 2.17 applies (essentially we reduce to linear equivalence, cf [3]). With a little more work, we can remove the assumption that g is C^1 — identical dynamics to a linear system implies the flow is linear and from this one can show that we can always choose g to be linear. □

Proof of theorem 2.9. Lemmas 2.16, 2.17, 2.18 give statements A,C,D and, as noted previously, statement B is trivial. Interchange $\mathcal{M}$ and $\mathcal{N}$ to obtain the reverse relations. □

2.1. **Algorithm for obtaining an output equivalence.** We conclude this section with some additional lemmas that are useful for giving an algorithm to determine an explicit output equivalence or domination. We continue to assume that $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$.

We introduce some new notation. Given $j \in \mathbf{p}$, let $\mathcal{M}^{-j}$ be the n identical cell network with $p-1$ input types and $\mathbb{A}(\mathcal{M}^{-j}) = \mathbb{A}(\mathcal{M}) \setminus \{M_j\}$.

Lemma 2.19. *If $j \in \mathbf{p}$ and $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-j})$ then $\mathcal{M} \sim_O M^{-j}$.*

Proof. While this follows from lemma 2.16, we give an explicit proof that $\mathcal{M} \prec_O M^{-j}$. Permuting inputs (lemma 2.10) we may and shall assume $j = p$. If $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M}^{-p})$, then there exists $\lambda_0, \cdots, \lambda_{p-1} \in \mathbb{Q}$ such that

$$M_p = \sum_{i=0}^{p-1} \lambda_i M_i.$$

Let $\mathcal{M}'$ be the n identical cell network with adjacency matrices I, M_p. Since $\mathbf{A}(\mathcal{M}') \subset \mathbf{A}(\mathcal{M}^{-p})$, $\mathcal{M}' \prec_O \mathcal{M}^{-p}$ by lemma 2.15. Suppose that $\mathcal{F}_1 \in \mathcal{M}'$ has model $f' : M \times M^{r_p} \to TM$. Then, by definition of output equivalence, there exists an invariant map $C : \mathbf{A}(r_p, r - r_p) \to \mathbb{Q}$ such that the map $\hat{g} : M \times \prod_{i=1}^{p-1} M^{r_i} \to TM$ defined by

$$\hat{g}(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_{p-1}) = \sum_{\gamma} C(\gamma) f_1(x_0; \overline{x_{\gamma(1)}, \cdots, x_{\gamma(r_p)}})$$

is the required model for the system $\mathcal{G} \in \mathcal{M}^{-p}$. Now suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$. Define g by

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_{p-1}) = \sum_{\gamma} C(\gamma) f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_{p-1}, \overline{x_{\gamma(1)}, \cdots, x_{\gamma(r_p)}})$$

It is clear that $g : M \times \prod_{i=1}^{p-1} M^{r_i} \to TM$ gives the model for the required system $\mathcal{F} \in \mathcal{M}^{-p}$ and so $\mathcal{M} \prec_O M^{-p}$. □

Lemma 2.20. *If $\mathcal{M}^*$ is a network obtained from $\mathcal{M}$ by removing input types so that*

(1) $\mathbf{A}(\mathcal{M}^*) = \mathbf{A}(\mathcal{M})$,

(2) $\mathbb{A}(\mathcal{M}^*)$ *is a linearly independent set,*

then $\mathcal{M}^* \sim_O \mathcal{M}$.

Proof. The result follows by repeated application of lemma 2.19. □

Now suppose $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ and that we want to determine an explicit output domination $\mathcal{M} \prec_O \mathcal{N}$.

Step 1: Choose a network $\mathcal{N}^*$ obtained from $\mathcal{N}$ by removing input types so that

(1) $\mathbf{A}(\mathcal{N}^*) = \mathbf{A}(\mathcal{N})$,

(2) $\mathbb{A}(\mathcal{N}^*)$ is a linearly independent set,

It suffices to find an explicit output domination $\mathcal{M} \prec_O \mathcal{N}^\star$ since every such output domination extends trivially to $\mathcal{M} \prec_O \mathcal{N}$.

Step 2: Choose a sequence $\mathcal{M} = \mathcal{M}^0, \mathcal{M}^1, \cdots, \mathcal{M}^k = \mathcal{M}^*$ so that each $\mathcal{M}^j$ is obtained from $\mathcal{M}^{j-1}$ by removal of an input type, and we have $\mathbf{A}(\mathcal{M}^*) = \mathbf{A}(\mathcal{M})$, and $\mathbb{A}(\mathcal{M}^*)$ is a linearly independent set as in step 1. We apply lemma 2.19 repeatedly to obtain an explicit domination $\mathcal{M} \prec_O \mathcal{M}^*$.

Step 3: Since $\mathcal{M}^*$ and $\mathcal{N}^*$ are both bases for the same subspace of $M(n;\mathbb{Q})$, both networks have the same number of input types and the same number $\ell \leq p, q$ of adjacency matrices. Using lemma 2.10, we can reorder the input types of $\mathcal{M}^*$ so that the first k adjacency matrices of $\mathcal{M}^*$ and $\mathcal{N}^*$ are pairwise equal and the remaining $\ell - k$ adjacency matrices are distinct. We then successively replace each adjacency matrix M^*_{k+i} by N^*_{k+i}, $1 \leq i \leq \ell - k$. At each stage we derive the corresponding output domination using lemma 2.15. The resulting chain $\mathcal{M} \prec_O \mathcal{M}^1 \prec_O \cdots \prec_O \mathcal{M}^* \prec_O \cdots \prec_O \mathcal{N}^*$ determines an output domination $\mathcal{M} \prec_O \mathcal{N}$.

3. Input Equivalence

We start by giving the definition of input equivalence which is a straightforward extension of that given in [1, §3] for networks with asymmetric inputs. Aside from assuming that models are defined on vector spaces V rather than manifolds M, we closely follow the notational conventions established in section 2. In particular, $\mathcal{M}$ and $\mathcal{N}$ will be coupled n identical cell networks. We fix an ordering of the cells of $\mathcal{N}$. Suppose cells in $\mathcal{N}$ have s inputs and q input types and let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in \mathcal{M}_{s_i}(n;\mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be the default connection matrix for $\mathcal{N}$.

We suppose cells in $\mathcal{M}$ have r inputs and p input types. Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in$

$\mathcal{M}_{r_i}(n;\mathbb{Z}^+),\ i\in\mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Denote the associated (default) connection matrix of $\mathcal{M}$ by $\mathfrak{m}=[\mathfrak{m}^1,\cdots,\mathfrak{m}^n]$.

Let $L=(L_1,\cdots,L_p)\in\prod_{i=1}^p M(r_i,1+\sum_{j=1}^q s_j;\mathbb{Q})$ and define the linear map $\mathbf{L}:V\times\prod_{i=1}^q V^{s_i}\to\prod_{i=1}^p V^{r_i}$ in the obvious (V-independent) way. Recall that $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h:G_{\mathcal{N}}\to G_{\mathcal{M}}$ such that

$$\mathbf{L}(\gamma(\mathbf{X}))=h(\gamma)\mathbf{L}(\mathbf{X}),\ \text{for all } \gamma\in G_{\mathcal{N}},\ \mathbf{X}\in V\times\prod_{i=1}^q V^{s_i}.$$

If $f:V\times\prod_{i=1}^p V^{r_i}\to V$ is $G_{\mathcal{M}}$-invariant, define $g:V\times\prod_{i=1}^q V^{s_i}\to V$ by

$$g(\mathbf{X}_0;\mathbf{X}_1,\cdots,\mathbf{X}_q)=f(\mathbf{X}_0;\mathbf{L}(\mathbf{X}_0;\mathbf{X}_1,\cdots,\mathbf{X}_q)). \quad (3.3)$$

Since $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant, g is $G_{\mathcal{N}}$-invariant. We write $f<^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})}g$, if

1. (3.3) is satisfied.
2. For $j\in\mathbf{n}$, we have $g(x_j;\mathbf{X}_{\mathfrak{n}_1^j},\cdots,\mathbf{X}_{\mathfrak{n}_q^j})=f(x_j;\mathbf{X}_{\mathfrak{m}_1^j},\cdots,\mathbf{X}_{\mathfrak{m}_p^j})$.

Definition 3.1. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks such that

(a) $n(\mathcal{M})=n(\mathcal{N})=n$.
(b) Cells in $\mathcal{M}$ have r inputs, cells in $\mathcal{N}$ have s inputs.
(c) Cells in $\mathcal{M}$ have p input types, cells in $\mathcal{N}$ have q input types.
(d) If we fix an ordering $\mathbf{C}_1,\ldots,\mathbf{C}_n$ of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n}=(\mathfrak{n}^1,\ldots,\mathfrak{n}^n)$.

We say $\mathcal{M}$ is *input dominated* by $\mathcal{N}$, denoted $\mathcal{M}\prec_I\mathcal{N}$, if there exist a linear map $\mathbf{L}$ and an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, such that for every $\mathcal{F}\in\mathcal{M}(\mathbb{L})$, there exists $\mathcal{G}\in\mathcal{N}(\mathbb{L})$ for which $f_{\mathcal{F}}<^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})}g_{\mathcal{G}}$. If $\mathcal{N}\prec_I\mathcal{M}$ and $\mathcal{M}\prec_I\mathcal{N}$, we say $\mathcal{M}$ and $\mathcal{N}$ are *input equivalent* and write $\mathcal{M}\sim_I\mathcal{N}$.

Remark 3.2. We write $\mathcal{M}\prec_{I,\mathbb{Z}}\mathcal{N}$ if $\mathcal{M}\prec_I\mathcal{N}$ and we can require the entries of $\mathbf{L}$ to lie in $\mathbb{Z}$). We similarly define $\mathcal{M}\sim_{I,\mathbb{Z}}\mathcal{N}$.

Lemma 3.3. *(Notation and assumptions as above). The following conditions are equivalent.*

1. $\mathcal{M}\prec_I\mathcal{N}$.
2. *There exists an ordering of the cells in $\mathcal{M}$ such that $\mathcal{M}_j\prec_I\mathcal{N}$, for all $j\in\mathbf{p}$.*

Proof. We omit the proof which is similar to that of lemma 2.8. □

As a consequence of lemma 3.3, it will be no loss of generality in what follows to assume that $\mathcal{M}$ has just one input type and so $p = 1$. Just as in our discussion of output equivalence, we simplify notation by setting $r_1 = r$. We then have $G_{\mathcal{M}} = S_r \approx S_1 \times S_r$.

Suppose that the linear map $\mathbf{L} : V \times \prod_{i=1}^{q} V^{s_i} \to V^r$ is defined by the matrix $L \in M(r, 1 + \sum_{i=1}^{q} s_i, \mathbb{Q})$. The map $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h : G_{\mathcal{N}} \to G_{\mathcal{M}} = S_r$ such that

$$\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X}),$$

for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^{q} V^{s_i}$.

Given a $G_{\mathcal{M}}$-invariant map $f : V \times V^r \to V$ and $G_{\mathcal{M},\mathcal{N}}$-equivariant linear map $\mathbf{L}$ as above, define the $G_{\mathcal{N}}$-invariant map $g : V \times \prod_{i=1}^{q} V^{s_i} \to V$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(\mathbf{X}_0; \mathbf{L}(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q)).$$

Let $L = [L_1, \cdots, L_r]$ where, $L_i \in \mathbb{Q} \times \prod_{i=1}^{q} \mathbb{Q}^{s_i}$ denotes the i^{th} row of L, $i \in \mathbf{r}$. Since $\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X})$ for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^{q} V^{s_i}$, we have $[L_1, \cdots, L_r](\gamma\mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X})$. That is,

$$[\gamma L_1, \cdots, \gamma L_r](\mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X}),$$

where, γL_i is defined using the natural permutation action of $G_{\mathcal{N}}$ on $\mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, $i \in \mathbf{r}$. This is true for all $\mathbf{X}$, hence

$$[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)[L_1, \cdots, L_r],$$

for all $\gamma \in G_{\mathcal{N}}$.

Definition 3.4. Suppose a finite group G acts on a non-empty set X. For $x \in X$, let $Gx = \{gx \mid g \in G\}$ denote the *G-orbit of x*.

Remark 3.5. We have $|Gx| = |G|/|G_x|$ where $G_x = \{g \in G \mid gx = x\}$ denotes the *isotropy subgroup* of G at x.

Theorem 3.6. *There exists $u(\leq r) \in \mathbb{N}$, $t_1, \cdots, t_u \in \mathbb{N}$, with $\sum_{i=1}^{u} t_i = r$ such that $\{L_1, \cdots, L_r\} = \bigcup_{i=1}^{u} G_{\mathcal{N}} L^i$ where $|G_{\mathcal{N}} L^i| = t_i$. (We allow $L^i = L^j$ for $i \neq j$, $i, j \in \mathbf{u}$.)*

Proof. $[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)[L_1, \cdots, L_r]$ for all $\gamma \in G_{\mathcal{N}}$. Therefore for each $i \in \mathbf{r}$, $\gamma L_i \in \{L_1, \cdots, L_r\}$ for all $\gamma \in G_{\mathcal{N}}$, so we have that the $G_{\mathcal{N}}$-orbit of L_i is contained in $\{L_1, \cdots, L_r\}$. Suppose $L_k = L_j$ for some $k \neq j$, then $\gamma L_k = \gamma L_j$ for all $\gamma \in G_{\mathcal{N}}$. So if an element is repeated m times then its full orbit is repeated m times. Therefore, there exist $u(\leq r) \in \mathbb{N}$, L_{j_i}, for $j_1, \cdots, j_u \in \mathbf{r}$ with $|G_{\mathcal{N}} L_{j_i}| = t_i$ and $\sum_{i=1}^{u} t_i = r$ such that $\{L_1, \cdots, L_r\} = \bigcup_{j=1}^{u} G_{\mathcal{N}} L_{j_i}$. Define $L^i = L_{j_i}$, $i \in \mathbf{u}$. $\square$

Remark 3.7. For each $i \in \mathbf{u}$, there are t_i choices for L_{j_i}.

From now on, we write the matrix of L in the form

$$\left[G_{\mathcal{N}}L^1, \cdots, G_{\mathcal{N}}L^u\right].$$

That is, we group rows according to the group orbits of $G_{\mathcal{N}}$. Note that this ordering is imposing a condition on the order of inputs of $\mathcal{M}$.

Example 3.8. Suppose $p, q = 1, s = s_1 = 2, r = 3$, $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Then we can take $L^1 = (0, 1, 2)$, $L^2 = (0, 1, 1)$. We have $t_1 = 2$ and $t_2 = 1$. If we write L in the form $[G_{\mathcal{N}}L^1, G_{\mathcal{N}}L^2]$, then $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

3.1. **Splittings and connection matrices.**

Definition 3.9. Let $P \in \mathcal{M}_k(n; \mathbb{Z}^+)$. A *splitting* $(P_1, \cdots, P_k)$ of P is a decomposition of P into a sum $P = P_1 + \cdots + P_k$ where each $P_j \in \mathcal{M}_1(n; \mathbb{Z}^+)$.

Suppose that the network $\mathcal{M}$ has connection matrix $\mathfrak{m}$, where $\mathfrak{m}$ is not necessarily the default connection matrix. Denote the adjacency matrices of $\mathcal{M}$ by $M_0 = I$ and M_1. The connection matrix $\mathfrak{m}$ naturally determines a unique splitting $M^1 + \cdots + M^r$ of M_1. Indeed, if we let $M^k = [m^k_{ij}]$, $k \in \mathbf{r}$, then $m^k_{ij} = 1$ if input k of cell j comes from cell i, else $m^k_{ij} = 0$. That is, $m^k_{ij} = 1$ iff $\mathfrak{m}^j_k = i$. Conversely, every splitting of M_1 uniquely determines a connection matrix $\mathfrak{m}$ for $\mathcal{M}$. All of this applies equally well if $\mathcal{M}$ has multiple input types.

Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ (not necessarily the default). Let $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$ denote the set of splittings of the adjacency matrices $\{N_1, \cdots, N_q\}$ determined by $\mathfrak{n}$. In particular, for $k \in \mathbf{q}$, we have $\mathbf{N}_k = (N_{k1}, \cdots, N_{ks_k})$ and $N_{k1} + \cdots + N_{ks_k} = N_k$.

Let $\mathbf{a} = (a_0; a_1, \cdots, a_q) \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$. We write $\mathbf{a} = (a_{ji})_{j \in \overline{\mathbf{q}}, i \in \mathbf{s_j}}$, where $a_j = (a_{j1}, \cdots, a_{js_j}) \in \mathbb{Q}^{s_j}$, $j \in \overline{\mathbf{q}}$. Suppose as above that $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$ is a set of splittings of the adjacency matrices $\{N_1, \cdots, N_q\}$ determined by $\mathfrak{n}$. We define

$$\mathbf{a} \star \mathbf{N} = a_0 N_0 + \sum_{j=1}^{q} \sum_{i=1}^{s_j} a_{ji} N_{ji} \in M(n, n; \mathbb{Q}).$$

Theorem 3.10. *(Notation and assumptions as above; in particular $p = 1$.) The following statements are equivalent*

(1) $\mathcal{M} \prec_I \mathcal{N}$.

(2) *Suppose that* $\mathfrak{n}$ *is a connection matrix for* $\mathcal{N}$*. There exist* $u \in \mathbb{N}$, $t_1, \cdots, t_u \in \mathbb{N}$, *with* $\sum_{i=1}^u t_i = r$, $M_1^i \in \mathcal{M}_{t_i}(n;\mathbb{Z}^+)$ *with* $\sum_{i=1}^u M_1^i = M_1$, $L^i \in \mathbb{Q} \times \prod_{v=1}^q \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$, *such that* $B_i = \{\mathbf{b} \star \mathbf{N}_i \mid b \in G_{\mathcal{N}} L^i\}$ *is a splitting of* M_1^i, *for all* $i \in \mathbf{u}$.

(3) *There exist* $u \in \mathbb{N}$, $t_1, \cdots, t_u \in \mathbb{N}$, *with* $\sum_{i=1}^u t_i = r$, $M_1^i \in \mathcal{M}_{t_i}(n;\mathbb{Z}^+)$ *with* $\sum_{i=1}^u M_1^i = M_1$, $L^i \in \mathbb{Q} \times \prod_{v=1}^q \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$ *such that for every connection matrix* $\mathfrak{n}$ *of* $\mathcal{N}$, $B_i = \{\mathbf{b}\star\mathbf{N}_i \mid b \in G_{\mathcal{N}} L^i\}$ *is a splitting of* M_1^i, *for all* $i \in \mathbf{u}$.

If either (2) or (3) are satisfied, then $\bigcup_{i=1}^u B_i$ *is a splitting of* M_1.

Proof. (1) $\Rightarrow$ (2). Suppose $\mathcal{M} \prec_I \mathcal{N}$. Then there is a linear transformation $\mathbf{L}$ with matrix $L = [G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u]$. Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ and denote the corresponding splittings of $N_1, \cdots, N_q$ by $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$. For each $j \in \mathbf{n}$, we have

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j}$$

where $\mathfrak{m}$ is a connection matrix for the network $\mathcal{M}$. Let

$$\mathfrak{m}^{ji} = (\mathfrak{m}^j_{1(T_{i-1}+1)}, \cdots, \mathfrak{m}^j_{1T_i}),$$

where $T_0 = 0$, $T_i = t_1 + \cdots + t_i$, $i \in \mathbf{u}$.
For each $i \in \mathbf{u}$, let $M_1^i \in \mathcal{M}_{t_i}(n;\mathbb{Z}^+)$ be the matrix corresponding to the connection matrix $[\mathfrak{m}^{1i}, \cdots, \mathfrak{m}^{ni}]$. If we define $B_i = \{\mathbf{b} \star \mathbf{N}_i \mid b \in G_{\mathcal{N}} L^i\}$, then B_i is a splitting of M_1^i.
(2) $\Rightarrow$ (3). Suppose statement (2) holds for the connection matrix $\mathfrak{n}$ and let $\widehat{\mathfrak{n}}$ be any other connection matrix for $\mathcal{N}$. Then for each $j \in \mathbf{n}$, $\widehat{\mathfrak{n}^j} = \gamma^j \mathfrak{n}^j$ for some $\gamma^j \in G_{\mathcal{N}}$ ($\gamma^j \mathfrak{n}^j$ is the natural action of $G_{\mathcal{N}}$ on $\{j\} \times \prod_{i=1}^q \mathbf{n}^{s_i}$). For $j \in \mathbf{n}$, let $\mathbf{N}^j$ denote the set of j^{th} columns of all matrices in $\mathbf{N}$. Since $\{b \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i\}$ is a splitting of M_1^i, $\{[\gamma^1(b) \star \mathbf{N}^1, \cdots, \gamma^n(b) \star \mathbf{N}^n] \mid b \in G_{\mathcal{N}} L^i\} = \{[b \star \gamma^1(\mathbf{N}^1), \cdots, b \star \gamma^n(\mathbf{N}^n)] \mid b \in G_{\mathcal{N}} L^i\}$ is a splitting of M_1^i, $i \in \mathbf{u}$. Thus statement (2) holds for $\widehat{\mathfrak{n}}$.
(3) $\Rightarrow$ (1). Take $L = [G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u]$. Fix a connection matrix $\mathfrak{n} = (\mathfrak{n}^1, \cdots, \mathfrak{n}^n)$ for the network $\mathcal{N}$ and denote the associated family of splittings of $N_1, \cdots, N_q$ by $\mathbf{N}$ as above. Since $\bigcup_{i=1}^u B_i$ is a splitting of M_1^i, we have a connection matrix $\mathfrak{m} = (\mathfrak{m}^i, \cdots, \mathfrak{m}^n)$, where $\mathfrak{m}^j = (\mathfrak{m}_1^j) \in \mathbf{n}^r$ is such that for all $j \in \mathbf{n}$,

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j}.$$

Thus for all $j \in \mathbf{n}$,

$$\begin{aligned} g(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) &= f(\mathbf{X}_j; \mathbf{L}(\mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j})) \\ &= f(\mathbf{X}_0; \mathbf{X}_{\mathfrak{m}_1^j}) \end{aligned}$$

This implies $\mathcal{M} \prec_I \mathcal{N}$. □

Remark 3.11. If in theorem 3.10, we have $\mathcal{M} \prec_I \mathcal{N}$ and we take the default connection matrix for $\mathcal{N}$, then the connection matrix on $\mathcal{M}$ given by (2) will generally *not* equal the default connection matrix of $\mathcal{M}$. For example, suppose that $\mathcal{N}$ is the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{M}$ is the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In this case we have $M_1 = N_0 + N_1$ and may easily check directly (see below) that $\mathcal{M} \prec_I \mathcal{N}$. If $\mathcal{F} \in \mathcal{M}$ has model $f : V \times V \to V$, then we define $\mathcal{G} \in \mathcal{N}$ either by $g(x; y) = f(x; \overline{x, y})$ or by $g(x; y) = f(x; \overline{y, x})$. Hence the only choices of L are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Neither of these choices gives the default connection matrix for $\mathcal{M}$.

Corollary 3.12. *(Notation and assumptions as above.) Suppose that* $M_1 \in \mathcal{M}_1(n; \mathbb{Z}^+)$*, then* $\mathcal{M} \prec_I \mathcal{N}$ *iff* $M_1 \in \mathbf{A}(\mathcal{N})$*.*

Proof. ($\Rightarrow$): Since $\mathcal{M} \prec_I \mathcal{N}$, there is a linear transformation $\mathbf{L}$ with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^q s_j, \mathbb{Q})$, where $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, such that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. Since L has only one row, $\mathcal{G}_\mathcal{N}\mathbf{a} = \{\mathbf{a}\}$. Therefore, for $j \in \mathbf{q}$, we may write $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$ where $\lambda_j \in \mathbb{Q}$. If we take $u = 1$, and $L^1 = \mathbf{a}$, then $M_1 = \sum_{j=0}^q \lambda_j N_j$.
($\Leftarrow$): Let $M_1 = \sum_{j=0}^q \lambda_j N_j$. Take $\mathbf{L}$ to be the linear transformation with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^q s_j, \mathbb{Q})$, $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, where $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$, $j \in \mathbf{q}$. It is straightforward to check that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Corollary 3.13. *(Notation and assumptions as above.) If* M_1 *has a splitting* $(Q_1, \cdots, Q_r)$ *such that* $\{Q_1, \cdots, Q_r\} \subseteq \mathbf{A}(\mathcal{N})$*, then* $\mathcal{M} \prec_I \mathcal{N}$*.*

Proof. Immediate from theorem 3.10. □

Corollary 3.14. *(Notation and assumptions as above.) Suppose that* $M_1 = \sum_{j=0}^q \alpha_j N_j$, $\alpha_j \in \mathbb{Z}^+$, $j \in \overline{\mathbf{q}}$*. Then* $\mathcal{M} \prec_I \mathcal{N}$*.*

Proof. For each $j \in \overline{\mathbf{q}}$, let $\mathbf{I}^j$ be the $\alpha_j s_j \times s_j$ matrix defined by

$$\mathbf{I}^j = \begin{bmatrix} I_{s_j} \\ \vdots \\ I_{s_j} \end{bmatrix}$$

(I_{s_j} denotes the identity $s_j \times s_j$ matrix.) Define the $(\sum_{j=0}^{q} \alpha_j s_j) \times (\sum_{j=0}^{q} s_j)$ matrix L by

$$L = \begin{bmatrix} \mathbf{I}^0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^1 & \mathbf{0} & \cdots & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I}^q \end{bmatrix}$$

It is straightforward to verify that $f \prec^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Corollary 3.15. *(Notation and assumptions as above.) If we can write* $M_1 = A + S$ *where* $A \in \mathbf{A}(\mathcal{N}, \mathbb{Z}^+)$*, and there exists a splitting* $(S_1, \cdots, S_t)$ *of* S *such that* $S_i \in \mathbf{A}(\mathcal{N})$, $i \in \mathbf{t}$*, then* $\mathcal{M} \prec_I \mathcal{N}$.

Proof. Combine the linear transformation matrices obtained for A using corollary 3.13 and S using corollary 3.14 to obtain a linear transformation $\mathbf{L}$ giving $f \prec^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Theorem 3.16. *(Notation and assumptions as above except that we allow* $p \geq 1$*.) The following statements are equivalent*

1. $\mathcal{M} \prec_I \mathcal{N}$.
2. *Suppose that* $\mathfrak{n}$ *is a connection matrix for* $\mathcal{N}$*. For* $j \in \mathbf{p}$ *there exist* $u_j \in \mathbb{N}$, $t^j_1, \cdots, t^j_{u_j} \in \mathbb{N}$, *with* $\sum_{i=1}^{u_j} t^j_i = r_j$, $M^i_j \in \mathcal{M}_{t^j_i}(n; \mathbb{Z}^+)$ *with* $\sum_{i=1}^{u_j} M^i_j = M_j$, $L^i_j \in \mathbb{Q} \times \prod_{v=1}^{q} \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$*, such that* $B_{ji} = \{\mathbf{b} \star \mathbf{N}_i \mid b \in G_{\mathcal{N}} L^i_j\}$ *is a splitting of* M^i_j*, for all* $i \in \mathbf{u}_j$.

Proof. The result is immediate from theorem 3.10 and lemma 3.3. □

Corollary 3.17. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be coupled identical cells networks with* $n(\mathcal{M}) = n(\mathcal{N}) = n$*. Assume cells in* $\mathcal{M}$ *have* r *inputs, cells in* $\mathcal{N}$ *have* s *inputs. Suppose that* $\mathcal{M}$ *has adjacency matrices* $M_0 = I, M_1, \cdots, M_p$ *and* $\mathcal{N}$ *has adjacency matrices* $N_0 = I, N_1, \cdots, N_q$*. We assume that for each* $i \in \mathbf{p}$ *either* $r_i = 1$ *or* $s_j > r_i > 1$*, for all* $j \in \mathbf{q}$*. Under these conditions the following statements are equivalent*

1. $\mathcal{M} \prec_I \mathcal{N}$.
2. *For all* $i \in \mathbf{p}$*, there exists a splitting* $(P_{i,1}, \cdots, P_{i,r_i})$ *of* M_i *such that* $P_{i,j} \in \mathbf{A}(\mathcal{N})$*, for all* $j \in \mathbf{r}_i$.

Proof. (Sketch.) (2) ⇒ (1) is trivial. In order to prove (1) ⇒ (2), we may assume $p = 1$. Set $r = r_1$. For every $\mathbf{a} \in \mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, $G_{\mathcal{N}}\mathbf{a}$ has one element or at least $\min_{j\in\mathbf{q}} s_j$ elements. Since $r < s_j$ for all $j \in \mathbf{q}$, we have $r < \min_{j\in\mathbf{q}} s_j$. Therefore L must be of the form $[L^1, \cdots, L^r]$ where $L^i \in \mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}\mathbf{1}^{s_j}$. □

Remark 3.18. If the network $\mathcal{M}$ has asymmetric inputs and $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$, hypothesis (2) of corollary 3.17 is automatically satisfied (and so we recover the result for networks with asymmetric inputs — see lemma [1, §3.13]). However, if $\mathcal{M}$ has symmetric inputs and $\mathbf{A}(\mathcal{M}) \subset \mathbf{A}(\mathcal{N})$, then it need not be the case that (2) is satisfied and so $\mathcal{M}$ may not be input dominated by $\mathcal{N}$, even if we assume linear phase spaces or scaling signalling. We give examples in the next section.

4. Examples

In addition to the Add-Subtract and scaling cells defined in [1], we introduce one further construction that we will use in some of the examples so as to simplify diagrams.

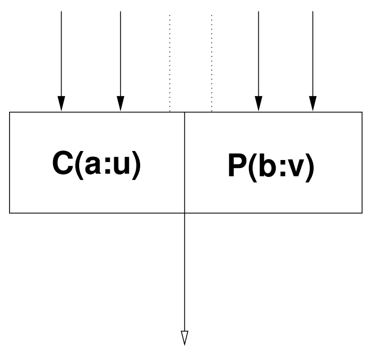

Figure 1. The choose and pick cell $\mathbf{C}(a, b : u, v)$

Let $\mathbf{C}$ be a cell with model $h : M \times M^k \to TM$. Suppose that $a, b, u, v \in \mathbb{Z}^+$ satisfy $u + v = k$ and $a \leq u$, $b \leq v$. The *choose and pick cell* $\mathbf{C}(a, b : u, v)$ is a collection of $\binom{v+b-1}{b}\binom{u}{a}$ class $\mathbf{C}$ cells whose outputs are added. More precisely, the cell $\mathbf{C}(a, b : u, v)$ has two components, denoted $C(a : u)$ and $P(b : v)$. Suppose $x_1, \cdots, x_u$ are the inputs to the $C(a : u)$ component and $y_1, \cdots, y_v$ are the inputs to the $P(b : v)$ component. Then the output of $\mathbf{C}(a, b : u, v)$ is defined to be

$$\sum_{\substack{1 \leq j_1 \leq \cdots \leq j_b \leq v \\ 1 \leq i_1 < \cdots < i_a \leq u}} h(x_0; y_{j_1}, \cdots, y_{j_b}, x_{i_1}, \cdots, x_{i_a}).$$

Note that the output of $\mathbf{C}(a, b : u, v)$ is symmetric in both $x_1, \cdots, x_u$ and $y_1, \cdots, y_v$. We use the symbol for $\mathbf{C}(a, b : u, v)$ shown in figure 1. In figure 2, we show the connections for the choose and pick cell $\mathbf{C}(1, 2 : 2, 2)$.

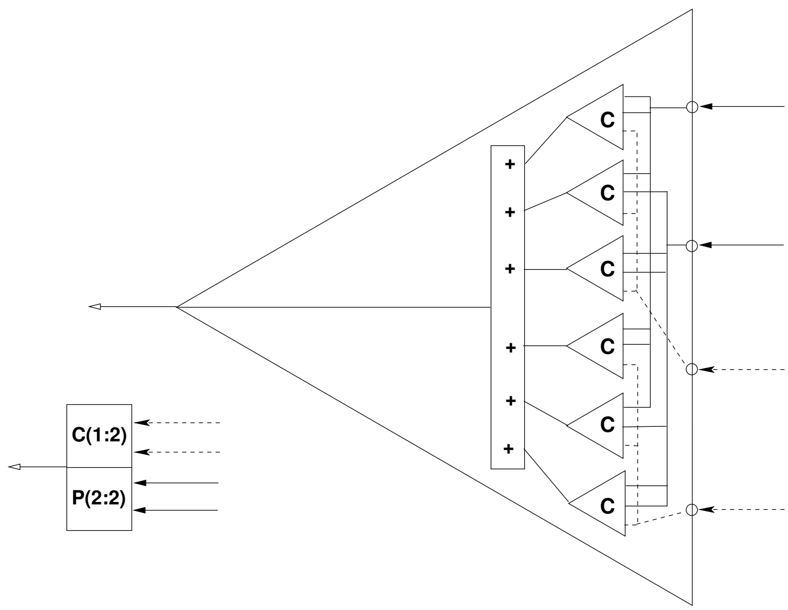

FIGURE 2. The choose and pick cell $\mathbf{C}(1,2:2,2)$.

Example 4.1. If $p = q = 1$, and $M_1 = aN_1$ for some $a \in \mathbb{N}$ (and so $r = as$) then $\mathcal{M} \prec_O \mathcal{N}$. Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^{as} \to TM$. Define $g : M \times M^s \to TM$ by

$$g(x_0; \overline{x_1, x_2, \cdots, x_s}) = f(x_0; \overline{x_1^a, x_2^a, \cdots, x_s^a}).$$

It is easy to check that g defines the required system $\mathcal{G} \in \mathcal{N}$.

Example 4.2. If $p = q = 1$, $N_1 = bS$, and $M_1 = aS$ for $S \in M_1(n; \mathbb{Z}^+)$, $a, b \in \mathbb{N}$, then $\mathcal{M} \prec_O \mathcal{N}$. Here $r = a$, $s = b$. Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^a \to TM$. Define $g : M \times M^b \to TM$ by

$$g(x_0; \overline{x_1, \cdots, x_b}) = \frac{1}{b}\left[f(x_0; x_1^a) + \cdots + g(x_0; x_b^a)\right],$$

where x^a signifies x repeated a-times. It is easy to check that g defines the required system $\mathcal{G} \in \mathcal{N}$.

Example 4.3. If $p = 1$, $M_1 \in M_1(n; \mathbb{Z}^+)$ and $M_1 = \sum_{j=0}^q \lambda_j N_j$, then $\mathcal{M} \prec_O \mathcal{N}$. Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M \to TM$. Define $g : M \times \prod_{j=1}^q M^{s_j} \to TM$ by

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = \lambda_0 f(x_0; x_0) + \sum_{j=1}^q \lambda_j \sum_{i=1}^{s_j} f(x_0; x_{ji})$$

where the $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denote variables corresponding to the inputs of type i, $i \in \mathbf{q}$. If we let $(\mathfrak{m}^1, \cdots, \mathfrak{m}^n)$ denote the connection matrix (vector) of $\mathcal{M}$, it follows from $M_1 = \sum_{j=0}^{q} \lambda_j N_j$ that for each $b \in \mathbf{n}$, we have

$$\begin{aligned} g(x_b; \mathbf{X}_{\mathfrak{n}_1^b}, \cdots, \mathbf{X}_{\mathfrak{n}_q^b}) &= \lambda_0 f(x_b; x_b) + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{s_j} f(x_0; x_{\mathfrak{n}_{ji}^b}) \\ &= f(x_b, x_{\mathfrak{m}^b}). \end{aligned}$$

Therefore g defines the required system $\mathcal{G} \in \mathcal{N}$.

Example 4.4. (Illustration of Lemma 2.12) Let $\mathcal{N}$ be the network with non-identity adjacency matrices $N_1 = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathcal{P}$ be the network with non-identity adjacency matrices $P_1 = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, $P_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. It is straightforward to check $P_1 = N_1 + N_2$, $P_2 = 2N_0$ and so $\mathbf{A}(\mathcal{P}) \subseteq \mathbf{A}(\mathcal{N})$. Hence, by lemma 2.12, $\mathcal{P} \prec_O \mathcal{N}$. Suppose that $\mathcal{F} \in \mathcal{P}$ has model $f : M \times M^6 \times M^2 \to TM$. Since $\mathcal{P} \prec_O \mathcal{N}$, there exists $\mathcal{G} \in \mathcal{N}$ with model g such that $\mathcal{F}$ is output dominated by $\mathcal{G}$. The relation between g and f is given by

$$g(x_0; \overline{x_1, \cdots, x_4}, \overline{x_5, x_6}) = f(x_0; \overline{x_1, \cdots, x_4, x_5, x_6}, \overline{x_0, x_0})$$

The differential equations for $\mathcal{F}$ and $\mathcal{G}$ are respectively given by

$$\begin{aligned} x_1' &= f(x_1; \overline{x_1, x_1, x_1, x_2, x_1, x_2}, \overline{x_1, x_1}) \\ x_2' &= f(x_2; \overline{x_2, x_2, x_3, x_3, x_2, x_2}, \overline{x_2, x_2}) \\ x_3' &= f(x_3; \overline{x_1, x_1, x_2, x_2, x_3, x_3}, \overline{x_3, x_3}) \end{aligned}$$

and

$$\begin{aligned} x_1' &= g(x_1; \overline{x_1, x_1, x_1, x_2}, \overline{x_1, x_2}) \\ x_2' &= g(x_2; \overline{x_2, x_2, x_3, x_3}, \overline{x_2, x_2}) \\ x_3' &= g(x_3; \overline{x_1, x_1, x_2, x_2}, \overline{x_3, x_3}). \end{aligned}$$

Example 4.5. (Illustration of lemma 2.13) Let $\mathcal{P}$ be the network of example 4.4 and $\mathcal{Q}$ be the network with non-identity adjacency matrix $Q_1 = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$. It is straightforward to check $Q_1 = P_1 - P_2$ and so $\mathbf{A}(\mathcal{Q}) \subseteq \mathbf{A}(\mathcal{P})$. Hence, by lemma 2.13, we have $\mathcal{Q} \prec_O \mathcal{P}$. Suppose that $\mathcal{F} \in \mathcal{Q}$ has model $f : M \times M^4 \to TM$. Then there exists $\mathcal{G} \in \mathcal{P}$ with model g such that $\mathcal{F}$ is output dominated by $\mathcal{G}$. The relation between f and g is given by

$$\begin{aligned} g(x_0,\overline{x_1,\cdots,x_6},\overline{x_7,x_8}) &= \sum_{1\le i_1<\cdots<i_4\le 6} f(x_0;\overline{x_{i_1},\cdots,x_{i_4}}) \\ &- \sum_{\substack{1\le i_1<\cdots<i_3\le 6\\ a\in A}} f(x_0;\overline{x_a,x_{i_1},x_{i_2},x_{i_3}}) \\ &+ \sum_{\substack{1\le i_1<i_2\le 6\\ b=(b_1,b_2)\in B}} f(x_0;\overline{x_{b_1},x_{b_2},x_{i_1},x_{i_2}}) \\ &- \sum_{\substack{1\le i_1\le 6\\ c=(c_1,c_2,c_3)\in C}} f(x_0;\overline{x_{c_1},x_{c_2},x_{c_3},x_{i_1}}) \\ &+ \sum_{d=(d_1,\cdots,d_4)\in D} f(x_0;\overline{x_{d_1},\cdots,x_{d_4}}), \end{aligned}$$

where

$$\begin{aligned} &A=\{(7),(8)\},\quad B=\{(7,7),(8,8),(7,8)\},\\ &C=\{(7,7,7),(7,7,8),(7,8,8),(8,8,8)\},\\ &D=\{(7,7,7,7),(7,7,7,8),(7,7,8,8),(7,8,8,8),(8,8,8,8)\}. \end{aligned}$$

Since $x_7 = x_8 = x_0$ for each cell, we can obtain a simplified expression by replacing x_7 and x_8 by x_0. Thus we may define the new cell class **D** for the network $\mathcal{P}$ as in figure 3.

The differential equations for $\mathcal{G}$ are given in example 4.4 and the differential equations for $\mathcal{F}$ are given by

$$\begin{aligned} x_1' &= f(x_1;\overline{x_1,x_1,x_2,x_2})\\ x_2' &= f(x_2;\overline{x_2,x_2,x_3,x_3})\\ x_3' &= f(x_3;\overline{x_1,x_1,x_2,x_2}) \end{aligned}$$

Example 4.6. (Illustration of lemma 2.14) Let $\mathcal{Q}$ be the network of example 4.5 and $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix}1&0&1\\1&1&1\\0&1&0\end{pmatrix}$. Since $M_1 = \frac{Q_1}{2}$, we have by lemma 2.14 that $\mathcal{M} \prec_O \mathcal{Q}$. Suppose that $\mathcal{F}\in\mathcal{M}$ has model $f : M\times M^2 \to TM$. The

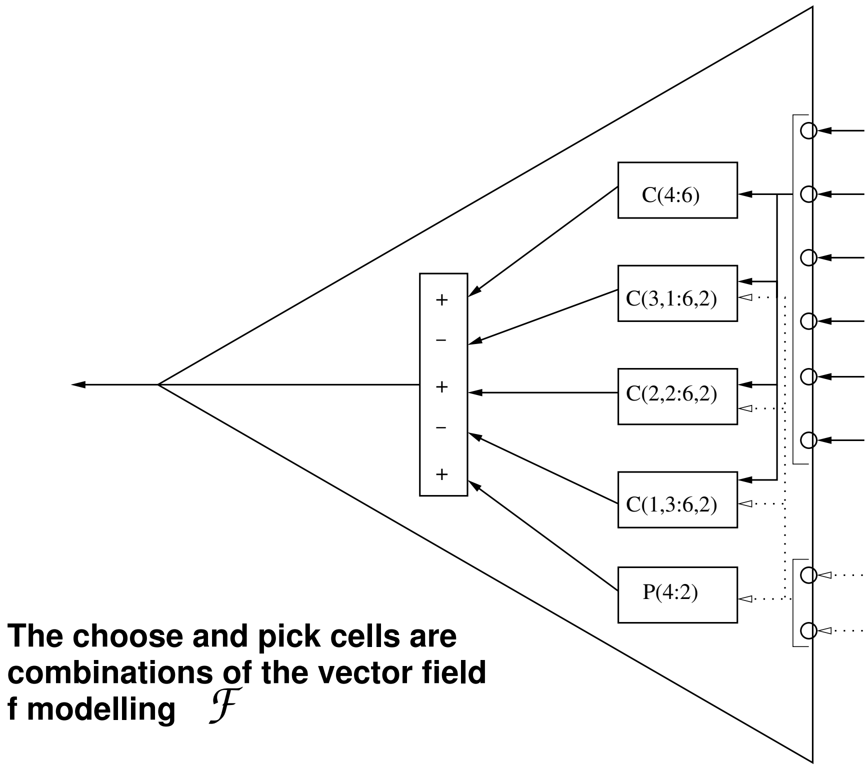

FIGURE 3. The cell **D**

differential equations for $\mathcal{F}$ are given by

$$\begin{aligned} x_1' &= f(x_1; \overline{x_1, x_2}) \\ x_2' &= f(x_2; \overline{x_2, x_3}) \\ x_3' &= f(x_3; \overline{x_1, x_2}) \end{aligned}$$

Since $\mathcal{M} \prec_O \mathcal{Q}$, there exists $\mathcal{G} \in \mathcal{Q}$ with model g such that $\mathcal{F}$ is output dominated by $\mathcal{G}$. The relation between f and g is given by

$$g(x_0; \overline{x_1, \cdots, x_4}) = \sum_{j \in \mathcal{J}} C_j \sum_{i_1, \cdots, i_s \in \mathbf{4}} f(x_0; \overline{x_{i_1}^{j_1}, \cdots, x_{i_s}^{j_s}})$$

where, $\mathcal{J} = \{(1,1), (2)\}$ (lemma 2.14). Setting $a = C_{(1,1)}, b = C_{(2)}$, we have

$$g(x_0; \overline{x_1, \cdots, x_4}) = a \sum_{i_1, i_2 \in \mathbf{4}} f(x_0; \overline{x_{i_1}, x_{i_2}}) + b \sum_{i_1 \in \mathbf{4}} f(x_0; \overline{x_{i_1}, x_{i_1}}).$$

After substituting $x_1 = x_2 = u$, $x_3 = x_4 = v$, we get the following terms: $f(x_0; \overline{u, u})$, $f(x_0; \overline{u, v})$, $f(x_0; \overline{v, v})$. The coefficient for (u, u) and (v, v) is $2a\binom{2}{2} + b\binom{2}{1}$ and for (u, v) is $2a\binom{2}{1}\binom{2}{1}$. Since we require $g(x_0; \overline{u, u, v, v}) = f(x_0; \overline{u, v})$, we obtain the system of linear equations

$$\begin{bmatrix} \binom{2}{1} & 2\binom{2}{2} \\ 0 & 2\binom{2}{1}\binom{2}{1} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving we obtain $a = \frac{1}{8}, b = -\frac{1}{8}$. We define the new cell class **D** for the network $\mathcal{Q}$ as in figure 4.

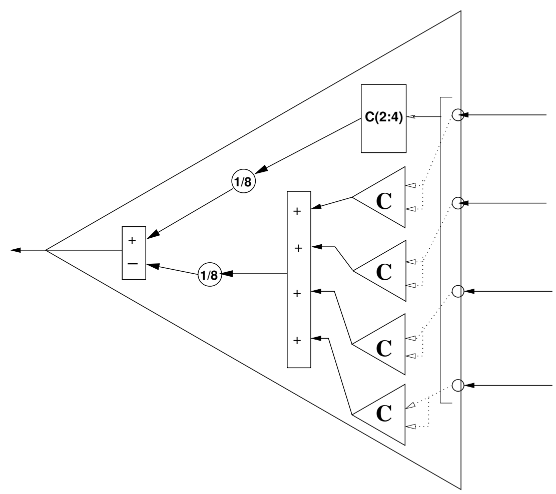

FIGURE 4. The cell **D**

Example 4.7. (Illustration of lemma 2.15.) Let $\mathcal{N}$ be the network defined in example 4.4 and $\mathcal{M}$ be the network of example 4.6. It is easy to check that $M_1 = \frac{N_1}{2} + \frac{N_2}{2} - N_0$. Following the notation of the proof of lemma 2.15, $\lambda = 2$, $P = N_1 + N_2$, $Q = 2N_0$. Suppose that $\mathcal{G} \in \mathcal{N}$ has model g and $\mathcal{F} \in \mathcal{M}$ has model f. The differential equations for $\mathcal{G}$ are given in example 4.4 and the differential equations for $\mathcal{F}$ are given in example 4.6. $\mathcal{P} \prec_O \mathcal{N}$ is proved in example 4.4, $\mathcal{Q} \prec_O \mathcal{P}$ is proved in example 4.5, $\mathcal{M} \prec_O \mathcal{Q}$ is proved in example 4.6. Since $\prec_O$ is a transitive relation, we see that $\mathcal{M} \prec_O \mathcal{N}$ and so we have an output relation between f and g.

Example 4.8. (Example of networks which are both input and output equivalent.) Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Note that $N_1 = 2M_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M})$. We will show that $\mathcal{M} \sim_O \mathcal{N}$ and $\mathcal{M} \sim_I \mathcal{N}$. Suppose that $\mathcal{G} \in \mathcal{N}$ has model g. Define

$$f(x_0;\overline{x_1,x_2}) = g(x_0;\overline{x_1,x_2,x_1,x_2})$$

Then f models a system $\mathcal{F} \in \mathcal{M}$ and it follows easily that $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{N} \prec_O \mathcal{M}$. Conversely, if we are given a system $\mathcal{F}$ in $\mathcal{M}$ with model f and we define g by

$$g(x_0;\overline{x_1,\cdots,x_4}) = \frac{1}{8}\sum_{1\le i\le j\le 4} f(x_0;\overline{x_i,x_j}) - \frac{1}{8}\sum_{1\le i\le 4} f(x_0;\overline{x_i,x_i})$$

then g models a system $\mathcal{G} \in \mathcal{N}$ and we see that $\mathcal{M} \prec_O \mathcal{N}$. If instead we define g by the input relation

$$g(x_0;\overline{x_1,x_2,x_3,x_4}) = f(x_0;x_0,\frac{x_1+x_2+x_3+x_4}{2} - x_0),$$

we deduce that g models $\mathcal{G} \in \mathcal{N}$ and that $\mathcal{M} \prec_I \mathcal{N}$.

Example 4.9. (Example where output domination holds, input domination fails.) Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix}1&1&1\\0&1&1\\1&0&0\end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix}2&2&2\\0&2&2\\2&0&0\end{pmatrix}$. We have $M_1 = N_1/2$. If $\mathcal{F} \in \mathcal{M}$ has model f, and we define g by

$$g(x_0;x_1,\cdots,x_4) = \frac{1}{8}\sum_{1\le i\le j\le 4} f(x_0;x_i,x_j) - \frac{1}{8}\sum_{1\le i\le 4} f(x_0;x_i,x_i),$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with identical dynamics to $\mathcal{F}$ and so $\mathcal{M} \prec_O \mathcal{N}$. On the other hand, suppose we take

$$g(x_0;x_1,\cdots,x_4) = f(x_0;\mathbf{L}(x_0;x_1,\cdots,x_4)),$$

where, $\mathbf{L} : V \times V^4 \to V^2$ is a linear map. As described in section 3, L must be of the form $\begin{pmatrix}a&b&b&b&b\\c&d&d&d&d\end{pmatrix}$ for some $a,b,c,d \in \mathbb{Q}$. It is easy to check that there is no choice of a,b,c,d for which $z' = g(z;x,y,x,y) = f(z;x,y)$. Hence $\mathcal{M} \not\prec_I \mathcal{N}$.

Example 4.10. (Example where output equivalence holds but input domination fails both ways.) Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 3 & 0 & 0 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix}$. We have $M_1 = \frac{3}{2}N_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$. By lemma 2.16, $\mathcal{M} \sim_O \mathcal{N}$. Suppose

$$g(x_0; x_1, \cdots, x_4) = f(x_0; \mathbf{L}(x_0; x_1, \cdots, x_4)),$$

where $\mathbf{L} : V \times V^4 \to V^6$ is a linear map. As described in section 3, L must either be of the form $[G_{\mathcal{N}} L^1, L^2, L^3]$ where $L^1 = (0; a, b, b, b)$, $L^2 = (0, c, c, c, c)$, $L^3 = (0, d, d, d, d)$ for some $a, b, c, d \in \mathbb{Q}$ or $[L^1, \cdots, L^6]$ where $L^i = (0; a_i, a_i, a_i, a_i)$ for some $a_i \in \mathbb{Q}$, $i \in \mathbf{6}$. It is easy to check that there is no choice of a, b, c, d, a_i for which $z' = g(z; x, y, x, y) = f(z; x, y, x, y, x, y)$. This shows $\mathcal{M} \not\prec_I \mathcal{N}$. Similarly, suppose that $\mathcal{G} \in \mathcal{N}$ has model g and define f by

$$f(x_0; x_1, \cdots, x_6) = g(x_0; \mathbf{L}(x_0; x_1, \cdots, x_6)),$$

where $\mathbf{L} : V \times V^6 \to V^4$ is a linear map. As described in section 3, L must be of the form $[L^1, \cdots, L^4]$ where $L^i = (0; a_i, a_i, a_i, a_i, a_i, a_i)$ for some $a_i \in \mathbb{Q}$, $i \in \mathbf{4}$. It is easy to check that there is no choice of a_i for which $z' = f(z; x, y, x, y, x, y) = g(z; x, y, x, y)$. This shows $\mathcal{N} \not\prec_I \mathcal{M}$.

Example 4.11. For every network $\mathcal{M}$ we have $\mathcal{M} \sim_O \mathcal{M}$. However, there may be many ways of achieving this self output equivalence. For example, consider the two cell network $\mathcal{M}$ with asymmetric inputs shown in figure 5.

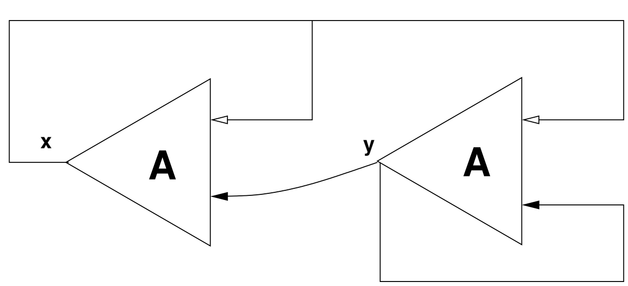

FIGURE 5. A two cell network $\mathcal{M}$ with asymmetric inputs

Suppose $\mathcal{F} \in \mathcal{M}$ has model f. It can be shown that the two-parameter family defined for $c, d \in \mathbb{R}$ by

$$\begin{aligned} f_{c,d}(x_0; x_1, x_2) = {} & cf(x_0; x_0, x_0) + df(x_0; x_0, x_1) \\ & - (c+d)f(x_0; x_0, x_2) - (c+d)f(x_0; x_1, x_0) \\ & + (1+c+d)f(x_0; x_1, x_2) + df(x_0; x_2, x_0) \\ & - df(x_0; x_2, x_1), \end{aligned}$$

gives all output equivalences $\mathcal{M} \sim_O \mathcal{M}$. For example, if we take $c = 0$, $d = -1/2$, then

$$\begin{aligned} g(x_0; x_1, x_2) &= f_{0,-1/2}(x_0; x_1, x_2) \\ &= \frac{1}{2}(-f(x_0; x_0, x_1) + f(x_0; x_0, x_2) + f(x_0; x_1, x_0) \\ &\qquad + f(x_0; x_1, x_2) - f(x_0; x_2, x_0) + f(x_0; x_2, x_1)). \end{aligned}$$

In particular, if we define the new cell class $\mathbf{A}^\star$ as in figure 6, then although the cell is *different* from the original cell $\mathbf{A}$, when it is incorporated in the network $\mathcal{M}$, it will give the same dynamics.

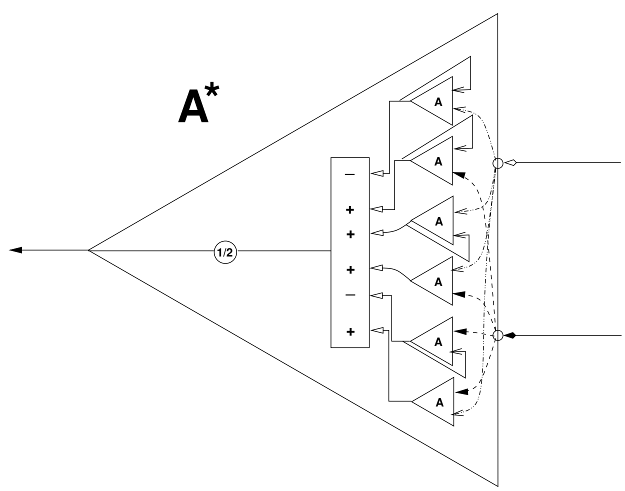

FIGURE 6. The cell $\mathbf{A}^\star$

This construction leads naturally to a number of observations and questions and we conclude by briefly describing some of these issues. First, to what extent can this process be reversed? That is, given a network of 'complex' cells, when is it equivalent to the same network but built of simpler cells? Secondly, is there a way of choosing the specific output equivalence so as to protect against failure of individual units comprising the new cells? For example, if we build the network $\mathcal{M}$ from the cells $\mathbf{A}^\star$, what is the effect on network dynamics of the failure of a single $\mathbf{A}$-cell in $\mathbf{A}^\star$? Is there an optimal way of choosing the output equivalence so as minimize the effect of failure of individual units? Are there potential applications to numerical analysis (for example, in the solution of partial differential equations)? There are also questions related to the effects of *inflation* [2] on $\mathbf{A}$-cells in $\mathbf{A}^\star$ as well as to extending the notion of input and output equivalence to allow for non-linearities and/or restricting to scalar signalling networks [1].

References

[1] N Agarwal and M Field. 'Dynamical equivalence of networks of coupled dynamical systems', preprint 2009.

[2] M Aguiar, A P S Dias, P Ashwin, and M Field. 'Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation', preprint, 2009.

[3] A P S Dias and I Stewart. 'Linear equivalence and ODE-equivalence for coupled cell networks', *Nonlinearity* **18** (2005), 1003–1020.

[4] M J Field. 'Combinatorial Dynamics', *Dynamical Systems* **19** (2004), 217–243.

Nikita Agarwal, Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA

E-mail address: nagarwal@math.uh.edu

Michael Field, Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA

E-mail address: mf@uh.edu