Math 1310
Section 4.1: Polynomial Functions and Their Graphs

A polynomial function is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$$

where $a_n \neq 0,\ a_0, a_1, \ldots, a_n$ are real numbers and *n* is a whole number.

The degree of the polynomial function is *n*. We call the term $a_n x^n$ the leading term, and a_n is called the leading coefficient.

$$P(0) = a_0$$

Our objectives in working with polynomial functions will be, first, to gather information about the graph of the function and, second, to use that information to generate a reasonably good graph without plotting a lot of points. In later examples, we'll use information given to us about the graph of a function to write its equation.

Graph Properties of Polynomial Functions

Let *P* be any *n*th degree polynomial function with real coefficients. The graph of *P* has the following properties.

1. *P* is continuous for all real numbers, so there are no breaks, holes, jumps in the graph.
2. The graph of *P* is a smooth curve with rounded corners and no sharp corners.
3. The graph of *P* has at most *n* *x*-intercepts.
4. The graph of *P* has at most *n* – 1 turning points.

Example 1: Given the following polynomial functions, state the leading term, the degree of the polynomial and the leading coefficient.

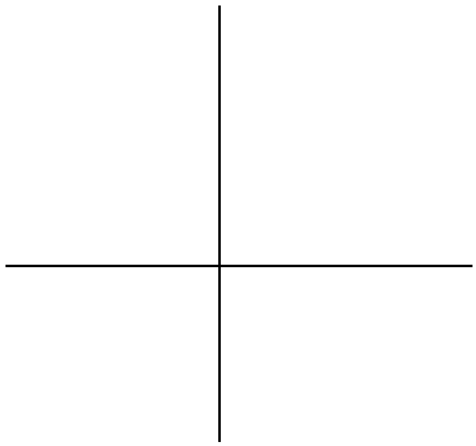
a. $P(x) = 6x^4 - 4x^3 + 7x - 2$

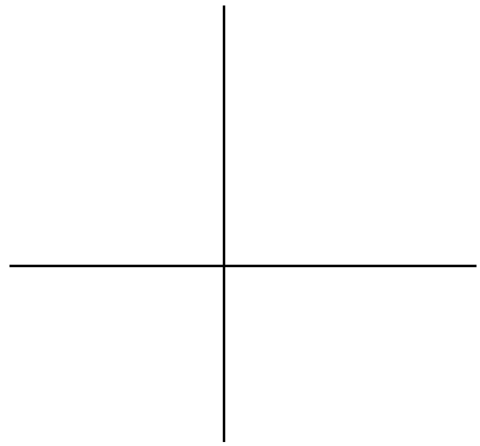
b. $P(x) = (3x + 4)(x + 1)^2(x + 5)^3$

We'll start with the shapes of the graphs of functions of the form $f\ (x) = x^n\ , n > 0$.

You should be familiar with the graphs of $f\ (x) = x^2$ and $g(x) = x^3$.

The graph of $f\ (x) = (x - a)^n\ , n > 0$, n is even, will look similar the graph of $f\ (x) = x^2$, and the graph of $f\ (x) = (x - a)^n\ , n > 0$, n is odd, will look similar the graph of $f\ (x) = x^3$.



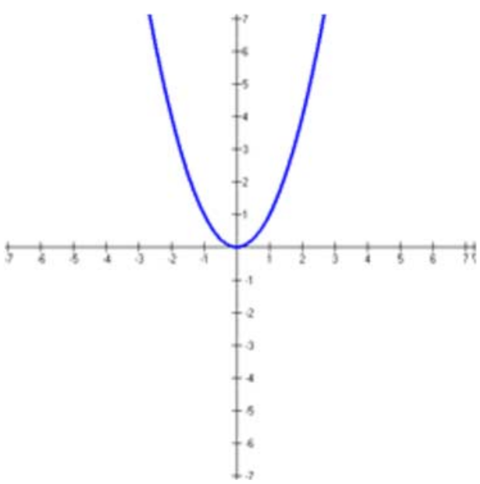


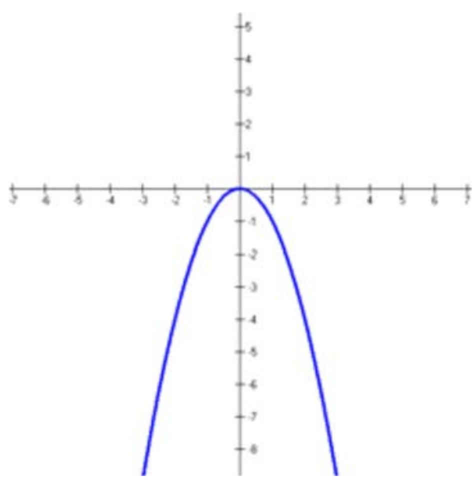
Next, you will need to be able to describe the end behavior of a function.

End Behavior of Polynomial Functions

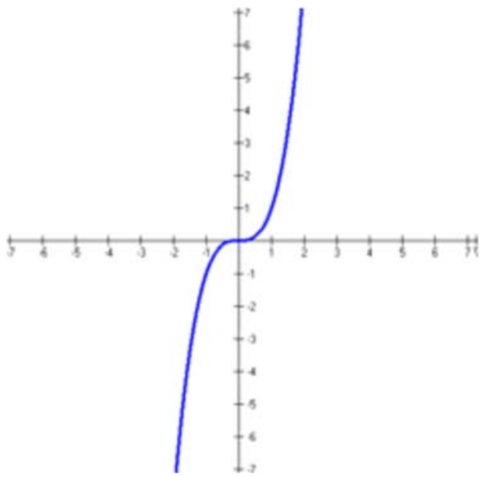
The behavior of a graph of a function to the far left or far right is called its **end behavior.**
The end behavior of a polynomial function is revealed by the leading term of the polynomial function.

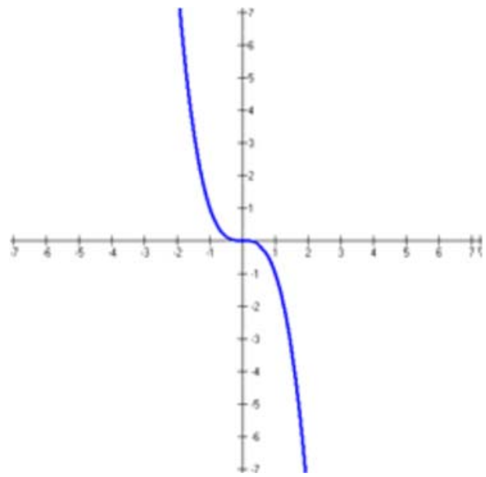
1. Even-degree polynomials look like . $y = \pm x^2$





2. Odd-degree polynomials look like. $y = \pm x^3$





Next, you should be able to find the *x* intercept(s) and the *y* intercept of a polynomial function.

Zeros of Polynomial Functions

You will need to set the function equal to zero and then use the Zero Product Property to find the *x*-intercept(s). That means if $ab = 0$, then either $a = 0$ or $b = 0$. To find the *y* intercept of a function, you will find $f(0)$.

Example 2: Find the zeros of:

a. $f(x) = x^4 - x^2$

b. $f(x) = -3x\left(x + \frac{1}{2}\right)(x - 4)^3$

In some problems, one or more of the factors will appear more than once when the function is factored. The power of a factor is called its multiplicity. So given $P(x) = x^3(x-3)^2(x+2)^1$, then the multiplicity of the first factor is 3, the multiplicity of the second factor is 2 and the multiplicity of the third factor is 1.

Description of the Behavior at Each *x*-intercept

1. Even Multiplicity: The graph touches the *x*-axis, but does not cross it. It looks like a parabola there.

2. Multiplicity of 1: The graph crosses the *x*-axis. It looks like a line there.

3. Odd Multiplicity greater than or equal 3: The graph crosses the *x*-axis. It looks like a cubic there.

You can use all of this information to generate the graph of a polynomial function.

- degree of the function
- end behavior of the function
- *x* and *y* intercepts (and multiplicities)
- behavior of the function through each of the *x* intercepts (zeros) of the function

Steps to graphing other polynomials:

1. Determine the **leading term**. Is the degree even or odd? Is the sign of the leading coefficient positive or negative?
2. Determine the **end behavior**. Which one of the 4 cases will it look like on the ends?
3. Factor the polynomial.
4. Make a table listing the factors, *x* intercepts, multiplicity, and describe the behavior at each *x* intercept.
5. Find the *y*- intercept.
6. Draw the graph, being careful to make a nice smooth curve with no sharp corners.

Note: without calculus or plotting lots of points, we don't have enough information to know how high or how low the turning points are

Example 3: Find the x and y intercepts. State the degree of the function. Sketch the graph of

$$f(x) = x^3 + 4x^2 + 4x$$

Example 4: Find the x and y intercepts. State the degree of the function. Sketch the graph of

$$P(x) = (x-3)^2(x+1)^5(x+2)^3$$

Example 5: Find the x and y intercepts. State the degree of the function. Sketch the graph of

$$g(x) = (3 - x)(x + 1)(x + 5)^2$$

Example 6: Write the equation of the cubic polynomial $P(x)$ that satisfies the following conditions: zeros at $x = 3$, $x = -1$, and $x = 4$ and passes through the point (-3, 7)

Example 7: Write the equation of the quartic function with *y* intercept 4 which is tangent to the *x* axis at the points (-1, 0) and (1, 0).