Section 4.4: Rational Function and their Graphs

The objective in this section will be to identify the important features of a rational function and then to use them to sketch an accurate graph of the function.

A **rational function** can be expressed as $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

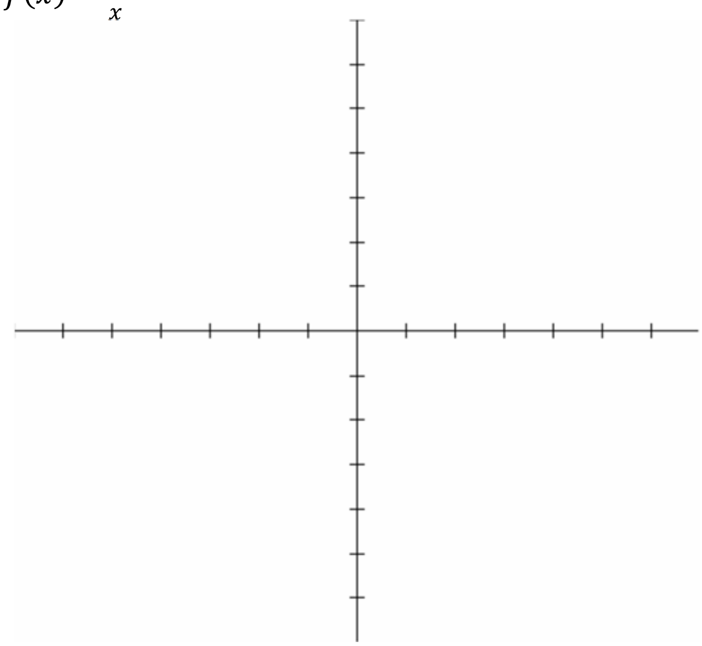
Example 1: Find the domain of $f(x) = \frac{x-2}{x^2-9}$

The features you will want to identify are:
- the locations of any holes in the graph of the function
- the locations of any vertical or horizontal asymptotes
- the locations of any *x* or *y* intercepts

Vertical Asymptote of Rational Functions

The line $x = a$ is a **vertical asymptote** of the graph of a function *f* if *f*(*x*) increases or decreases without bound as *x* approaches a.

Basic example is $f(x) = \frac{1}{x}$



Locating Vertical Asymptotes and Holes

Factor the numerator and denominator. Look at each factor in the denominator.

- If a factor cancels with a factor in the numerator, then there is a hole where that factor equals zero.
- If a factor does not cancel, then there is a vertical asymptote where that factor equals zero.

Example 2: Find any vertical asymptote(s) and/or hole(s) of $f(x) = \dfrac{x^2 - 3x - 10}{x^2 - x - 6}$

Example 3: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 + 3x - 4}{x^2 - 9}$$

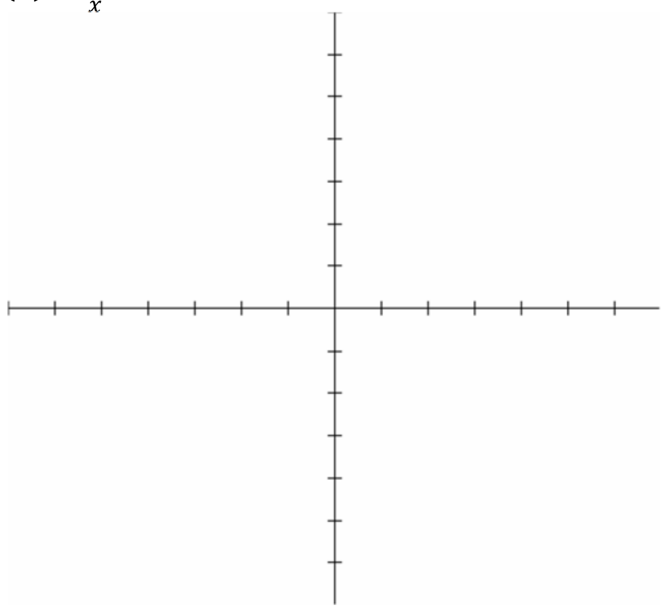
Example 4: Find any vertical asymptote(s) and/or hole(s) of

$$f(x) = \frac{x^2 - 16}{x^2 - 2x - 8}$$

Horizontal Asymptote of Rational Functions

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

Again we look at $f(x) = \frac{1}{x}$



Horizontal asymptotes really have to do with what happens to the y-values as x becomes very large or very small. If the y-values approach a particular number at the far left and far right ends of the graph, then the function has a horizontal asymptote.

Note: A rational function may have several vertical asymptotes, but only at most one horizontal asymptote. Also, a graph cannot cross a vertical asymptote, but may cross a horizontal asymptote.

Locating Horizontal Asymptotes

To find the location of any horizontal asymptote, determine the degree of the numerator and the degree of the denominator. Then

- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is $y = 0$.
- If the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is $y = \frac{a}{b}$, where a and b are the leading coefficients of the numerator and denominator
- If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

Example 5: Find the horizontal, if there is one

$$f(x) = \frac{x+2}{x^2+6x+9}$$

Example 6: Find the horizontal, if there is one

$$f(x) = \frac{3x^4 + 12x^2 + 12}{x^4 + 7x^3 + 10}$$

Example 7: Find the horizontal, if there is one

$$f(x) = \frac{x^2 + 3x + 2}{3x + 6}$$

Steps to Graphing a Rational Function

1. Factor numerator and denominator. If a factor in the numerator cancels with a factor in the denominator then there is a **hole** in the graph when that cancelled factor equal zero.
2. Find x-intercept(s) by setting numerator equal to zero.
3. Find y-intercept (if there is one) by substituting $x = 0$ in the function.
4. Find horizontal asymptote (if there is one).
5. Find vertical asymptote, if any, by setting the denominator equal to zero.
6. Use the x-intercept(s) and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.
7. Graph! *Except for the breaks at the vertical asymptotes, the graph should be a nice smooth curve with no sharp corners.*

Example 8: Sketch the graph of

$$f(x) = \frac{1}{x-1}$$

Example 9: Sketch the graph of

$$f(x) = \frac{x^2 + x - 6}{x^2 + 3x - 10}$$

Example 10: Sketch the graph of

$$f(x) = \frac{x^2 + 5}{x^2 - 4}$$