Example 5: Use right endpoints and 4 subdivisions of the interval to approximate the area under \( f(x) = 2x^2 + 1 \) on the interval \([0, 2] \).

\[
\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}
\]

1. \([0, \frac{1}{2}]\), \( x_1 = \frac{1}{2} \)
2. \([\frac{1}{2}, 1] \), \( x_2 = 1 \)
3. \([1, \frac{3}{2}] \), \( x_3 = \frac{3}{2} \)
4. \([\frac{3}{2}, 2] \), \( x_4 = 2 \)

\[
A = \left( f(x_0) + f(x_1) + \ldots + f(x_n) \right) \cdot \Delta x
\]

\[
= \left( 9 + 1 + \frac{9}{4} + \frac{5}{8} + \frac{57}{8} \right) \cdot \frac{1}{2}
\]

\[
= \left( \frac{11b}{4} \right) \cdot \frac{1}{2} = \frac{11b}{16} = \frac{29}{4} = 7.25
\]

Example 6: Use midpoints and 4 subdivisions of the interval to approximate the area under \( f(x) = 2x^2 + 1 \) on the interval \([0, 2] \).

\[
\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}
\]

1. \([0, \frac{1}{2}] \), \( x_1 = \frac{\sqrt{2}+10}{2} = 0.75 \)
2. \([\frac{1}{2}, 1] \), \( x_2 = \frac{13+4}{2} = 0.75 \)
3. \([1, \frac{3}{2}] \), \( x_3 = 1.25 \)
4. \([\frac{3}{2}, 2] \), \( x_4 = 1.75 \)

\[
A = \left( f(x_{mid1}) + f(x_{mid2}) + f(x_{mid3}) + f(x_{mid4}) \right) \cdot \Delta x
\]

\[
= \left( 9 + \frac{7}{8} + \frac{57}{8} + \frac{57}{8} \right) \cdot \frac{1}{2}
\]

\[
= \left( \frac{11b}{4} \right) \cdot \frac{1}{2} = \frac{11b}{16} = \frac{29}{4} = 7.25
\]
Example 7: Suppose \( f(x) = 1 + 3x \). Approximate the area under the graph of \( f \) on the interval \([0, 12]\) using 6 subdivisions and left endpoints.

\[
\Delta x = \frac{b-a}{n} = \frac{12-0}{6} = 2
\]

\[
\begin{align*}
0 & \quad 2 & \quad 4 & \quad 6 & \quad 8 & \quad 10 & \quad 12 \\
0,2 & \quad x_1 = 0 \\
2,4 & \quad x_2 = 2 \\
4,6 & \quad x_3 = 4 \\
6,8 & \quad x_4 = 6 \\
8,10 & \quad x_5 = 8 \\
10,12 & \quad x_6 = 10
\end{align*}
\]

\[
A = \left( f(0) + f(2) + f(4) + f(6) + f(8) + f(10) \right) \cdot 2
\]

\[
= (1 + 7 + 13 + 19 + 25 + 31) \cdot 2
\]

\[
= (96) \cdot 2
\]

\[
= 192
\]

The Definite Integral

Let \( f \) be defined on \([a, b]\). If \( \lim_{n \to \infty} \left( f(x_1) + f(x_2) + \ldots + f(x_n) \right) \Delta x \) exists for all choices of representative points in the \( n \) subintervals of \([a, b]\) of equal width \( \Delta x = \frac{b-a}{n} \), then this limit is called the **definite integral of \( f \) from \( a \) to \( b \)**. The definite integral is noted by

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \left( f(x_1) + f(x_2) + \ldots + f(x_n) \right) \Delta x.
\]

The number \( a \) is called the **lower limit** of integration and the number \( b \) is called the **upper limit** of integration.

A function is said to be **integrable** on \([a, b]\) if it is continuous on the interval \([a, b]\).
The definite integral of a nonnegative function:

The definite integral of a general function:

From this section, you should be able to
- Explain the procedure used to approximate area under a curve
- Use Riemann sums to approximate the area under a curve using right endpoints, left endpoints or midpoints
- Explain what we mean by definite integral of a non-negative function or a general function.
Math 1314
Lesson 19
The Fundamental Theorem of Calculus

In the last lesson, we approximated the area under a curve by drawing rectangles, computing the area of each rectangle and then adding up their areas. We saw that the actual area was found as we let the number of rectangles get arbitrarily large. Computing area in this manner is very tedious, so we need another way to find the area.

The fundamental theorem of calculus allows us to do just this. It establishes a relationship between the antiderivative of a function and its definite integral.

The Fundamental Theorem of Calculus

Let \( f \) be a continuous function on \([a, b]\). Then
\[
\int_a^b f(x)dx = F(b) - F(a)
\]
where \( F(x) \) is any antiderivative of \( f \). If you are interested in seeing why this works, see the link for the “proof” of the fundamental theorem of calculus on Marjorie Marks’ class notes page.

Example 1: Suppose \( f(x) = 2x + 1 \). Find the area under the graph of \( f \) from \( x = 1 \) to \( x = 3 \).

\[
\int_1^3 (2x+1) dx = x^2+x+C
\]

\[
F(3) - F(1) = (3^2+3+C) - (1^2+1+C) = 10
\]

Example 2: Evaluate \( \int_0^2 (6x - 4e^x)dx \)

The antiderivative is \( 2x^3 - 4e^x \).

\[
F(2) - F(0) = (2(2)^3 - 4e^2) - (2(0)^3 - 4e^0) = 20 - 4e^2
\]

\[
16 - 4e^2 - (-4) = 20 - 4e^2
\]
Example 3: Evaluate: \( \int_0^3 (3x^2 + 4x - 7) \, dx \)

Find antiderivative:

\[
\begin{align*}
F(x) &= x^3 + 2x^2 - 7x \\
\left. F(x) \right|_0^3 &= (3^3 + 2(3)^2 - 7(3)) - (0^3 + 2(0)^2 - 7(0)) \\
&= (27 + 18 - 21) - (0 + 0 - 0) \\
&= 24 - (-4) \\
&= 28
\end{align*}
\]

Popper 1: Evaluate

\[
\int_0^5 x \, dx
\]

a. \( \frac{5}{2} \)
b. \( 5 \)
c. \( \frac{25}{2} \)
d. \( 25 \)
Example 4: Evaluate: \[ \int_1^4 \left( \frac{1}{x} - \frac{1}{x^2} \right) \, dx \]

\[ \left. \ln |x| + \frac{1}{x} \right|_1^4 \]

\[ \ln 4 + \frac{1}{4} - (\ln 1 + 1) \]

\[ \ln 4 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4} \]

Example 5: Evaluate: \[ \int_2^5 \frac{2x^2 - 4x + 6}{x} \, dx \]

\[ \left. x^2 - 4x + 6 \ln |x| \right|_2^5 \]

\[ (5^2 - 4(5) + 6 \ln 5) - (2^2 - 4(2) + 6 \ln 2) \]

\[ (25 - 20 + 6 \ln 5) - (4 - 8 + 6 \ln 2) \]

\[ 5 + 6 \ln 5 - (4 + 6 \ln 2) \]

\[ 9 + 6 \ln 5 - 6 \ln 2 = 9 + 6(\ln 5 - \ln 2) \]

Popper 2: Evaluate:

\[ \int_1^3 \frac{3}{x} \, dx \]

a. 2
b. 9
c. 6
d. 3
Example 6: Evaluate: \( \int_0^3 (e^x - x + 1) \, dx \)

\[
e^x - \frac{x^2}{2} + x \bigg|_0^3
\]

\[
= \left( e^3 - \frac{3^2}{2} + 3 \right) - \left( e^0 - \frac{(0)^2}{2} + 0 \right)
\]

\[
= \left( e^3 - \frac{9}{2} + \frac{3}{2} \right) - \left( 1 - 0 + 0 \right)
\]

\[
e^3 - \frac{3}{2} - 1 = e^3 - \frac{5}{2}
\]

Example 7: Evaluate: \( \int_0^2 (x^2 - 3)(x - 1) \, dx \)