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NAVIER–STOKES EQUATIONS IN ROTATION FORM: A ROBUST MULTIGRID SOLVER FOR THE VELOCITY PROBLEM*

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Abstract. The topic of this paper is motivated by the Navier–Stokes equations in *rotation* form. Linearization and application of an implicit time stepping scheme results in a linear stationary problem of Oseen type. In well-known solution techniques for this problem such as the Uzawa (or Schur complement) method, a subproblem consisting of a coupled nonsymmetric system of linear equations of diffusion-reaction type must be solved to update the velocity vector field. In this paper we analyze a standard finite element method for the discretization of this coupled system, and we introduce and analyze a multigrid solver for the discrete problem. Both for the discretization method and the multigrid solver the question of robustness with respect to the amount of diffusion and variation in the convection field is addressed. We prove stability results and discretization error bounds for the Galerkin finite element method. We present a convergence analysis of the multigrid method which shows the robustness of the solver. Results of numerical experiments are presented which illustrate the stability of the discretization method and the robustness of the multigrid solver.

Key words. finite elements, multigrid, convection-diffusion, Navier–Stokes equations, rotation form, vorticity

AMS subject classifications. 65N30, 65N55, 76D17, 35J55

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1. Introduction.

The incompressible Navier–Stokes problem written in velocity-pressure variables has several equivalent formulations. Very popular is the *convection* form of the problem: find velocity $\mathbf{u}(t,\mathbf{x})$ and kinematic pressure $p(t,\mathbf{x})$ such that

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in} \quad \Omega\times(0,T], \\ \operatorname{div}\mathbf{u} = 0 \quad &\text{in} \quad \Omega\times(0,T], \end{aligned} \tag{1.1}$$

with given force field $\mathbf{f}$ and viscosity $\nu > 0$. Suitable boundary and initial conditions have to be added to (1.1). One alternative to (1.1) is the *rotation* form of the Navier–Stokes problem:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \nabla P = \mathbf{f} \quad &\text{in} \quad \Omega\times(0,T], \\ \operatorname{div}\mathbf{u} = 0 \quad &\text{in} \quad \Omega\times(0,T], \end{aligned} \tag{1.2}$$

which results from (1.1) after replacing the kinematic pressure by the Bernoulli (or dynamic, or total; cf., e.g., [18]) pressure $P = p + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$ and using the identity $(\mathbf{u}\cdot\nabla)\mathbf{u} = (\operatorname{curl}\mathbf{u})\times\mathbf{u} + \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u})$. In the three-dimensional case $\times$ stands for the vector product and $\operatorname{curl}\mathbf{u} := \nabla\times\mathbf{u}$. In two dimensions, $\operatorname{curl}\mathbf{u} := -\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$ and

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$a \times \mathbf{u} := (-au_2, au_1)^T$ for a scalar a. Linearization and application of an implicit time stepping scheme to (1.2) results in an Oseen-type problem in which the equations are of the form

$$\begin{aligned} -\nu\Delta\mathbf{u} + \mathbf{w} \times \mathbf{u} + \alpha\mathbf{u} + \nabla P &= \mathbf{f} \quad \text{in} \quad \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \end{aligned} \tag{1.3}$$

with $\alpha \geq 0$ and $\mathbf{w} = \operatorname{curl}\mathbf{a}$, where $\mathbf{a}$ is a known approximation of $\mathbf{u}$. Note that the above linearization of $(\operatorname{curl}\mathbf{u}) \times \mathbf{u}$ ensures the ellipticity of (1.3) in a certain sense (cf. section 2). One strategy to solve (1.3) is an Uzawa-type algorithm, in which a Schur complement problem $\mathbf{S}_{\mathrm{rot}}P = \tilde{\mathbf{g}}$ for the pressure has to be solved. The Schur complement operator has the formal representation $\mathbf{S}_{\mathrm{rot}} = -\operatorname{div}(-\nu\Delta + \mathbf{w} \times + \alpha I)^{-1}\nabla$. The operator $(-\nu\Delta + \mathbf{w} \times + \alpha I)^{-1}$ in this Schur complement is the solution operator of the problem

$$\begin{aligned} -\nu\Delta\mathbf{u} + \mathbf{w} \times \mathbf{u} + \alpha\mathbf{u} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

where, for simplicity, we used homogeneous Dirichlet boundary conditions. The exact solution of (1.4) can be replaced by a suitable approximation like in the inexact Uzawa method [3] or in block preconditioners for (1.3) (see, e.g., [11], [19]).

Linearization and application of an implicit time stepping scheme to the convection form (1.1) result in equations as in (1.3) with $\mathbf{w} \times \mathbf{u}$ replaced by $(\mathbf{a} \cdot \nabla)\mathbf{u}$. The Uzawa technique applied to this linear stationary problem for $\mathbf{u}$ and p corresponds to a Schur complement problem with operator $\mathbf{S}_{\mathrm{conv}} = -\operatorname{div}(-\nu\Delta + \mathbf{a} \cdot \nabla + \alpha I)^{-1}\nabla$. The operator $(-\nu\Delta + \mathbf{a} \cdot \nabla + \alpha I)^{-1}$ in this Schur complement is the solution operator of decoupled convection-diffusion(-reaction) problems. Hence in this approach an efficient solver for convection-diffusion equations is of major importance. In the setting of this paper we are particularly interested in finite element discretization methods and multigrid solvers for the discrete problem. There is extensive literature on these solution techniques for convection-diffusion problems; see, e.g., [1], [4], [9], [14], [15], [16], [20], [21], [23], and the references therein. Important topics are appropriate stabilization techniques for the finite element discretization and robustness of the multigrid solvers for convection dominated problems.

In this paper we study the problem (1.4), which can be seen as the counterpart, for the Navier–Stokes equations in rotation form, of the convection-diffusion problems that correspond to the Navier–Stokes problem in convection form. Note that, opposite to the convection-diffusion problems, the problem (1.4) is a coupled system. In this paper we restrict ourselves to the two-dimensional case, since for this case we are able to give complete error analyses for a finite element discretization and a multigrid solver. However, the methodology (see [12]) and all multigrid tools can be extended to the three-dimensional case as well. We allow $\alpha = 0$, which corresponds to the linearization of a *stationary* Navier–Stokes problem in rotation form. We will prove that, under certain reasonable assumptions on the rotation function $\mathbf{w}$, the standard Galerkin finite element discretization method, without any stabilization, is a useful method (see Theorem 3.2 and Remark 3.2). The bounds for the discretization error that are shown to hold are similar to finite element error bounds for scalar linear *reaction-diffusion* problems (as, e.g., in [17], [22]). We consider a multigrid solver for the discrete problem that results from the Galerkin discretization of (1.4) with standard conforming finite elements. It is proved that a multigrid W-cycle method with a canonical prolongation and restriction and a block Richardson smoother is a robust solver for this problem, in the sense that its contraction number (in the Euclidean norm) is bounded by a constant smaller than one independent of all relevant parameters. Although to prove a robust convergence of the multigrid method we need more restrictive assumptions on $\mathbf{w}$, numerical experiments demonstrate good performance of the method, even if such assumptions do not hold. Such a theoretical robustness result is not known for multigrid applied to convection-diffusion problems. Moreover, in the multigrid solver we do not need so-called robust smoothers or matrix-dependent prolongations and restrictions, which are believed to be important for robustness of multigrid applied to convection-diffusion problems. We will show results of numerical experiments that illustrate the stability of the discretization method and the robustness of the multigrid solver. Both in the analysis and the numerical experiments it can be observed that the problem (1.4) resembles a scalar reaction-diffusion problem. Note that from the numerical solution point of view reaction-diffusion equations are believed to be simpler than convection-diffusion equations.

Recently, in [12], a new preconditioning technique for a discretization of the Schur complement operator $\mathbf{S}_{\mathrm{rot}}$ has been introduced, which has good robustness properties with respect to variation in ν and in the mesh size parameter. In this paper we consider only the inner solution operator that appears in the Schur complement operator. Of course, a stabilization may be needed in the outer iterations for (1.3). This subject is addressed in [10], where it is shown that a Petrov–Galerkin-type stabilization method for (1.3) yields optimal error bounds. The possible impact to (1.4) of additional terms resulting from stabilized finite element method for (1.3) is not considered in this paper. Generally, such terms enhance ellipticity of (1.4).

The results in [12], [10], and in the present paper show that for the application of coupled (pressure-velocity) solvers and implicit schemes the rotation form of the Navier–Stokes equations has interesting advantages compared to the convection form. Some numerical experiments with a low order finite element method for rotation form of the incompressible Navier–Stokes equations and comparision with the convection form can be found in [13]. However, relatively little is known about the numerical solution of the Navier–Stokes equations in rotation form, and we believe that this topic deserves further research.

The remainder of the paper is organized as follows. In section 2 notation and assumptions are introduced. Furthermore, continuity and regularity results for the continuous problem are proved. In section 3 the finite element method is treated. We prove discretization error bounds in a problem dependent norm and in the L_2-norm. In section 4 a multigrid solver for the discrete problem is introduced. A convergence analysis is presented that is based on smoothing and approximation properties. In section 5 we show results of a few numerical experiments.

2. Preliminaries and a priori estimates. Let Ω be a convex polygonal domain in $\mathbb{R}^2$. This assumption on Ω will be needed to obtain sufficient regularity, which strongly simplifies the multigrid convergence theory based on the smoothing and approximation property. However, multigrid methods are known to preserve their typical fast convergence, if this assumption is violated.

By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the scalar product and the corresponding norm in $L_2(\Omega)^n, n=1,2$. The standard norm in the Sobolev space $H^k(\Omega)^2$ is denoted by $\|\cdot\|_k$. For $\mathbf{u}=(u_1,u_2),\ \mathbf{v}=(v_1,v_2)\in L_2(\Omega)^2$ we have $(\mathbf{u},\mathbf{v})=(u_1,v_1)+(u_2,v_2)$. The norm on the space $L_\infty(\Omega)$ is denoted by $\|\cdot\|_\infty$.

For a scalar a and vector $\mathbf{v}$ we define the vector product $a\times\mathbf{v}:=(-av_2,av_1)^T$.

We consider the variational formulation of (1.4) in the two-dimensional case: for given $\nu > 0,\ \alpha > 0,\ w \in L_\infty(\Omega),\ \mathbf{f} \in L_2(\Omega)^2$, determine $\mathbf{u} \in \mathbf{U} := H_0^1(\Omega)^2$ such that

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U}, \tag{2.1}$$

where

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v}) + \alpha(\mathbf{u},\mathbf{v}) + (w \times \mathbf{u},\mathbf{v}) \quad \text{for } \mathbf{u},\mathbf{v} \in \mathbf{U}.$$

Here we use the notation $(\nabla\mathbf{u},\nabla\mathbf{v}) := \sum_{i=1}^2 (\nabla u_i, \nabla v_i) = \sum_{i,j=1}^2 (\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j})$.

Throughout the paper we use C to denote some generic strictly positive constant independent of ν, α, and w .

The definition of the vector product implies $(w \times \mathbf{u},\mathbf{v}) = -(w \times \mathbf{v},\mathbf{u})$ for all $\mathbf{u},\mathbf{v} \in L_2(\Omega)^2$, and thus the bilinear form $a(\cdot,\cdot)$ is elliptic:

$$C\,\nu\|\mathbf{u}\|_1^2 \le a(\mathbf{u},\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbf{U}\ .$$

Using $\|w \times \mathbf{u}\| \le \|w\|_\infty \|\mathbf{u}\|$ we obtain the continuity of the bilinear form:

$$a(\mathbf{u},\mathbf{v}) \le (\nu + \alpha + \|w\|_\infty)\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{U}. \tag{2.2}$$

From the Lax–Milgram lemma it follows that the variational problem (2.1) has a unique solution.

For the analysis below we introduce a parameter dependent norm on $\mathbf{U}$:

$$|||\mathbf{u}|||_\tau = \left(\nu\|\nabla\mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2 + \frac{\tau}{\|w\|_\infty}\|w \times \mathbf{u}\|^2\right)^{\frac{1}{2}}, \quad \tau \ge 0.$$

If $w = 0$, then the third term on the right-hand side is dropped. The constant appearing in the Friedrichs inequality is denoted by C_F:

$$\|\varphi\| \le C_F\|\nabla\varphi\| \quad \text{for all } \varphi \in H_0^1(\Omega).$$

The domain Ω is such that for any $g \in L_2(\Omega)$ the solution of the variational problem

$$\text{find } \varphi \in H_0^1(\Omega) \text{ such that } (\nabla\varphi,\nabla v) = (g,v) \ \text{ for all } \ v \in H_0^1(\Omega) \tag{2.3}$$

is an element of $H^2(\Omega)$ and satisfies the regularity estimate $\|\varphi\|_2 \le C_P\|g\|$.

For the analysis in the remainder of this paper the following three conditions are formulated. We denote $c_w := \text{ess inf}_\Omega |w|$.

(A1) Condition (A1) is satisfied if $\alpha + c_w > 0$ and

$$\eta := \frac{\|w\|_\infty}{\alpha + c_w} \le C.$$

(A2) Condition (A2) is satisfied if

$$w(\mathbf{x}) \ge 0 \text{ a.e. in } \Omega \ \text{ or } w(\mathbf{x}) \le 0 \text{ a.e. in } \Omega.$$

(A3) Condition (A3) is fulfilled if $\nabla w \in L_q(\Omega)^2$ for some $q > 2$ and

$$\|\nabla w\|_{L_q} \le C\,\|w\|_\infty.$$

If w is a finite element function, then C is assumed to be independent of h.

In the analysis below it will be explicitly stated which of these conditions are assumed.

Remark 2.1. (A2) holds, for example, if w stems from the effect of Coriolis forces (cf., e.g., [6]); (A1) holds if w is continuous and does not have any zeros in Ω or if in a time stepping scheme we have lower bound for α: $0 < \alpha_{\min} \le \alpha$. □

Note that $(|w|\mathbf{u}, \mathbf{u}) = (|w| \times \mathbf{u}, 1 \times \mathbf{u}) \ge 0$, and thus we have for $\mathbf{u} \in L_2(\Omega)^2$

$$c_w \|\mathbf{u}\|^2 \le (|w| \times \mathbf{u}, 1 \times \mathbf{u}). \tag{2.4}$$

Using $(|w| \times \mathbf{u}, 1 \times \mathbf{u}) \le \||w| \times \mathbf{u}\|\|1 \times \mathbf{u}\| = \|w \times \mathbf{u}\|\|\mathbf{u}\|$ we get

$$(\alpha + c_w)\|\mathbf{u}\| \le \|w \times \mathbf{u}\| + \alpha\|\mathbf{u}\|. \tag{2.5}$$

The inequalities (2.4) and (2.5) are used in the analysis below.

2.1. Analysis of the continuous problem. In this section we will derive a regularity result (Theorem 2.1) and a continuity result (Lemma 2.2). In the latter, opposite to the result in (2.2), the problem dependent norm $|||\cdot|||_\tau$ is used. The continuity result is used in the derivation of the discretization error bounds in section 3.

THEOREM 2.1. *For* $\mathbf{f} \in L_2(\Omega)^2$ *let* $\mathbf{u} \in \mathbf{U}$ *be the solution of problem* (2.1). *Then* $\mathbf{u}$ *is an element of* $H^2(\Omega)^2$ *and the estimates*

$$\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2 \le c(\nu, \alpha)\|\mathbf{f}\|^2 \ , \tag{2.6}$$

$$\nu^2\|\mathbf{u}\|_2^2 + C_P^2\|w \times \mathbf{u}\|^2 \le 2C_P^2\big(4 + 2c(\nu, \alpha)^2\|w\|_\infty^2\big)\|\mathbf{f}\|^2 \tag{2.7}$$

hold, with $c(\nu, \alpha) = \frac{C_F^2}{\nu + C_F^2\alpha}$. *If conditions* (A1) *and* (A3) *are satisfied, then*

$$\nu^2\|\mathbf{u}\|_2^2 + \nu(\|w\|_\infty + \alpha)\|\nabla \mathbf{u}\|^2 + \alpha^2\|\mathbf{u}\|^2 + \|w \times \mathbf{u}\|^2 \le C\|\mathbf{f}\|^2 \tag{2.8}$$

with a constant C *independent of* $\mathbf{f}$, ν, α, *and* w.

Proof. Define $\tilde{\mathbf{f}} = \mathbf{f} - w \times \mathbf{u} - \alpha\mathbf{u}$. Note that $\tilde{\mathbf{f}} \in L_2(\Omega)^2$ and $(\nabla \mathbf{u}, \nabla \mathbf{v}) = -\frac{1}{\nu}(\tilde{\mathbf{f}}, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{U}$. Hence, due to the regularity result for the Poisson equation (2.3), we have $\mathbf{u} \in H^2(\Omega)^2$ and

$$\|\mathbf{u}\|_2 \le \frac{C_P}{\nu}\|\tilde{\mathbf{f}}\| \le \frac{C_P}{\nu}(\|\mathbf{f}\| + \|w \times \mathbf{u}\| + \alpha\|\mathbf{u}\|). \tag{2.9}$$

Note that $\|\mathbf{u}\|^2 = c(\nu, \alpha)(\nu C_F^{-2} + \alpha)\|\mathbf{u}\|^2 \le c(\nu, \alpha)(\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2)$. Using this and taking $\mathbf{v} = \mathbf{u}$ in (2.1) we get

$$\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2 \le \|\mathbf{f}\|\|\mathbf{u}\| \le \|\mathbf{f}\|c(\nu, \alpha)^{\frac{1}{2}}(\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2)^{\frac{1}{2}}, \tag{2.10}$$

and thus the result in (2.6) holds. We also have, using (2.6),

$$\|w \times \mathbf{u}\|^2 \le \|w\|_\infty^2\|\mathbf{u}\|^2 \le c(\nu, \alpha)\|w\|_\infty^2(\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2) \le c(\nu, \alpha)^2\|w\|_\infty^2\|\mathbf{f}\|^2. \tag{2.11}$$

Combining this estimate with (2.9), and noting that $\alpha\|\mathbf{u}\| \le \|\mathbf{f}\|$, yields

$$\begin{aligned}
\nu^2\|\mathbf{u}\|_2^2 + C_P^2\|w \times \mathbf{u}\|^2 &\le C_P^2(\|\mathbf{f}\| + c(\nu, \alpha)\|w\|_\infty\|\mathbf{f}\| + \|\mathbf{f}\|)^2 + C_P^2 c(\nu, \alpha)^2\|w\|_\infty^2\|\mathbf{f}\|^2 \\
&= C_P^2((2 + c(\nu, \alpha)\|w\|_\infty)^2 + c(\nu, \alpha)^2\|w\|_\infty^2)\|\mathbf{f}\|^2 \\
&\le 2C_P^2(3 + 2c(\nu, \alpha)^2\|w\|_\infty^2)\|\mathbf{f}\|^2,
\end{aligned}$$

and thus the estimate (2.7) is proved.

Now assume the conditions (A1) and (A3) to be valid. Since $\mathbf{f} \in L_2(\Omega)^2$ and $\mathbf{u} \in H^2(\Omega)^2$, (1.4) is satisfied in a strong sense, and thus $\|-\nu\Delta\mathbf{u}+\alpha\mathbf{u}+w\times\mathbf{u}\| = \|\mathbf{f}\|$ holds. Taking the square of this identity and noting that $(\mathbf{u}, w\times\mathbf{u}) = 0$ results in

$$\nu^2\|\Delta\mathbf{u}\|^2 + 2\nu\alpha\|\nabla\mathbf{u}\|^2 + \alpha^2\|\mathbf{u}\|^2 + 2\nu(\nabla\mathbf{u}, \nabla(w\times\mathbf{u})) + \|w\times\mathbf{u}\|^2 = \|\mathbf{f}\|^2. \tag{2.12}$$

A simple computation yields $(\nabla\mathbf{u}, \nabla(w\times\mathbf{u})) = -(\nabla u_1, u_2\nabla w) + (\nabla u_2, u_1\nabla w)$ and

$$|(\nabla\mathbf{u}, \nabla(w\times\mathbf{u}))| \le \|\nabla\mathbf{u}\|(\|u_1\nabla w\|^2 + \|u_2\nabla w\|^2)^{\frac{1}{2}}. \tag{2.13}$$

Take q as in (A3) and define $\tilde{q} = \frac{1}{2}q$. The Hölder inequality with $\frac{1}{p}+\frac{1}{\tilde{q}} = 1$ and the injection $H_1(\Omega) \hookrightarrow L_{2p}(\Omega)$ yields, for $i = 1, 2$,

$$\begin{aligned}\|u_i\nabla w\| &= (u_i^2, \nabla w\cdot\nabla w)^{\frac{1}{2}} \le \|u_i\|_{L_{2p}}\|\nabla w\cdot\nabla w\|_{L_{\tilde{q}}}^{\frac{1}{2}} \\ &\le C\|\nabla u_i\|\|\nabla w\|_{L_q} \le C\|\nabla u_i\|\|w\|_\infty.\end{aligned} \tag{2.14}$$

In the last inequality in (2.14) we used (A3). The combination of (2.13) and (2.14) yields

$$2\nu|(\nabla\mathbf{u}, \nabla(w\times\mathbf{u}))| \le \bar{c}\,\nu\|w\|_\infty\,\|\nabla\mathbf{u}\|^2.$$

From this result and (2.12) we obtain

$$\nu^2\|\Delta\mathbf{u}\|^2 + 2\nu\alpha\|\nabla\mathbf{u}\|^2 + \alpha^2\|\mathbf{u}\|^2 + \|w\times\mathbf{u}\|^2 \le \|\mathbf{f}\|^2 + \bar{c}\,\nu\|w\|_\infty\,\|\nabla\mathbf{u}\|^2. \tag{2.15}$$

From (2.1) and (2.5) it follows that, for $\delta > 0$,

$$\begin{aligned}\nu\|\nabla\mathbf{u}\|^2 \le \|\mathbf{f}\|\,\|\mathbf{u}\| &= \frac{1}{\sqrt{\delta}(\alpha+c_w)}\|\mathbf{f}\|\sqrt{\delta}(\alpha+c_w)\|\mathbf{u}\| \\ &\le \frac{\|\mathbf{f}\|^2}{2\delta(\alpha+c_w)^2} + \delta(\alpha^2\|\mathbf{u}\|^2 + \|w\times\mathbf{u}\|^2).\end{aligned} \tag{2.16}$$

If we set $\delta = (4\,\bar{c}\,\|w\|_\infty)^{-1}$ and multiply (2.16) with $\frac{1}{2\delta}$ we obtain

$$2\bar{c}\nu\|w\|_\infty\|\nabla\mathbf{u}\|^2 \le \bar{c}^2\frac{\|w\|_\infty^2}{(\alpha+c_w)^2}\|\mathbf{f}\|^2 + \frac{1}{2}\alpha^2\|\mathbf{u}\|^2 + \frac{1}{2}\|w\times\mathbf{u}\|^2\ .$$

Adding this to (2.15) yields

$$\begin{aligned}&\nu^2\|\Delta\mathbf{u}\|^2 + \nu(\bar{c}\|w\|_\infty + 2\alpha)\|\nabla\mathbf{u}\|^2 + \alpha^2\|\mathbf{u}\|^2 + \|w\times\mathbf{u}\|^2 \\ &\quad\le \Big(1 + \bar{c}^2\frac{\|w\|_\infty^2}{(\alpha+c_w)^2}\Big)\|\mathbf{f}\|^2 + \frac{1}{2}\alpha^2\|\mathbf{u}\|^2 + \frac{1}{2}\|w\times\mathbf{u}\|^2.\end{aligned}$$

Using assumption (A1), i.e., $\frac{\|w\|_\infty^2}{(\alpha+c_w)^2} = \eta^2 \le C$ and $\|\mathbf{u}\|_2 \le C_P\|\Delta\mathbf{u}\|$, the result in (2.8) follows. ∎

Note that in (2.6) and (2.7) with $\alpha = 0$ we have regularity estimates of the form $\|\mathbf{u}\|_1 = O(\nu^{-1})$ and $\|\mathbf{u}\|_2 = O(\nu^{-2})$, which show a similar behavior as regularity results for convection-diffusion problems of the form $-\nu\Delta u+\mathbf{a}\cdot\nabla u = f$ (cf. [16]). The result in (2.8), which holds if conditions (A1) and (A3) are satisfied, yields regularity estimates of the form $\|\mathbf{u}\|_1 = O(\nu^{-1/2})$ and $\|\mathbf{u}\|_2 = O(\nu^{-1})$. These bounds show a behavior that is typical for the solution of reaction-diffusion problems of the form $-\nu\Delta u + bu = f$ if $b > 0$ (cf. [17]). In section 4.2 the regularity result (2.8) will be used in the convergence analysis of the multigrid method.

LEMMA 2.2. *Take $\tau > 0$. The following holds:*

$$a(\mathbf{v},\mathbf{u}) \le C_\tau\, |||\mathbf{v}|||_\tau \Big(\nu\|\nabla\mathbf{u}\|^2 + (\alpha + \|w\|_\infty)\|\mathbf{u}\|^2\Big)^{\frac{1}{2}} \quad \text{for all } \mathbf{v},\mathbf{u} \in \mathbf{U}. \tag{2.17}$$

If condition (A1) *is satisfied, then*

$$a(\mathbf{v},\mathbf{u}) \le C_\tau\, |||\mathbf{v}|||_\tau\, |||\mathbf{u}|||_\tau \quad \text{for all } \mathbf{v},\mathbf{u} \in \mathbf{U}. \tag{2.18}$$

The constants C_τ may depend on τ.

Proof. For $\mathbf{v},\mathbf{u} \in \mathbf{U}$ we have

$$\begin{aligned} a(\mathbf{v},\mathbf{u}) &= \nu(\nabla\mathbf{v},\nabla\mathbf{u}) + \alpha(\mathbf{v},\mathbf{u}) + (w\times\mathbf{v},\mathbf{u}) \\ &\le \nu\|\nabla\mathbf{v}\|\|\nabla\mathbf{u}\| + \alpha\|\mathbf{v}\|\|\mathbf{u}\| + \|w\times\mathbf{v}\|\|\mathbf{u}\|. \end{aligned} \tag{2.19}$$

We define $\kappa := \tau\|w\|_\infty^{-1}$. If we use $\|w\times\mathbf{v}\|\|\mathbf{u}\| = (\kappa^{\frac{1}{2}}\|w\times\mathbf{v}\|)(\kappa^{-\frac{1}{2}}\|\mathbf{u}\|)$ and apply the Cauchy–Schwarz inequality in (2.19) we obtain

$$\begin{aligned} a(\mathbf{v},\mathbf{u}) &\le \Big(\nu\|\nabla\mathbf{v}\|^2 + \alpha\|\mathbf{v}\|^2 + \kappa\|w\times\mathbf{v}\|^2\Big)^{\frac{1}{2}} \Big(\nu\|\nabla\mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2 + \kappa^{-1}\|\mathbf{u}\|^2\Big)^{\frac{1}{2}} \\ &\le C_\tau|||\mathbf{v}|||_\tau \Big(\nu\|\nabla\mathbf{u}\|^2 + (\alpha + \|w\|_\infty)\|\mathbf{u}\|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{2.20}$$

and thus the result in (2.17) holds. If condition (A1) is satisfied we get, using (2.5),

$$\begin{aligned} \|w\times\mathbf{v}\|\,\|\mathbf{u}\| &\le \|w\times\mathbf{v}\|\frac{1}{\alpha + c_w}(\alpha\|\mathbf{u}\| + \|w\times\mathbf{u}\|) \\ &\le \kappa^{\frac{1}{2}}\|w\times\mathbf{v}\|\frac{\kappa^{-\frac{1}{2}}(\alpha^{\frac{1}{2}} + \kappa^{-\frac{1}{2}})}{\alpha + c_w}(\alpha^{\frac{1}{2}}\|\mathbf{u}\| + \kappa^{\frac{1}{2}}\|w\times\mathbf{u}\|) \\ &\le C_\tau(\kappa^{\frac{1}{2}}\|w\times\mathbf{v}\|)(\alpha\|\mathbf{u}\|^2 + \kappa\|w\times\mathbf{u}\|^2)^{\frac{1}{2}}. \end{aligned} \tag{2.21}$$

In the last inequality in (2.21) we used condition (A1):

$$\frac{\kappa^{-\frac{1}{2}}(\alpha^{\frac{1}{2}} + \kappa^{-\frac{1}{2}})}{\alpha + c_w} \le \frac{\frac{3}{2}\kappa^{-1} + \frac{1}{2}\alpha}{\alpha + c_w} \le \frac{3}{2\tau}\eta + \frac{\alpha}{2(\alpha + c_w)} \le C_\tau.$$

From the results in (2.19), (2.21), and the Cauchy–Schwarz inequality, we obtain (2.18). ∎

3. Finite element method. In this section we apply a standard finite element method to the problem (2.1) and derive bounds for the discretization error.

Let $(\mathcal{T}_h)$ be a quasi-uniform family of triangulations of Ω, with mesh size parameter h, and $\mathbf{U}_h \subset \mathbf{U}$ be a finite element subspace of $\mathbf{U}$, consisting of piecewise polynomials of degree $k \in \mathbb{N}$. The finite element Galerkin discretization of the problem (2.1) is as follows: Find $\mathbf{u}_h \in \mathbf{U}_h$ such that

$$a(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h. \tag{3.1}$$

To measure the effect of different terms in (1.4) we introduce mesh numbers:[1]

$$\mathrm{Ek}_h = \frac{\nu}{\|w\|_\infty h^2}, \qquad \mathrm{D}_h = \frac{\alpha h^2}{\nu}.$$

First we prove the stability of $a(\mathbf{u},\mathbf{v})$ on $\mathbf{U}_h$. Below we use the inverse inequality

$$\|\nabla \mathbf{v}_h\| < \mu_u h^{-1}\|\mathbf{v}_h\| \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h.$$

The L_2-orthogonal projection $\mathrm{P}_h : L_2(\Omega)^2 \to \mathbf{U}_h$ is defined by

$$(\mathrm{P}_h \mathbf{u}, \mathbf{v}_h) = (\mathbf{u}, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h. \tag{3.2}$$

We will assume the following approximation property of the spaces $\mathbf{U}_h$ (cf., e.g., [5]): their exists interpolation operator $I_h : \mathbf{U} \to \mathbf{U}_h$ such that

$$\|\mathbf{u} - I_h\mathbf{u}\| \le Ch^m\|\mathbf{u}\|_m, \quad m = 0,1,2 \quad \text{for } \mathbf{u} \in \mathbf{U}\cap H^m(\Omega)^2, \tag{3.3}$$

$$\|\mathbf{u} - I_h\mathbf{u}\|_1 \le Ch^{m-1}\|\mathbf{u}\|_m, \quad m = 1,2 \quad \text{for } \mathbf{u} \in \mathbf{U}\cap H^m(\Omega)^2. \tag{3.4}$$

In (3.3) we use the notation $H^0(\Omega)^2 := L_2(\Omega)^2$ and $\|\cdot\|_0 := \|\cdot\|$.

LEMMA 3.1. *Assume that conditions* (A1) *and* (A2) *are fulfilled. If* $\mathrm{Ek}_h > 1$ *and* $\mathrm{D}_h < 1$, *condition* (A3) *is also assumed. Then there exists some* $\tau \in (0,1]$ *such that*

$$\inf_{\mathbf{u}_h\in\mathbf{U}_h} \sup_{\mathbf{v}_h\in\mathbf{U}_h} \frac{a(\mathbf{u}_h,\mathbf{v}_h)}{|||\mathbf{u}_h|||_\tau\, |||\mathbf{v}_h|||_\tau} \ge C > 0. \tag{3.5}$$

Proof. Take a fixed $\mathbf{u}_h \in \mathbf{U}_h$. Note that

$$\begin{aligned}(w\times\mathbf{u}_h, \mathrm{P}_h(w\times\mathbf{u}_h)) &= (\mathrm{P}_h(w\times\mathbf{u}_h), \mathrm{P}_h(w\times\mathbf{u}_h)),\\ (\mathbf{u}_h, \mathrm{P}_h(w\times\mathbf{u}_h)) &= 0.\end{aligned}$$

Using (2.4) and condition (A2) it follows that

$$\begin{aligned}c_w\|\mathbf{u}_h\|^2 &\le (|w|\times\mathbf{u}_h, 1\times\mathbf{u}_h) = (\mathrm{P}_h(|w|\times\mathbf{u}_h), 1\times\mathbf{u}_h)\\ &= (|\mathrm{P}_h(w\times\mathbf{u}_h)|, 1\times\mathbf{u}_h) \le \|\mathrm{P}_h(w\times\mathbf{u}_h)\|\|\mathbf{u}_h\|,\end{aligned}$$

and thus

$$(\alpha + c_w)\|\mathbf{u}_h\| \le \alpha\|\mathbf{u}_h\| + \|\mathrm{P}_h(w\times\mathbf{u}_h)\|. \tag{3.6}$$

We take

$$\tau = \min\{1, \mu_u^{-2}, \tilde{c}^{-1}\}, \tag{3.7}$$

where $\tilde{c}$ is a constant (independent of all parameters) that will occur in the proof. Let $\kappa := \tau\|w\|_\infty^{-1}$. Using (3.6) we obtain

$$\begin{aligned}\alpha\|\mathbf{u}_h\|^2 + \kappa\|w\times\mathbf{u}_h\|^2 &\le (\alpha + \kappa\|w\|_\infty^2)\|\mathbf{u}_h\|^2\\ &\le \frac{2(\alpha+\kappa\|w\|_\infty^2)}{(\alpha+c_w)^2}(\alpha^2\|\mathbf{u}_h\|^2 + \|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2)\\ &\le \frac{2(\alpha+\kappa\|w\|_\infty^2)(\alpha+\kappa^{-1})}{(\alpha+c_w)^2}(\alpha\|\mathbf{u}_h\|^2 + \kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2).\end{aligned}$$

[1]The abbreviation and definition of Ek is chosen to be consistent with the definition of the Ekman number in the theory of rotating flows. However, the latter is only a particular case ($w = \text{const}$).

Note that $\tau^{-1}+\tau \le \max\{1,\mu_u^2,\tilde{c}\}+1 \le C$ and thus, using condition (A1),

$$\begin{aligned}\frac{(\alpha+\kappa\|w\|_\infty^2)(\alpha+\kappa^{-1})}{(\alpha+c_w)^2} &= \frac{\alpha^2+(\tau^{-1}+\tau)\alpha\|w\|_\infty+\|w\|_\infty^2}{(\alpha+c_w)^2}\\ &\le C\frac{\alpha^2+\|w\|_\infty^2}{(\alpha+c_w)^2} \le C(1+\eta^2)\le C.\end{aligned}$$

Hence,

$$\alpha\|\mathbf{u}_h\|^2+\kappa\|w\times\mathbf{u}_h\|^2 \le C(\alpha\|\mathbf{u}_h\|^2+\kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2). \tag{3.8}$$

To prove (3.5) we choose $\mathbf{v}_h=\mathbf{u}_h+\kappa\mathrm{P}_h(w\times\mathbf{u}_h)$. Then

$$\begin{aligned}a(\mathbf{u}_h,\mathbf{v}_h) &= \nu\|\nabla\mathbf{u}_h\|^2+\alpha\|\mathbf{u}_h\|^2+\nu\kappa(\nabla\mathbf{u}_h,\nabla\mathrm{P}_h(w\times\mathbf{u}_h))+\kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2\\ &\ge \nu\|\nabla\mathbf{u}_h\|^2+\alpha\|\mathbf{u}_h\|^2-\nu\kappa\|\nabla\mathbf{u}_h\|\,\|\nabla\mathrm{P}_h(w\times\mathbf{u}_h)\|+\kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2.\end{aligned} \tag{3.9}$$

For the estimation of the term $\|\nabla\mathrm{P}_h(w\times\mathbf{u}_h)\|$ we distinguish three cases: $\mathrm{Ek}_h\le 1$ (case 1), $\mathrm{D}_h\ge 1$ (case 2), and $\mathrm{Ek}_h>1$ and $\mathrm{D}_h<1$ (case 3).
In case 1 we have

$$\begin{aligned}(\nu\kappa)^{\frac12}\|\nabla\mathrm{P}_h(w\times\mathbf{u}_h)\| &\le \left(\frac{\nu\tau\mu_u^2}{\|w\|_\infty h^2}\right)^{\frac12}\|\mathrm{P}_h(w\times\mathbf{u}_h)\|\\ &= (\mathrm{Ek}_h\tau\mu_u^2)^{\frac12}\|\mathrm{P}_h(w\times\mathbf{u}_h)\|\le\|\mathrm{P}_h(w\times\mathbf{u}_h)\|.\end{aligned} \tag{3.10}$$

Using this in (3.9) and applying the Cauchy–Schwarz inequality, we get

$$a(\mathbf{u}_h,\mathbf{v}_h)\ge\frac12\nu\|\nabla\mathbf{u}_h\|^2+\alpha\|\mathbf{u}_h\|^2+\frac12\kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2. \tag{3.11}$$

In case 2 we have

$$\begin{aligned}\nu^{\frac12}\kappa\|\nabla\mathrm{P}_h(w\times\mathbf{u}_h)\| &\le \nu^{\frac12}\kappa\mu_u h^{-1}\|w\|_\infty\|\mathbf{u}\| = \tau\mu_u\mathrm{D}_h^{-\frac12}\alpha^{\frac12}\|\mathbf{u}\|\\ &\le \tau^{\frac12}\mu_u\mathrm{D}_h^{-\frac12}\alpha^{\frac12}\|\mathbf{u}\|\le\alpha^{\frac12}\|\mathbf{u}\|.\end{aligned} \tag{3.12}$$

Using this in (3.9) and applying the Cauchy–Schwarz inequality, we get

$$a(\mathbf{u}_h,\mathbf{v}_h)\ge\frac12\nu\|\nabla\mathbf{u}_h\|^2+\frac12\alpha\|\mathbf{u}_h\|^2+\kappa\|\mathrm{P}_h(w\times\mathbf{u}_h)\|^2. \tag{3.13}$$

For case 3 first note that, using condition (A3) and the result in (2.14) it follows that

$$\begin{aligned}\|\nabla(w\times\mathbf{u}_h)\|^2 &= \sum_{i=1}^2\|(u_h)_i\nabla w\|^2+\|w\nabla(u_h)_i\|^2+2((u_h)_i\nabla w,w\nabla(u_h)_i)\\ &\le 2\sum_{i=1}^2\|(u_h)_i\nabla w\|^2+\|w\nabla(u_h)_i\|^2\le c_1\|w\|_\infty^2\|\nabla\mathbf{u}_h\|^2.\end{aligned}$$

We use that the L_2-orthogonal projection is bounded in the H^1-norm (cf. [2]):

$$\|\mathrm{P}_h\mathbf{u}\|_1\le c_2\|\mathbf{u}\|_1 \qquad \text{for } \mathbf{u}\in\mathbf{U}.$$

For the constant $\tilde{c}$ in (3.7) we take $\tilde{c} = 2c_2\sqrt{c_1}$ and then obtain

$$\kappa\|\nabla \mathrm{P}_h(w\times \mathbf{u}_h)\| \le c_2\kappa\|\nabla(w\times \mathbf{u}_h)\| \le c_2\sqrt{c_1}\kappa\|w\|_\infty\|\nabla \mathbf{u}_h\| \le \frac{1}{2}\|\nabla \mathbf{u}_h\|. \tag{3.14}$$

Using this in (3.9) results in

$$a(\mathbf{u}_h,\mathbf{v}_h) \ge \frac{1}{2}\nu\|\nabla \mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2 + \kappa\|\mathrm{P}_h(w\times \mathbf{u}_h)\|^2. \tag{3.15}$$

The combination of (3.11), (3.13), (3.15) with (3.8) proves that

$$a(\mathbf{u}_h,\mathbf{v}_h) \ge C|||\mathbf{u}_h|||_\tau^2 \tag{3.16}$$

holds. The results in (3.10), (3.12), and (3.14) imply

$$\nu\kappa^2\|\nabla \mathrm{P}_h(w\times \mathbf{u}_h)\|^2 \le |||\mathbf{u}_h|||_\tau^2 \ .$$

Using this it follows that

$$\begin{aligned}
|||\mathbf{v}_h|||_\tau^2 &= \nu\|\nabla(\mathbf{u}_h + \kappa \mathrm{P}_h(w\times \mathbf{u}_h))\|^2 + \alpha\|\mathbf{u}_h + \kappa\mathrm{P}_h(w\times \mathbf{u}_h)\|^2 \\
&\quad + \kappa\|\mathrm{P}_h(w\times \mathbf{u}_h + \kappa w\times \mathrm{P}_h(w\times \mathbf{u}_h))\|^2 \\
&\le 2(\nu\|\nabla \mathbf{u}_h\|^2 + \nu\kappa^2\|\nabla\mathrm{P}_h(w\times \mathbf{u}_h)\|^2) + \alpha\|\mathbf{u}_h\|^2 + \kappa^2\alpha\|\mathrm{P}_h(w\times \mathbf{u}_h)\|^2 \\
&\quad + 2\kappa(\|\mathrm{P}_h(w\times \mathbf{u}_h)\|^2 + \kappa^2\|\mathrm{P}_h(w\times \mathrm{P}_h(w\times \mathbf{u}_h))\|^2) \\
&\le 2\nu\|\nabla \mathbf{u}_h\|^2 + 2|||\mathbf{u}_h|||_\tau^2 + \alpha(1+\tau^2)\|\mathbf{u}_h\|^2 + 2\kappa(1+\tau^2)\|\mathrm{P}_h(w\times \mathbf{u}_h)\|^2 \\
&\le 2\nu\|\nabla \mathbf{u}_h\|^2 + 2\alpha\|\mathbf{u}_h\|^2 + 4\kappa\|\mathrm{P}_h(w\times \mathbf{u}_h)\|^2 + 2|||\mathbf{u}_h|||_\tau^2 \\
&\le 6|||\mathbf{u}_h|||_\tau^2.
\end{aligned}$$

The combination of the latter estimate and (3.16) completes the proof. ◻

Remark 3.1. Note that τ in Lemma 3.1 does not depend on ν, α, or w.

Remark 3.2. Using the mesh-dependent norm

$$|||\mathbf{u}|||_{\tau,h} = \left(\nu\|\nabla \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2 + \frac{\tau}{\|w\|_\infty}\|\mathrm{P}_h(w\times \mathbf{u})\|^2\right)^{\frac{1}{2}} \tag{3.17}$$

the stability of $a(\cdot,\cdot)$ on $\mathbf{U}_h$ can be proved without assumption (A1) and (A2) on w, since estimate (3.8) is not needed. Moreover, continuity of $a(\cdot,\cdot)$ on $\mathbf{U}_h\times\mathbf{U}$ in the mesh-dependent norm (3.17) can be proved without the assumptions (A1), (A2). This then results in satisfactory discretization error bounds in the norm $|||\cdot|||_{\tau,h}$. (See the treatment of the Oseen problem in [10].) However, for a certain duality argument in the proof of the approximation property in the multigrid convergence analysis (see Theorem 3.3 and section 4) we need the continuity of $a(\cdot,\cdot)$ on $\mathbf{U}\times\mathbf{U}$, and then the mesh-dependent norm becomes inconvenient.

We now derive discretization error bounds for the finite element method using standard arguments based on Galerkin orthogonality, stability, continuity, and approximation properties of the finite element spaces.

THEOREM 3.2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solution of* (2.1) *and* (3.1), *respectively. Let the assumptions of Lemma* 3.1 *be fulfilled and take* $\tau\in(0,1]$ *as in Lemma* 3.1. *Then the following inequalities hold:*

$$|||\mathbf{u}-\mathbf{u}_h|||_\tau \le C_\tau\, h^j(\nu^{\frac{1}{2}}\|\mathbf{u}\|_{j+1} + (\alpha^{\frac{1}{2}} + \|w\|_\infty^{\frac{1}{2}})\|\mathbf{u}\|_j), \quad j=0,1, \tag{3.18}$$

$$|||\mathbf{u}-\mathbf{u}_h|||_\tau \le C_\tau\, h(\nu^{\frac{1}{2}} + (\alpha^{\frac{1}{2}} + \|w\|_\infty^{\frac{1}{2}})h)\|\mathbf{u}\|_2. \tag{3.19}$$

The constants C_τ *are independent of* ν, α, w, $\mathbf{u}$, *and* h *but may depend on* τ.

Proof. Let $\hat{\mathbf{u}}_h$ be an arbitrary function in $\mathbf{U}_h$. Take τ as in Lemma 3.1. Then there exists $\mathbf{v}_h \in \mathbf{U}_h$ such that

$$C|||\mathbf{u}_h - \hat{\mathbf{u}}_h|||_\tau \, |||\mathbf{v}_h|||_\tau \le a(\mathbf{u}_h - \hat{\mathbf{u}}_h, \mathbf{v}_h).$$

Using Galerkin orthogonality and the continuity result in (2.18) we obtain

$$a(\mathbf{u}_h - \hat{\mathbf{u}}_h, \mathbf{v}_h) = a(\mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{v}_h) \le C_\tau |||\mathbf{u} - \hat{\mathbf{u}}_h|||_\tau |||\mathbf{v}_h|||_\tau.$$

Hence,

$$|||\mathbf{u}_h - \hat{\mathbf{u}}_h|||_\tau \le C_\tau |||\mathbf{u} - \hat{\mathbf{u}}_h|||_\tau \tag{3.20}$$

holds. From the triangle inequality and (3.20) it follows that

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_\tau^2 &\le C_\tau |||\mathbf{u} - \hat{\mathbf{u}}_h|||_\tau^2 \\ &\le C_\tau \left(\nu \|\nabla(\mathbf{u} - \hat{\mathbf{u}}_h)\|^2 + \alpha \|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 + \frac{\tau}{\|w\|_\infty} \|w \times (\mathbf{u} - \hat{\mathbf{u}}_h)\|^2 \right) \\ &\le C_\tau \left(\nu \|\mathbf{u} - \hat{\mathbf{u}}_h\|_1^2 + (\alpha + \tau \|w\|_\infty) \|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 \right). \end{aligned} \tag{3.21}$$

According to (3.3) and (3.4) $\hat{\mathbf{u}}_h = I_h \mathbf{u}$ can be taken such that

$$\|\mathbf{u} - \hat{\mathbf{u}}_h\|_1^2 \le C h^{2j} \|\mathbf{u}\|_{j+1}^2, \quad \|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 \le C h^{2j} \|\mathbf{u}\|_j^2, \quad j = 0, 1.$$

Using this in (3.21) proves the result in (3.18). If we use the inequalities

$$\|\mathbf{u} - \hat{\mathbf{u}}_h\|_1^2 \le C h^2 \|\mathbf{u}\|_2^2, \quad \|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 \le C h^4 \|\mathbf{u}\|_2^2,$$

in (3.21) we get the result in (3.19). ∎

Note that $\|w\|_\infty$ occurs in the estimates (3.18)–(3.19) in a similar way as α, which measures the *reaction*.

We now prove a discretization error bound in the L_2-norm. This result will play an important role in the convergence analysis of the multigrid method.

THEOREM 3.3. *Assume that the conditions* (A1), (A2), *and* (A3) *are fulfilled. For* $\mathbf{f} \in L_2(\Omega)^2$ *let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of* (2.1) *and* (3.1), *respectively. Then*

$$\|\mathbf{u} - \mathbf{u}_h\| \le C \min\left\{ \frac{h^2}{\nu}, \frac{1}{\alpha + \|w\|_\infty} \right\} \|\mathbf{f}\| \tag{3.22}$$

holds with a constant C *independent of* $\nu, \alpha, w, h,$ *and* $\mathbf{f}$.

Proof. Take $\mathbf{f} \in L_2(\Omega)^2$ and let $\mathbf{u}$, $\mathbf{u}_h$ be the solutions of (2.1) and (3.1), respectively. From (3.18) and the regularity estimate (2.8) it follows that

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_\tau &\le C_\tau h \left(\nu^{\frac{1}{2}} \|\mathbf{u}\|_2 + (\alpha^{\frac{1}{2}} + \|w\|_\infty^{\frac{1}{2}}) \|\mathbf{u}\|_1 \right) \\ &\le C_\tau \frac{h}{\sqrt{\nu}} \left(\nu^2 \|\mathbf{u}\|_2^2 + \nu(\alpha + \|w\|_\infty) \|\nabla \mathbf{u}\|^2 \right)^{\frac{1}{2}} \le C_\tau \frac{h}{\sqrt{\nu}} \|\mathbf{f}\|. \end{aligned} \tag{3.23}$$

We now apply a duality argument. For this we introduce the adjoint bilinear form

$$a^*(\mathbf{u}, \mathbf{v}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) - (w \times \mathbf{u}, \mathbf{v}) \quad \text{ for } \mathbf{u}, \mathbf{v} \in \mathbf{U},$$

and the adjoint problem

$$\text{find} \quad \tilde{\mathbf{u}} \in \mathbf{U} \ \text{ such that } \ a^*(\tilde{\mathbf{u}}, \mathbf{v}) = (\tilde{\mathbf{f}}, \mathbf{v}) \qquad \text{for all } \mathbf{v} \in \mathbf{U},$$

with $\tilde{\mathbf{f}} := \mathbf{u} - \mathbf{u}_h \in \mathbf{U} \subset L_2(\Omega)^2$. Let $\tilde{\mathbf{u}}_h \in \mathbf{U}_h$ be the discrete solution of the adjoint problem, i.e., $a^*(\tilde{\mathbf{u}}_h, \mathbf{v}_h) = (\tilde{\mathbf{f}}, \mathbf{v}_h)$ for all $\mathbf{v}_h \in \mathbf{U}_h$. Note that $a^*(\cdot,\cdot)$ equals $a(\cdot,\cdot)$ if, in $a(\cdot,\cdot)$, we replace w by $-w$. The results in Lemma 3.1 and Theorem 3.2 do not depend on sign(w) and thus hold for the adjoint problem, too. Moreover, since the choice of τ in Lemma 3.1 does not depend on w (cf. Remark 3.1), the estimate (3.23) holds for the original and the adjoint problem with the same τ value. Using this discretization error bound for the original and adjoint problem and the continuity result of Lemma 2.2 we obtain

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|^2 &= (\tilde{\mathbf{f}}, \tilde{\mathbf{f}}) = a^*(\tilde{\mathbf{u}}, \tilde{\mathbf{f}}) = a(\tilde{\mathbf{f}}, \tilde{\mathbf{u}}) = a(\mathbf{u} - \mathbf{u}_h, \tilde{\mathbf{u}}) = a(\mathbf{u} - \mathbf{u}_h, \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h)\\ &\le C_\tau |||\mathbf{u} - \mathbf{u}_h|||_\tau |||\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h|||_\tau \le C_\tau \frac{h^2}{\nu} \|\mathbf{f}\| \, \|\tilde{\mathbf{f}}\| = C_\tau \frac{h^2}{\nu} \|\mathbf{f}\| \, \|\mathbf{u} - \mathbf{u}_h\|.\end{aligned}$$

Hence, $\|\mathbf{u} - \mathbf{u}_h\| \le C_\tau \frac{h^2}{\nu} \|\mathbf{f}\|$ holds, which proves the first bound in (3.22). For the second bound we note that from (2.5) and (A1) it follows that

(3.24)

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\| &\le \frac{1}{\alpha + c_w} (\alpha \|\mathbf{u} - \mathbf{u}_h\| + \|w \times (\mathbf{u} - \mathbf{u}_h)\|)\\ &\le \frac{1}{\alpha + \|w\|_\infty} \frac{\alpha + \|w\|_\infty}{\alpha + c_w} \left(\alpha^{\frac12} + \frac{\|w\|_\infty^{\frac12}}{\tau^{\frac12}} \right) \left(\alpha^{\frac12} \|\mathbf{u} - \mathbf{u}_h\| + \frac{\tau^{\frac12}}{\|w\|_\infty^{\frac12}} \|w \times (\mathbf{u} - \mathbf{u}_h)\| \right)\\ &\le \frac{2}{\alpha + \|w\|_\infty} (1+\eta) \tau^{-\frac12} (\alpha^{\frac12} \tau^{\frac12} + \|w\|_\infty^{\frac12}) |||\mathbf{u} - \mathbf{u}_h|||_\tau\\ &\le C_\tau \frac{1}{\alpha + \|w\|_\infty} (\alpha^{\frac12} + \|w\|_\infty^{\frac12}) |||\mathbf{u} - \mathbf{u}_h|||_\tau.\end{aligned}$$

Finally, note that due to (3.18) with $j = 0$ and the results in (2.5), (2.8) we get

$$\begin{aligned}(\alpha^{\frac12} + \|w\|_\infty^{\frac12}) |||\mathbf{u} - \mathbf{u}_h|||_\tau &\le (\alpha^{\frac12} + \|w\|_\infty^{\frac12})(\nu^{\frac12} \|\mathbf{u}\|_1 + (\alpha^{\frac12} + \|w\|_\infty^{\frac12}) \|\mathbf{u}\|)\\ &\le \nu^{\frac12} (\alpha^{\frac12} + \|w\|_\infty^{\frac12}) \|\mathbf{u}\|_1 + 2(\alpha + \|w\|_\infty) \|\mathbf{u}\|\\ &\le \nu^{\frac12} (\alpha^{\frac12} + \|w\|_\infty^{\frac12}) \|\mathbf{u}\|_1 + 2(1+\eta)(\|w \times \mathbf{u}\| + \alpha \|\mathbf{u}\|)\\ &\le C\big(\nu(\alpha + \|w\|_\infty) \|\nabla \mathbf{u}\|^2 + \alpha^2 \|\mathbf{u}\|^2 + \|w \times \mathbf{u}\|^2\big)^{\frac12}\\ &\le C \|\mathbf{f}\|.\end{aligned}$$

This in combination with (3.24) yields the second bound in (3.22). ∎

4. A solver for the discrete problem. For the approximate solution of the discrete problem we apply a multigrid method. The method and its convergence analysis will be presented in a matrix-vector form as in Hackbusch [8].

4.1. Multigrid components. For the application of the multigrid solver we assume that the quasi-uniform family of triangulations of Ω results from a *global regular refinement* technique. This yields a hierarchy of nested finite element spaces

$$\mathbf{U}_0 \subset \mathbf{U}_1 \subset \cdots \subset \mathbf{U}_k \subset \cdots \subset \mathbf{U}.$$

The corresponding mesh size parameter is denoted by h_k and satisfies

$$c_0 2^{-k} \le h_k/h_0 \le c_1 2^{-k}$$

with positive constants c_0 and c_1 independent of k. Note that $\mathbf{U}_k = U_k \times U_k$, where U_k is a standard conforming finite element space consisting of scalar functions. For the matrix-vector formulation of the discrete problem we use the standard nodal basis in U_k, denoted by $\{\phi_i\}_{1\le i\le n_k}$, and the isomorphism

$$P_k : \mathbb{R}^{n_k} \to U_k, \qquad P_k x = \sum_{i=1}^{n_k} x_i \phi_i.$$

For the product space $\mathbf{U}_k = U_k \times U_k$ we use the isomorphism

$$\mathbf{P}_k : X_k := \mathbb{R}^{2n_k} \to \mathbf{U}_k, \quad \mathbf{P}_k \mathbf{x} = \mathbf{P}_k \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = P_k x^1 \times P_k x^2, \quad x^i \in \mathbb{R}^{n_k}, \quad i = 1, 2.$$

On $\mathbb{R}^{n_k}$ and X_k we use scaled Euclidean scalar products: $\langle x, y\rangle_k = h_k^2 \sum_{i=1}^{n_k} x_i y_i$ for $x, y \in \mathbb{R}^{n_k}$ and $\langle \mathbf{x}, \mathbf{y}\rangle_k = \langle x^1, y^1\rangle_k + \langle x^2, y^2\rangle_k$ for $\mathbf{x},\ \mathbf{y} \in X_k$. The corresponding norms are denoted by $\|\cdot\|$. The adjoint $\mathbf{P}_k^* :\ \mathbf{U}_k \to X_k$ satisfies $(\mathbf{P}_k\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \mathbf{P}_k^*\mathbf{v}\rangle_k$ for all $\mathbf{x} \in X_k,\ \mathbf{v} \in \mathbf{U}_k$. Note that the following norm equivalence holds:

$$C^{-1}\|\mathbf{x}\| \le \|\mathbf{P}_k\mathbf{x}\| \le C\|\mathbf{x}\| \quad \text{for all } \mathbf{x} \in X_k, \tag{4.1}$$

with a constant C independent of k. The stiffness matrix $L_k : \mathbb{R}^{2n_k} \to \mathbb{R}^{2n_k}$ on level k is defined by

$$\langle L_k\mathbf{x}, \mathbf{y}\rangle_k = a(\mathbf{P}_k\mathbf{x}, \mathbf{P}_k\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in X_k. \tag{4.2}$$

This matrix has the block structure

$$L_k = \begin{pmatrix} \nu A + \alpha M & -M_w \\ M_w & \nu A + \alpha M \end{pmatrix},$$

with

$$\begin{aligned} \langle Ax, y\rangle_k &= (\nabla P_k x, \nabla P_k y), \quad \langle Mx, y\rangle_k = (P_k x, P_k y), \\ \langle M_w x, y\rangle_k &= (w P_k x, P_k y) \end{aligned} \tag{4.3}$$

for all $x, y \in \mathbb{R}^{n_k}$. Note that A is a stiffness matrix for a single (velocity) component, M is a mass matrix, and M_w is of mass matrix type corresponding to the bilinear form $[x, y] \to (wx, y)$. The latter is not necessarily a scalar product. The matrices A, M, M_w are symmetric and A and M are positive definite.

For the prolongation and restriction in the multigrid algorithm we use the canonical choice:

$$\begin{aligned} p_k &:\ X_{k-1} \to X_k, \qquad p_k = \mathbf{P}_k^{-1}\mathbf{P}_{k-1}, \\ r_k &:\ X_k \to X_{k-1}, \qquad r_k = \mathbf{P}_{k-1}^*(\mathbf{P}_k^*)^{-1} = \left(\tfrac{h_k}{h_{k-1}}\right)^2 p_k^T. \end{aligned} \tag{4.4}$$

Consider a smoother of the form

$$\mathbf{x}^{\text{new}} = \mathbf{x}^{\text{old}} - W_k^{-1}(L_k\mathbf{x}^{\text{old}} - \mathbf{b}) \quad \text{for } \mathbf{x}^{\text{old}}, \mathbf{b} \in X_k$$

with the corresponding iteration matrix denoted by $S_k = I - W_k^{-1}L_k$.

The damped block Jacobi method corresponds to

$$W_k = \omega^{-1} \begin{pmatrix} \mathrm{diag}(\nu A + \alpha M) & -\mathrm{diag}(M_w) \\ \mathrm{diag}(M_w) & \mathrm{diag}(\nu A + \alpha M) \end{pmatrix}, \tag{4.5}$$

with a damping parameter $\omega \in (0, 1]$. This type of smoother will be used in our numerical experiments in section 5. In the convergence analysis of the multigrid method we consider a smoother of block Richardson type:

$$W_k = \begin{pmatrix} \beta_1 I & -\beta_2 I \\ \beta_2 I & \beta_1 I \end{pmatrix}, \tag{4.6}$$

where I is the identity matrix and β_1, β_2 are suitable scaling factors. With the components defined above, a standard multigrid algorithm with μ_1 pre- and μ_2 postsmoothing iterations can be formulated (cf. [8]) with an iteration matrix M_k on level k that satisfies the recursion

$$\begin{aligned} &M_0(\mu_1, \mu_2) = 0, \\ &M_k(\mu_1, \mu_2) = S_k^{\mu_2} \left(I - p_k (I - M_{k-1}^{\gamma}) L_{k-1}^{-1} r_k L_k \right) S_k^{\mu_1}, \quad k = 1, 2, \dots . \end{aligned}$$

The choices $\gamma = 1$ and $\gamma = 2$ correspond to the V- and W-cycle, respectively. For analysis of this multigrid method we use the framework of [7], [8] based on the approximation and smoothing property. In sections 4.2 and 4.3 we will prove the following approximation and smoothing properties:

$$\|L_k^{-1} - p_k \, L_{k-1}^{-1} r_k\| \le C \left(\frac{\nu}{h^2} + \alpha + \|w\|_\infty \right)^{-1}, \tag{4.7}$$

$$\|L_k S_k^{\mu_1}\| \le \frac{C}{\sqrt{\mu_1}} \left(\frac{\nu}{h^2} + \alpha + \|w\|_\infty \right). \tag{4.8}$$

As a direct consequence of (4.7) and (4.8) one obtains a bound for the contraction number of the two-grid method:

$$\|(I - p_k L_{k-1}^{-1} r_k L_k) S_k^{\mu_1}\| \le \frac{C}{\sqrt{\mu_1}} \ . \tag{4.9}$$

Using the analysis in [8, Theorem 10.6.25] the convergence of the multigrid W-cycle can be obtained as a consequence of the approximation and smoothing property. In section 4.3 we will prove $\|S_k\| \le 1$. Using this and (4.7), (4.8), Theorem 10.6.25 from [8] yields the following result.

THEOREM 4.1. *Assume* (A1)–(A3) *hold; then for any* $\psi \in (0, 1)$ *there exists* $\bar{\mu}_0 > 0$ *independent of the problem parameters* ν, α *and the level number* k *such that for the contraction number of the multigrid W-cycle with smoothing* (4.6) *we have*

$$\|M_k(\mu, 0)\| \le \psi \quad \textit{for all } \mu \ge \bar{\mu}_0. \qquad \square$$

This proves the robustness of the multigrid W-cycle with respect to variation in the problem parameters ν *and* α *and the mesh size* h_k.

This robustness is confirmed by the numerical experiments in section 5.

4.2. Approximation property. The analysis of the approximation property is as in [7], [8]. The key ingredient is the finite element error bound in Theorem 3.3.

THEOREM 4.2. *Let the assumptions* (A1)–(A3) *be valid; then*

$$\|L_k^{-1} - p_k\, L_{k-1}^{-1} r_k\| \le C\, \left(\frac{\nu}{h_k^2} + \alpha + \|w\|_\infty\right)^{-1} \le C\|L_k\|^{-1}. \tag{4.10}$$

Proof. Take $\mathbf{y}_k \in X_k$. The constants C that appear in the proof do not depend on $\nu, \alpha, \mathbf{y}_k$, or k. Let $\mathbf{s}^* \in \mathbf{U}$, $\mathbf{s}_k \in \mathbf{U}_k$, and $\mathbf{s}_{k-1} \in \mathbf{U}_{k-1}$ be such that

$$\begin{aligned} a(\mathbf{s}^*, \mathbf{v}) &= ((\mathbf{P}_k^*)^{-1}\mathbf{y}_k, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U}, \\ a(\mathbf{s}_k, \mathbf{v}) &= ((\mathbf{P}_k^*)^{-1}\mathbf{y}_k, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U}_k, \\ a(\mathbf{s}_{k-1}, \mathbf{v}) &= ((\mathbf{P}_k^*)^{-1}\mathbf{y}_k, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U}_{k-1}. \end{aligned}$$

Putting $\mathbf{f} = (\mathbf{P}_k^*)^{-1}\mathbf{y}_k \in L_2(\Omega)^2$ in Theorem 3.3, we obtain

$$\|\mathbf{s}^* - \mathbf{s}_l\| \le C \min\left\{\frac{h_l^2}{\nu}, \frac{1}{\alpha + \|w\|_\infty}\right\} \|(\mathbf{P}_k^*)^{-1}\mathbf{y}_k\| \quad \text{for } l \in \{k-1, k\}.$$

Due to $h_{k-1} \le c h_k$ this yields

$$\|\mathbf{s}_k - \mathbf{s}_{k-1}\| \le C \min\left\{\frac{h_k^2}{\nu}, \frac{1}{\alpha + \|w\|_\infty}\right\} \|(\mathbf{P}_k^*)^{-1}\mathbf{y}_k\|.$$

From (4.2) and (4.4) it follows that $\mathbf{s}_k = \mathbf{P}_k L_k^{-1}\mathbf{y}_k$ and $\mathbf{s}_{k-1} = \mathbf{P}_{k-1} L_{k-1}^{-1} r_k \mathbf{y}_k$. Thus, using (4.1), we get

$$\begin{aligned} \|(L_k^{-1} - p_k L_{k-1}^{-1} r_k)\mathbf{y}_k\| &\le C\|\mathbf{P}_k L_k^{-1}\mathbf{y}_k - \mathbf{P}_{k-1} L_{k-1}^{-1} r_k \mathbf{y}_k\| = C\|\mathbf{s}_k - \mathbf{s}_{k-1}\| \\ &\le C\, \min\left\{\frac{h_k^2}{\nu}, \frac{1}{\alpha + \|w\|_\infty}\right\} \|(\mathbf{P}_k^*)^{-1}\mathbf{y}_k\| \\ &\le C\, \min\left\{\frac{h_k^2}{\nu}, \frac{1}{\alpha + \|w\|_\infty}\right\} \|\mathbf{y}_k\|. \end{aligned}$$

Note that $\min\{\frac{1}{p}, \frac{1}{q}\} \le \frac{2}{p+q}$ for all $p, q > 0$. Hence the first inequality in (4.10) is proved. For the second inequality in (4.10) we note that

$$\begin{aligned} \|L_k\| &= \left\| \begin{pmatrix} \nu A + \alpha M & \emptyset \\ \emptyset & \nu A + \alpha M \end{pmatrix} + \begin{pmatrix} \emptyset & -M_w \\ M_w & \emptyset \end{pmatrix} \right\| \\ &\le \|\nu A + \alpha M\| + \|M_w\| \le \nu\|A\| + (\alpha + \|w\|_\infty)\|M\|. \end{aligned}$$

Using $\|A\| \le C h_k^{-2}$ and $\|M\| \le C$ we obtain $\|L_k\| \le C(\nu h_k^{-2} + \alpha + \|w\|_\infty)$. ◻

4.3. Smoothing property. Let a_1, m_1 be positive constants independent of ν, α, and k such that for spectral radius of the matrices in (4.3) we have

$$\rho(A) \le \frac{a_1}{h_k^2}, \qquad \rho(M) \le m_1.$$

Furthermore, let $w_{\min} = \text{ess inf}_\Omega\, w$ and $w_{\max} = \text{ess sup}_\Omega\, w$ and define

$$C_w = \begin{cases} w_{\max} & \text{if } w_{\max} \ge -w_{\min}, \\ w_{\min} & \text{if } w_{\max} < -w_{\min}. \end{cases}$$

Note that $|C_w| = \|w\|_\infty$. In the analysis below we use the following elementary result.

LEMMA 4.3. *Assume that for $B \in \mathbb{R}^{n\times n}$ and $\Lambda \in (0,\infty)$ we have $B^TB \le \Lambda(B+B^T)$. Then $\|I-\omega B\| \le 1$ holds for any $\omega \in [0, \frac{1}{\Lambda}]$.*

This result follows from

$$\begin{aligned} 0 \le (I-\omega B)^T(I-\omega B) &= I - \omega(B+B^T) + \omega^2 B^TB \\ &\le I - \omega(1-\omega\Lambda)(B+B^T) \le I. \qquad \square \end{aligned}$$

Using this lemma we prove that the contraction number of the block Richardson method is bounded by 1.

LEMMA 4.4. *Assume that* (A1) *and* (A2) *are satisfied. Consider the block Richardson method with W_k as in* (4.6) *and*

$$\beta_1 = \frac{\nu a_1}{h_k^2} + \alpha\kappa_1 m_1, \quad \beta_2 = \kappa_2 C_w, \quad \textit{with constants}$$

$$\kappa_1 \ge 2(1+\eta^2), \quad \kappa_2 \ge 4m_1\eta. \tag{4.11}$$

Then the following inequality holds:

$$\|I - W_k^{-1}L_k\| \le 1.$$

Proof. A straightforward computation yields

$$W_k^{-1}L_k = R_1 + R_2, \qquad \text{with} \tag{4.12}$$

$$R_1 = \frac{\nu}{\beta_1^2+\beta_2^2}\begin{pmatrix} \beta_1 A & \beta_2 A \\ -\beta_2 A & \beta_1 A \end{pmatrix},$$

$$R_2 = \frac{1}{\beta_1^2+\beta_2^2}\begin{pmatrix} \beta_1\alpha M + \beta_2 M_w & \beta_2\alpha M - \beta_1 M_w \\ -\beta_2\alpha M + \beta_1 M_w & \beta_1\alpha M + \beta_2 M_w \end{pmatrix}.$$

From

$$\frac{1}{2}(R_1^T + R_1) = \frac{\nu\beta_1}{\beta_1^2+\beta_2^2}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \qquad R_1^TR_1 = \frac{\nu^2}{\beta_1^2+\beta_2^2}\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}$$

it follows that

$$R_1^TR_1 \le \frac{1}{2}(R_1^T + R_1) \;\Leftrightarrow\; \nu A \le \beta_1 I \;\Leftrightarrow\; \nu A \le \left(\frac{\nu a_1}{h_k^2} + \alpha\kappa_1 m_1\right) I.$$

The last inequality holds, due to $\rho(A) \le \frac{a_1}{h_k^2}$ and $\alpha\kappa_1 m_1 \ge 0$. Application of Lemma 4.3 yields

$$\|I - 2R_1\| \le 1\ . \tag{4.13}$$

For the matrix R_2 we obtain

$$\frac{1}{2}(R_2^T + R_2) = \frac{1}{\beta_1^2+\beta_2^2}\begin{pmatrix} \beta_1\alpha M + \beta_2 M_w & \emptyset \\ \emptyset & \beta_1\alpha M + \beta_2 M_w \end{pmatrix},$$

$$R_2^TR_2 = \frac{1}{\beta_1^2+\beta_2^2}\begin{pmatrix} \alpha^2M^2 + M_w^2 & \alpha(M_wM - MM_w) \\ -\alpha(M_wM - MM_w) & \alpha^2M^2 + M_w^2 \end{pmatrix}.$$

We use the notation $\hat{M} = \beta_1\alpha M + \beta_2 M_w$. Note that $R_2^TR_2 \le \frac{1}{2}(R_2^T + R_2)$ holds if the following two conditions are satisfied:

$$\alpha^2M^2 + M_w^2 \le \frac{1}{2}\hat{M}, \tag{4.14}$$

$$\alpha|\langle (M_wM - MM_w)x, y\rangle_k| \le \frac{1}{4}\Big(\langle \hat{M}x, x\rangle_k + \langle \hat{M}y, y\rangle_k\Big), \tag{4.15}$$

for all $x, y \in \mathbb{R}^{n_k}$. We first consider (4.14). We have $M_w^2 \leq \|w\|_\infty^2 M^2 \leq m_1 \|w\|_\infty^2 M$. Due to (A2) the matrix M_w is definite and $C_w M_w$ is positive definite; moreover, $C_w M_w \geq |C_w| c_w M = \|w\|_\infty c_w M$. Using this we obtain

$$\alpha^2 M^2 + M_w^2 \leq (m_1 \alpha^2 + m_1 \|w\|_\infty^2) M,$$
$$\frac{1}{2}\hat{M} \geq \frac{1}{2}(\kappa_1 m_1 \alpha^2 M + \kappa_2 C_w M_w) \geq \frac{1}{2}(\kappa_1 m_1 \alpha^2 + \kappa_2 \|w\|_\infty c_w) M.$$

Hence, (4.14) is fulfilled if the inequality

$$m_1 \alpha^2 + m_1 \|w\|_\infty^2 \leq \frac{1}{2}(\kappa_1 m_1 \alpha^2 + \kappa_2 \|w\|_\infty c_w)$$

holds. Substitution of $\|w\|_\infty = \eta(\alpha + c_w)$ and rearranging terms results in the equivalent inequality

$$\alpha^2 m_1 \left(\frac{1}{2}\kappa_1 - (1+\eta^2)\right) + \alpha c_w \eta \left(\frac{1}{2}\kappa_2 - 2m_1\eta\right) + \eta c_w^2 \left(\frac{1}{2}\kappa_2 - m_1 \eta\right) \geq 0.$$

This inequality holds for κ_1, κ_2 as in (4.11). Hence, with κ_1, κ_2 as in (4.11) the condition (4.14) is fulfilled. To prove (4.15) we note that

$$\begin{aligned}
\alpha|\langle (M_w M - M M_w)x, y\rangle_k| &\leq \alpha(\langle |M_w M x, y\rangle_k| + \alpha|\langle M M_w x, y\rangle_k|, \\
\alpha|\langle M_w M x, y\rangle_k| = \alpha|\langle Mx, M_w y\rangle_k| &\leq \tfrac{1}{2}\left(\alpha^2\langle M^2 x, x\rangle_k + \langle M_w^2 y, y\rangle_k\right), \\
\alpha|\langle M M_w x, y\rangle_k| = \alpha|\langle M_w x, My\rangle_k| &\leq \tfrac{1}{2}\left(\langle M_w^2 x, x\rangle_k + \alpha^2\langle M^2 y, y\rangle_k\right).
\end{aligned}$$

Thus (4.15) follows from (4.14). We conclude that (4.15) and (4.14) are satisfied for κ_1, κ_2 as in (4.11). Hence, $R_2^T R_2 \leq \frac{1}{2}(R_2^T + R_2)$ holds. And due to Lemma 4.3

$$\|I - 2R_2\| \leq 1. \tag{4.16}$$

Finally, (4.12), (4.13), and (4.16) yield

$$\|I - W_k^{-1} L_k\| = \|I - (R_1 + R_2)\| \leq \frac{1}{2}\|I - 2R_1\| + \frac{1}{2}\|I - 2R_2\| \leq 1. \qquad \square$$

THEOREM 4.5. *Assume that* (A1) *and* (A2) *are satisfied. Consider the block Richardson method with* W_k *as in* (4.6) *and*

$$\beta_1 = 2\left(\frac{\nu a_1}{h_k^2} + \alpha \kappa_1 m_1\right), \qquad \beta_2 = 2\kappa_2 C_w,$$

with constants κ_1, κ_2 *from* (4.11). *Then the following estimate holds:*

$$\|L_k S_k^{\mu_1}\| \leq \frac{C}{\sqrt{\mu_1}}\left(\frac{\nu}{h^2} + \alpha + \|w\|_\infty\right), \quad \mu_1 = 1, 2, \ldots\ . \tag{4.17}$$

Proof. From Lemma 4.4 we obtain

$$\|I - 2W_k^{-1} L_k\| \leq 1. \tag{4.18}$$

Furthermore,

$$\begin{aligned}
\|W_k\| &= \rho\left(\begin{pmatrix} \beta_1 I & -\beta_2 I \\ \beta_2 I & \beta_1 I \end{pmatrix}\begin{pmatrix} \beta_1 I & \beta_2 I \\ -\beta_2 I & \beta_1 I \end{pmatrix}\right)^{\frac{1}{2}} \\
&= (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} \leq \beta_1 + \beta_2 \leq C\left(\frac{\nu}{h^2} + \alpha + \|w\|_\infty\right).
\end{aligned} \tag{4.19}$$

From (4.18) and (4.19) and Theorem 10.6.8 in [8] the result in (4.17) follows. $\square$

5. Numerical results. In this section results of a few numerical experiments related to the accuracy of the discretization method and the convergence behavior of the multigrid solver are presented. For the discretization we use linear conforming finite elements on a uniform triangulation of the unit square. The mesh size parameter is $h = h_k = 2^{-k},\ k = 4, 5, \dots, 9$.

In our experiments we consider problems with an a priori known continuous solution $\mathbf{u} \in H^2(\Omega)^2 \cap \mathbf{U}$ to the problem (2.1). Discretization errors are measured as follows. Let $\hat{\mathbf{u}}_h \in \mathbf{U}_h$ be the nodal interpolant of the continuous solution $\mathbf{u}$ and $\mathbf{u}_h \in \mathbf{U}_h$ be the solution of the discrete problem. As a measure for the discretization error we take

$$err(\mathbf{u}, h, \nu) = \frac{\|\hat{\mathbf{u}}_h - \mathbf{u}_h\|}{\|\mathbf{f}\|}. \tag{5.1}$$

For the iterative solution of the discrete problem a multigrid V-cycle is applied. The prolongations and restrictions in this multigrid method are the canonical ones, as in (4.4). For the smoother a damped block Jacobi method as in (4.5) is used. Thus for each pair of nodal values of $\{u_1, u_2\}$ a 2×2 linear system is solved. The damping parameter ω in each smoothing step is determined in a dynamic way based on a residual minimization criterion: We set $\omega = (\mathbf{q}, \mathbf{q})/(\mathbf{q}, \mathbf{r})$, where for grid level k

$$\mathbf{r} = \bar{W}_k^{-1}(L_k \mathbf{x}^{old} - b), \qquad \mathbf{q} = \bar{W}_k^{-1} L_k \mathbf{r},$$

and $\bar{W}_k$ equals W_k from (4.5) for $\omega = 1$.

We always use two pre- and two postsmoothing iterations. For the starting vector in the iterative solver we take $\mathbf{u}^0 = 0$. The iterations are stopped as soon as the residual, in the Euclidean norm, is at least a factor 10^9 smaller than the starting residual.

We consider test problems with different choices for w. Note that in the setting of a (linearized) Navier–Stokes problem $w = \operatorname{curl} \mathbf{v} = -\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y}$, where $\mathbf{v} = (v_1(x, y), v_2(x, y))$ is an approximation of the flow field. In Experiment I we consider a problem which corresponds to a flow with rotating vortices. In Experiment II we take a flow field $\mathbf{v}$ with a parabolic boundary layer behavior. Both in Experiment I and Experiment II the right-hand side is taken such that the continuous solution $\mathbf{u}$ equals the flow field $\mathbf{v}$. This seems a reasonable choice if the problem (2.1) results from a linearized Navier–Stokes problem. Finally, in Experiment III a flow $\mathbf{v}$ which exhibits an internal layer behavior is considered.

In all the experiments we present results for the case $\alpha = 0$. For $\alpha > 0$ in our numerical experiments we always observed better results than for $\alpha = 0$, both with respect to the discretization error and with respect to the multigrid convergence.

Experiment Ia. We take $\mathbf{v}_r = (v_1, v_2)$, with

$$\begin{aligned} v_1(x, y) &= 4(2y - 1)x(1 - x), \\ v_2(x, y) &= -4(2x - 1)y(1 - y), \end{aligned} \tag{5.2}$$

and $w = \operatorname{curl} \mathbf{v}_r$. This type of convection $\mathbf{v}_r$ simulates a rotating vortex. For this w the conditions (A2) and (A3) are fulfilled. Related to (A1) we note that $\|w\|_\infty = \mathcal{O}(1)$ and $c_w = 0$. However, based on the fact that w equals zero only at the corner points of the domain, one could say that (A1) is "almost" fulfilled. For several values of h and ν the quantity $err(\mathbf{u}, h, \nu)$ is given in Table 5.1.

In Figure 5.1 the differences $(u_1 - (u_h)_1)(0.5, y)$ and $(\frac{\partial u_1}{\partial y} - \frac{\partial (u_h)_1}{\partial y})(0.5, y)$ between (the derivatives of) the first components of the continuous and finite element solution

TABLE 5.1
$err(\mathbf{u}, h, \nu)$ for Experiment Ia.

	h					
ν	1/16	1/32	1/64	1/128	1/256	1/512
1	4.5e-4	1.1e-4	2.8e-5	7.2e-6	1.8e-6	4.5e-7
1e-2	8.6e-3	2.1e-3	5.2e-4	1.3e-4	3.3e-5	8.2e-6
1e-4	1.0e-2	2.7e-3	7.0e-4	1.7e-4	4.4e-5	1.1e-5
1e-6	1.0e-2	2.7e-3	7.7e-4	2.1e-4	5.4e-5	1.3e-5
1e-8	1.0e-2	2.7e-3	7.7e-4	2.1e-4	5.9e-5	1.6e-5

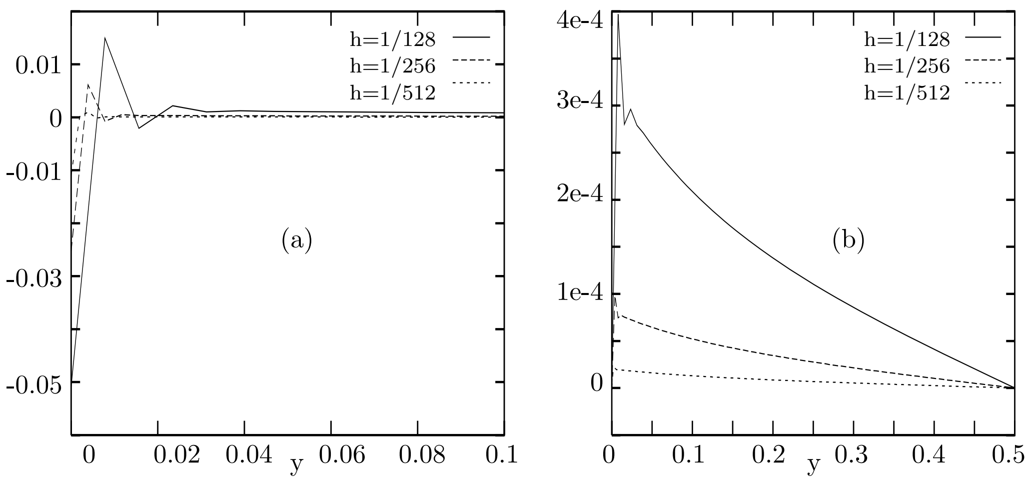

FIG. 5.1. *Discretization error in Experiment* Ia*;* $\nu = 10^{-6}$, $x = 0.5$ (a) *in y-derivative,* (b) *in solution.*

TABLE 5.2
V-cycle convergence for Experiment Ia.

	h				
ν	1/32	1/64	1/128	1/256	1/512
1	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)
1e-2	**11**(0.14)	**11**(0.14)	**11**(0.14)	**11**(0.15)	**11**(0.15)
1e-4	**6**(0.03)	**7**(0.05)	**9**(0.10)	**11**(0.14)	**11**(0.15)
1e-6	**5**(0.01)	**5**(0.01)	**5**(0.01)	**7**(0.04)	**7**(0.05)
1e-8	**5**(0.01)	**5**(0.01)	**5**(0.01)	**5**(0.01)	**5**(0.01)

Number of iterations and average reduction factor

are plotted for the case $\nu = 10^{-6}$. Because of the symmetry the error in the solution is shown only on half of the interval (Figure 5.1b) and the error in the solution derivative only on the interval $[0, 0.1]$ near the boundary (Figure 5.1a). The numerical boundary layer, typical for reaction-diffusion problems with dominating reaction terms, is clearly seen. Results for the convergence behavior of the multigrid method are shown in Table 5.2.

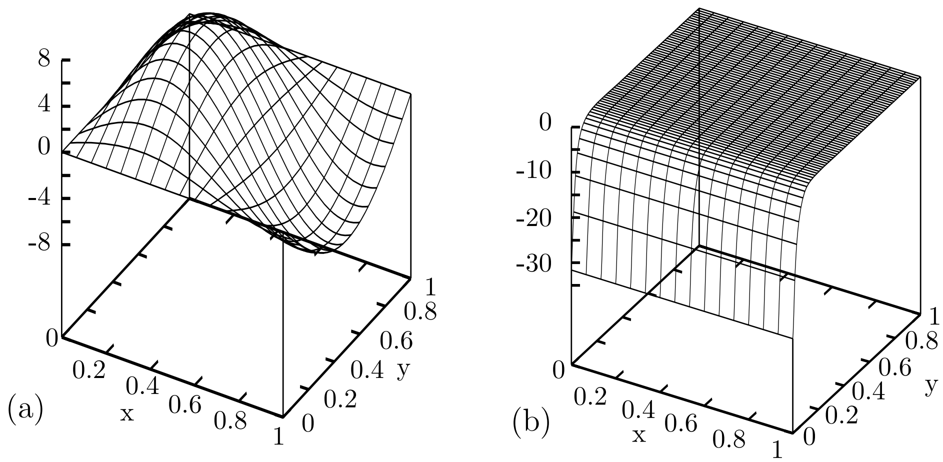

FIG. 5.2. (a) *Function* w *in Experiment* Ib*;* (b) *function* w *in Experiment* II*,* $\nu = 10^{-3}$.

TABLE 5.3
$err(\mathbf{u}, h, \nu)$ *for Experiment* Ib.

	h					
ν	1/16	1/32	1/64	1/128	1/256	1/512
1	1.9e-3	4.9e-4	1.2e-4	3.0e-5	7.5e-6	1.9e-6
1e-2	1.5e-2	3.6e-3	9.0e-4	2.3e-4	5.7e-5	1.4e-5
1e-4	4.8e-2	7.1e-3	1.8e-3	4.5e-4	1.1e-4	2.9e-5
1e-6	1.4e-1	7.8e-2	1.0e-2	9.5e-4	2.3e-4	5.7e-5
1e-8	1.4e-1	9.7e-2	6.7e-2	2.9e-2	2.0e-3	1.4e-4

Experiment Ib. We take $\mathbf{v}_R = (v_1, v_2)$, with

$$\begin{aligned} v_1(x,y) &= \frac{1}{\psi}\sin(\psi\pi x)\cos(\pi y), \\ v_2(x,y) &= -\cos(\psi\pi x)\sin(\pi y), \end{aligned} \tag{5.3}$$

and $w = \operatorname{curl} \mathbf{v}_R$. This models a flow with two vortices rotating in opposite directions. Note that the conditions (A1) and (A2) are not fulfilled. For the parameter ψ we choose $\psi = 1.6$. One vortex lies entirely in the computational domain, the second one only partially. The (vorticity) function w for this problem is plotted in Figure 5.2(a). Note the change of sign for w at $x = 0.625$. The error in the discrete solution shown in Table 5.3 is larger compared to example Ia (which might correspond to the strong violation of the conditions (A1) and (A2)). In Figure 5.3 the difference $(u_1 - (u_h)_1)(0.5, y)$ is plotted for $\nu = 10^{-6}$. Note that some local oscillations in the error are observed in the neighborhood of $x = 0.625$, i.e., where condition (A1) is *locally* violated. The results for the convergence behavior of the multigrid method are very similar to those in Table 5.2 for Experiment Ia.

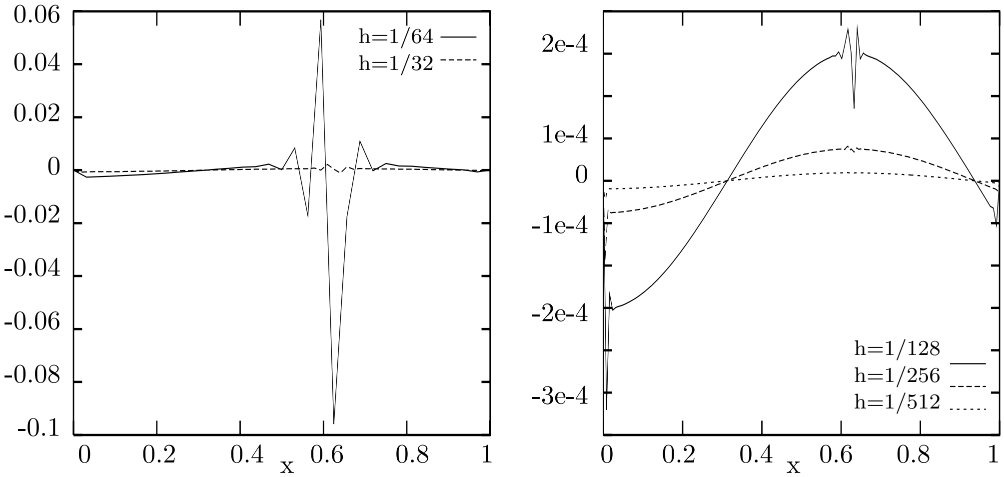

FIG. 5.3. *Error in finite element solutions in Experiment* Ib; $\nu = 10^{-6}$, $y = 0.5$.

TABLE 5.4
$err(\mathbf{u}, h, \nu)$ *for Experiment* II.

	h					
ν	1/16	1/32	1/64	1/128	1/256	1/512
1	7.4e-6	1.8e-6	4.5e-7	1.1e-7	2.8e-8	7.0e-9
1e-2	3.7e-3	8.6e-3	2.1e-4	5.3e-5	1.3e-5	2.2e-6
1e-4	4.2e-2	2.4e-2	3.1e-3	6.8e-4	1.6e-4	4.1e-5
1e-6	1.2e-2	1.2e-2	1.2e-2	1.2e-2	1.0e-2	8.0e-4
1e-8	3.9e-3	3.7e-3	3.7e-3	3.7e-3	3.6e-3	3.6e-3

Experiment II. We take $\mathbf{v}_l = (v_1, v_2)$, with

$$\begin{aligned} v_1(x,y) &= 1 - \exp(-y/\sqrt{\nu}), \\ v_2(x,y) &= 0, \end{aligned} \tag{5.4}$$

and $w = \operatorname{curl} \mathbf{v}_l$. This models a parabolic boundary layer behavior in the velocity field. The width of the layer is proportional to $\sqrt{\nu}$. Note that $\|w\|_\infty = O(\nu^{-1/2})$. The vorticity is of $\nu^{-\frac{1}{2}}$ magnitude near the boundary and decays exponentially outside the layer (see Figure 5.2(b)). As before, we take $\mathbf{f}$ such that the continuous solution equals the flow field: $\mathbf{u} = \mathbf{v}_l$. Results for the discretization error are given in Table 5.4. The L_2 norm of $\mathbf{f}$ is $O(\nu^{-\frac{1}{4}})$ for $\nu \to 0$; therefore one has to use a proper scaling of the values from Table 5.4 (e.g., multiplying by 10 for $\nu = 10^{-4}$) to obtain the absolute value of the error $\|\hat{\mathbf{u}}_h - \mathbf{u}_h\|$ (cf. (5.1)).

In Figure 5.4 we plot $u_1(0.5, y)$ and $(u_h)_1(0.5, y)$ for the cases $\nu = 10^{-3}$ and $\nu = 10^{-4}$ and for several h values. The finite element solution is a poor approximation to the continuous one if the boundary layer is not resolved: $h > \nu^{\frac{1}{2}}$. However, for $h \sim \nu^{\frac{1}{2}}$ the results are quite good, although both the *mesh* Reynolds numbers and Ek_h^{-1} are very large (e.g., $\approx 10^2$ for $\nu = 10^{-4}$). Moreover, no global oscillations are observed even for very coarse meshes. We expect that a significant improvement can be obtained if this simple full Galerkin discretization is combined with local grid

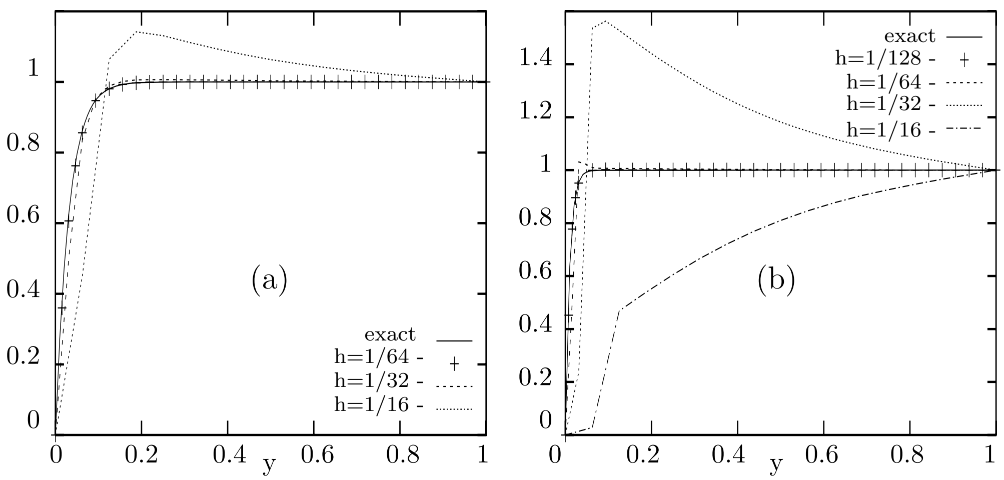

FIG. 5.4. *Exact and discrete solutions in Experiment* II*;* $x = 0.5$*:* (a) $\nu = 10^{-3}$*;* (b) $\nu = 10^{-4}$.

TABLE 5.5
V-cycle convergence for Experiment II.

	h				
ν	1/32	1/64	1/128	1/256	1/512
1	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)
1e-2	**12**(0.16)	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)
1e-4	**18**(0.30)	**17**(0.29)	**16**(0.26)	**14**(0.22)	**13**(0.19)
1e-6	**23**(0.40)	**29**(0.48)	**29**(0.49)	**28**(0.41)	**29**(0.48)
1e-8	**15**(0.24)	**19**(0.33)	**23**(0.40)	**28**(0.47)	**25**(0.43)

Number of iterations and average reduction factor

refinement in the boundary layer. In Table 5.5 numerical results for the multigrid method are presented. Note that assumptions (A1) and (A2) were also violated in this experiment. Hence our convergence analysis of the multigrid method does not apply here. One reason for the deterioration of multigrid convergence compared to the case Ib could be weaker regularity of the function w.

Experiment III. In this experiment we try to model the presence of an internal layer. To this end, for the convection field we take the model of the Euler flow (extreme case if $\nu \to 0$), where the tangential velocity component is discontinuous on some line in the interior of the domain. Hence the flow, potential a.e., has a vorticity concentrated on this line (so-called vortex sheet). We take $w = \operatorname{curl} \mathbf{v}_d$, with $\mathbf{v}_d = (v_1, v_2)$, and, for a given constant ψ,

$$\begin{cases} v_1(x,y) = \cos\psi \\ v_2(x,y) = \sin\psi \end{cases} \quad \text{if } \cos\psi > (x - 0.25)\sin\psi,$$

$$\begin{cases} v_1(x,y) = 0 \\ v_2(x,y) = 0 \end{cases} \quad \text{if } \cos\psi \le (x - 0.25)\sin\psi.$$

Using the parameter ψ one can vary the angle under which the layer enters the domain. We set $\psi = \pi/3$ so the grid is not aligned to the layer. For the discrete velocity $\mathbf{v}_h^d \in$

TABLE 5.6
V-cycle convergence for Experiment III.

	h				
ν	1/32	1/64	1/128	1/256	1/512
1	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)	**11**(0.15)
1e-2	**13**(0.20)	**13**(0.19)	**14**(0.22)	**14**(0.21)	**13**(0.19)
1e-4	**19**(0.33)	**19**(0.34)	**20**(0.35)	**21**(0.36)	**22**(0.38)
1e-6	**17**(0.29)	**20**(0.36)	**24**(0.42)	**28**(0.47)	**30**(0.50)
1e-8	**17**(0.29)	**20**(0.35)	**24**(0.42)	**28**(0.48)	**32**(0.53)

Number of iterations and average reduction factor

$\mathbf{U}_h$ we take the nodal interpolant of $\mathbf{v}_d$, and set $w = \operatorname{curl} \mathbf{v}_h^d$, obtaining a piecewise constant function w, which is essentially mesh-dependent due to the discontinuity of $\mathbf{v}_d$ ($\|w\|_\infty = O(h^{-1})$). Results for the convergence behavior of the multigrid method are given in Table 5.6.

Since discontinuous solutions are generally not allowed for viscous motions and our given data are mesh-dependent, we do not consider discretization errors in this example.

5.1. Discussion of numerical results. Recall that the analysis in the previous sections yields, for the case $\alpha = 0$,

$$err(\mathbf{u}, h, \nu) \le c \min\{\nu^{-1} h^2, \|w\|_\infty^{-1}\} \tag{5.5}$$

under certain assumptions on w. These assumptions are "almost valid" for the problem Ia and do not hold for the problems Ib and II.

The results of the numerical experiments indeed show the $O(h^2)$ behavior of $err(\mathbf{u}, h, \nu)$ unless ν is very small. In the latter case the second, ν- and h-independent, upper bound for $err(\mathbf{u}, h, \nu)$ in (5.5) is observed and $O(h^2)$ convergence is recovered for smaller h. For fixed h and $\nu \to 0$ a growth of the error is observed (up to some limit). In the experiments Ia,b this growth appears to be less than $O(\nu^{-1})$, indicating that the ν-dependence in (5.5) might be somewhat pessimistic for these cases.

Although in the last two examples the multigrid convergence for a small values of ν is somewhat worse, the multigrid V-cycle with block Jacobi smoothing appears to be a very robust solver. The convergence rates for realistic values of viscosity (in laminar flows $1 - 10^{-4}$) are excellent.

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