European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau (eds.)
O. Nevanlinna and R. Rannacher (assoc. eds.)
Jyväskylä, 24–28 July 2004

A STOKES INTERFACE PROBLEM: STABILITY, FINITE ELEMENT ANALYSIS AND A ROBUST SOLVER

Maxim A. Olshanskii[⋆], **Arnold Reusken**[†]

[⋆] Department of Mechanics and Mathematics
Moscow State University, Moscow 119899, Russia
e-mail: Maxim.Olshanskii@mtu-net.ru

[†] Institut für Geometrie und Praktische Mathematik
RWTH-Aachen, D-52056 Aachen, Germany
e-mail: reusken@igpm.rwth-aachen.de

Key words: Stokes equations, interface problem, two-phase flows, finite elements, preconditioning.

Abstract. *We consider a stationary Stokes problem with a piecewise constant viscosity coefficient. For the variational formulation of this problem in* $\mathbf{H}_0^1 \times L_2^0$ *we prove a well-posedness result in which the constants are uniform with respect to the jump in the viscosity coefficient. For a standard discretization with a pair of LBB stable finite element spaces we prove an infsup stability result uniform with respect to the jump in the viscosity coefficient. From this we derive an estimate for the discretization error. We introduce a robust preconditioner for the Schur complement of the discrete system. Results of numerical experiments are presented.*

1 Introduction

In many numerical simulations of two-phase flows a so-called one-fluid approach is used. In such a method the two phases are modeled by a single set of conservation laws for the whole computational domain. Such an approach leads to Navier-Stokes equations with discontinuous density and viscosity coefficients. The effect of surface tension can be taken into account by using a special localized force term at the interface. The latter approach is known as the continuum surface force (CSF) model, cf. [1]. One well-known technique for capturing the unknown interface is based on the level set method, cf. [16, 13, 8] and the references therein. If in such a setting one has highly viscous flows then the Stokes equations with discontinuous viscosity are a reasonable model problem.

In this paper we treat the following Stokes problem on a bounded connected domain $\Omega \subset \mathbb{R}^d$: Find a velocity $\mathbf{u}$ and a pressure p such that

$$
\begin{aligned}
-\mathrm{div}\,(\nu(\mathbf{x})\nabla\mathbf{u}) + \nabla p &= \mathbf{f} \quad \text{in } \Omega, && (1)\\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega, && (2)\\
\mathbf{u} &= 0 \quad \text{on } \partial\Omega, && (3)
\end{aligned}
$$

with a *piecewise constant viscosity*:

$$
\nu = \begin{cases} 1 & \text{in } \Omega_1 \\ \varepsilon & \text{in } \Omega_2, \quad \varepsilon \in (0,1]. \end{cases}
$$

The subdomains $\Omega_1,\ \Omega_2$ are assumed to be Lipschitz and such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. By Γ we denote the interface between the subdomains: $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. We assume that

$$
\mathrm{meas}(\partial\Omega_1 \cap \partial\Omega) > 0 \tag{4}
$$

holds. For the variational formulation of the interface problem we use the standard spaces $\mathbf{V} = H_0^1(\Omega)^d$ and

$$
Q := L_0^2(\Omega) = \{\, p \in L^2(\Omega) \mid \int_\Omega p\,dx = 0 \,\} .
$$

The variational problem is as follows: given $\mathbf{f} \in \mathbf{V}'$ find $\{\mathbf{u}, p\} \in \mathbf{V} \times Q$ such that

$$
\begin{aligned}
(\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (\mathrm{div}\,\mathbf{v}, p) &= \mathbf{f}(\mathbf{v}) \qquad \text{for } \ \mathbf{v} \in \mathbf{V},\\
(\mathrm{div}\,\mathbf{u}, q) &= 0 \qquad\quad\ \ \text{for } \ q \in Q .
\end{aligned} \tag{5}
$$

Here and in the remainder the L^2 scalar product and associated norm are denoted by $(\cdot,\cdot)$, $\|\cdot\|$, respectively. The bilinear form $(\nu\nabla\cdot, \nabla\cdot)$ defines a scalar product on $\mathbf{V}$. We use the norm induced by this scalar product:

$$
\|\mathbf{u}\|_{\mathbf{V}} := (\nu\nabla\mathbf{u}, \nabla\mathbf{u})^{\frac{1}{2}} \qquad \text{for } \ \mathbf{u} \in \mathbf{V}
$$

On Q, apart from the standard L^2 scalar product we will also use a weighted L^2 scalar product:

$$(p,q)_\nu := \int_\Omega \nu^{-1}\, p\, q\, dx = (\nu^{-1}p, q) \qquad \text{for} \;\; p,q \in Q \;, \tag{6}$$

and the norm $\|p\|_\nu := (p,p)_\nu^{\frac{1}{2}}$.

In this paper we analyze the Stokes problem with discontinuous viscosity given in (5). For pure diffusion problems (Poisson equation) with a discontinuous diffusion coefficient one can find analysis of discretization methods [2, 3, 6, 9, 17], error estimators [14, 4] and iterative solvers [5, 7, 15, 20] in the literature. For the Stokes interface problem however, much less is known. First theoretical results are presented in [12]. In that paper we analyzed the Stokes interface problem in the space $\mathbf{V} \times M$, with $M := \{\, p \in L^2(\Omega) \mid (p,1)_\nu = 0 \,\}$. Note that in the space M we use the nonstandard orthogonality condition $(p,1)_\nu = 0$. In [12] we proved uniform well-posedness and a uniform discrete infsup result with respect to the norms $\|\cdot\|_{\mathbf{V}}$ (in $\mathbf{V}$) and $\|\cdot\|_\nu$ (in M). In the present paper we derive similar results *but now for the more standard space* $Q = L_0^2(\Omega)$ instead of M. The main results of this paper (theorems 1,2 and 4) were already formulated in [12], however, *without proofs*. In the present paper we give a complete analysis including the proofs. Although the main ideas in this paper are the same as for the case $\mathbf{V} \times M$ in [12] there are significant differences both in the analysis and in the results, due to the fact that in the space Q we need a norm different from $\|\cdot\|_\nu$ (namely the one defined in (10) below). Further comments on the difference between the results in the space $\mathbf{V} \times M$ and in $\mathbf{V} \times Q$ are given in remark 2 at the end of section 3 and in remark 3 at the end of section 4.

In section 2 we introduce an appropriate norm on Q and prove a continuity and an infsup result that are uniform with respect to the parameter ε. Using standard arguments this then yields uniform well-posedness of the continuous Stokes problem.

In section 3 we consider the discrete variational problem in a pair of finite element spaces ($Q_h \subset Q$, $\mathbf{V}_h \subset \mathbf{V}$) that are assumed to be LBB stable. As a main result of this paper we present a discrete infsup result that is uniform with respect to the parameters h (mesh size) and ε. This result is used to derive a (sharp) uniform bound for the discretization error. In section 4 we derive a robust preconditioner for the Schur complement. In combination with known results on block-preconditioning and on multigrid this then implies optimality results for certain iterative methods. For a preconditioned MINRES method we present results of numerical experiments in section 5.

2 Uniform well-posedness of the variational problem

In this section we analyze the variational problem (5). We need some preliminaries. Let $\tilde{p}$ be the piecewise constant function

$$\tilde{p} = \begin{cases} \;\;|\Omega_1|^{-1} & \text{on } \Omega_1 \\ -|\Omega_2|^{-1} & \text{on } \Omega_2. \end{cases} \tag{7}$$

Since $(\tilde{p}, 1) = 0$, we have $\tilde{p} \in Q$. Consider the one-dimensional subspace $Q_0 := \text{span}\{\tilde{p}\}$ of Q and an L^2-orthogonal decomposition $Q = Q_0 \oplus Q_0^\perp$. For $p \in Q$ we use the notation

$$p = p_0 + p_0^\perp \ , \quad p_0 \in Q_0,\ p_0^\perp \in Q_0^\perp \tag{8}$$

One easily checks that

$$Q_0^\perp = \{\, p \in Q \mid (p,1)_{\Omega_1} = (p,1)_{\Omega_2} = 0 \,\}, \tag{9}$$

For functions in $Q_0^\perp$ we use the ν-norm from (6). On Q we introduce the norm

$$\|p\|_Q := (\|p_0\|^2 + \|p_0^\perp\|_\nu^2)^{\frac{1}{2}}. \tag{10}$$

The following uniform continuity result holds for the bilinear form $(\operatorname{div}\cdot,\cdot)$:

Lemma 1 *There exists a constant C independent of ε such that*

$$|(\operatorname{div}\mathbf{u}, p)| \le C\|\mathbf{u}\|_{\mathbf{V}}\|p\|_Q \quad \textit{for all} \ \ \mathbf{u} \in \mathbf{V},\ p \in Q$$

Proof. We decompose $p = p_0 + p_0^\perp$. For the component $p_0^\perp$ the Cauchy inequality gives for any $\mathbf{u} \in \mathbf{V}$:

$$(\operatorname{div}\mathbf{u}, p_0^\perp) \le \|\nu^{\frac{1}{2}}\operatorname{div}\mathbf{u}\|\|p_0^\perp\|_\nu \le \sqrt{d}\|\mathbf{u}\|_{\mathbf{V}}\|p_0^\perp\|_\nu. \tag{11}$$

Let $\mathbf{u}|_\Gamma$ denote the trace of $\mathbf{u}$ on Γ. By a trace theorem one gets $\mathbf{u} \in \mathbf{H}^{\frac{1}{2}}(\Gamma) \hookrightarrow \mathbf{L}_2(\Gamma)$. Moreover,

$$|\int_\Gamma \mathbf{u}{\cdot}\mathbf{n}\, d\,s| \le (\int_\Gamma \mathbf{u}^2\, d\,s)^{\frac{1}{2}}(\int_\Gamma 1\, d\,s)^{\frac{1}{2}} \le c_1\,\|\mathbf{u}|_\Gamma\|_{L_2(\Gamma)}.$$

For the component p_0 (constant in each subdomain) and for arbitrary $\mathbf{u} \in \mathbf{V}$ we have due to the Stokes formula:

$$\begin{aligned} |(\operatorname{div}\mathbf{u}, p_0)| &= |\int_\Gamma \mathbf{u}{\cdot}\mathbf{n}\, d\,s| \left(\|p_0\|_{L_\infty(\Omega_1)} + \|p_0\|_{L_\infty(\Omega_2)}\right) \\ &\le c_1\|\mathbf{u}|_\Gamma\|_{L_2(\Gamma)} \left(\frac{1}{\sqrt{|\Omega_1|}}\|p_0\|_{\Omega_1} + \frac{1}{\sqrt{|\Omega_2|}}\|p_0\|_{\Omega_2}\right) \\ &\le c_2\,\|\mathbf{u}\|_{H^1(\Omega_1)}\|p_0\| \le c_3\|\nabla\mathbf{u}\|_{\Omega_1}\|p_0\| \le c_3\|\mathbf{u}\|_{\mathbf{V}}\|p_0\|. \end{aligned} \tag{12}$$

For the estimate $\|\mathbf{u}\|_{H^1(\Omega_1)} \le c\,\|\nabla\mathbf{u}\|_{\Omega_1}$ we used assumption (4). Taking a sum of (12) and (11) completes the proof. $\square$

In the next theorem we prove a uniform infsup property corresponding to the problem (5). It generalizes the well-known Nečas inequality:

$$c(\Omega)\|p\| \le \|\nabla p\|_{-1} := \sup_{\mathbf{u}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{u}, p)}{\|\nabla\mathbf{u}\|} \quad \forall\ p \in Q,$$

with $c(\Omega) > 0$. We will need an equivalent form of this Nečas inequality: for any $p \in Q$ there exists $\mathbf{u} \in \mathbf{V}$ such that

$$\|p\|^2 = (\operatorname{div}\mathbf{u}, p) \quad \text{and} \quad c(\Omega)\|\nabla\mathbf{u}\| \le \|p\|. \tag{13}$$

Theorem 1 *There exists a constant $c > 0$ independent of ε such that*

$$\sup_{\mathbf{u}\in\mathbf{V}} \frac{(\operatorname{div}\mathbf{u}, p)}{\|\mathbf{u}\|_{\mathbf{V}}} \geq c\,\|p\|_Q \quad \textit{for all} \;\; p \in Q.$$

Proof. Fix an arbitrary $p \in Q$. First consider the component $p_0^\perp$ from (8). Since $p_0^\perp|_{\Omega_k} \in L^2(\Omega_k)$ and $(p_0^\perp, 1)_{\Omega_k} = 0$ for $k = 1, 2$, we can apply the Nečas inequality in the form (13) in each subdomain. Thus there exists a function $\mathbf{u}_1 \in H_0^1(\Omega_1)^d$ such that the following relations hold with a constant $c(\Omega_1) > 0$:

$$\|p_0^\perp\|^2_{\Omega_1} = (\operatorname{div}\mathbf{u}_1, p_0^\perp)_{\Omega_1} \quad \text{and} \quad c(\Omega_1)\|\nabla\mathbf{u}_1\|_{\Omega_1} \leq \|p_0^\perp\|_{\Omega_1} \tag{14}$$

Similarly, using a scaling argument, it follows that there exists $\mathbf{u}_2 \in H_0^1(\Omega_2)^d$ such that

$$\|\varepsilon^{-\frac{1}{2}} p_0^\perp\|^2_{\Omega_2} = (\operatorname{div}\mathbf{u}_2, p_0^\perp)_{\Omega_2} \;\; \text{and} \;\; c(\Omega_2)\|\varepsilon^{\frac{1}{2}}\nabla\mathbf{u}_2\|_{\Omega_2} \leq \|\varepsilon^{-\frac{1}{2}} p_0^\perp\|_{\Omega_2}, \tag{15}$$

with $c(\Omega_2) > 0$. Extending $\mathbf{u}_1$ and $\mathbf{u}_2$ by zero on the whole domain Ω and taking a sum of (14) and (15) we get

$$\|p_0^\perp\|^2_\nu = (\operatorname{div}\tilde{\mathbf{u}}, p_0^\perp) \quad \text{and} \quad c_1\|\tilde{\mathbf{u}}\|_{\mathbf{V}} \leq \|p_0^\perp\|_\nu, \quad \tilde{\mathbf{u}} := \mathbf{u}_1 + \mathbf{u}_2,$$

with $c_1 = \min\{c(\Omega_1), c(\Omega_2)\}$. For the component p_0 we use the Nečas inequality (13): there exists $\bar{\mathbf{u}} \in H_0^1(\Omega)^d$ such that

$$\|p_0\|^2 = (\operatorname{div}\bar{\mathbf{u}}, p_0) \quad \text{and} \quad c(\Omega)\|\nabla\bar{\mathbf{u}}\| \leq \|p_0\|.$$

We also have

$$(\operatorname{div}\tilde{\mathbf{u}}, p_0) = 0, \quad \|p_0^\perp\|_\nu \geq \|p_0^\perp\|, \quad \|\bar{\mathbf{u}}\|_{\mathbf{V}} \leq \|\nabla\bar{\mathbf{u}}\|, \quad \|\operatorname{div}\bar{\mathbf{u}}\| \leq \sqrt{d}\|\nabla\bar{\mathbf{u}}\|.$$

Thus for any $\alpha > 0$ and $\delta > 0$ we get

$$\begin{aligned}
(\operatorname{div}(\tilde{\mathbf{u}} + \alpha\bar{\mathbf{u}}), p) &= \|p_0^\perp\|^2_\nu + \alpha\|p_0\|^2 + \alpha(\operatorname{div}\bar{\mathbf{u}}, p_0^\perp) \\
&\geq \|p_0^\perp\|^2_\nu + \alpha\|p_0\|^2 - \alpha\sqrt{d}\,c(\Omega)^{-1}\|p_0\|\|p_0^\perp\| \\
&\geq (1 - \frac{\delta\alpha\sqrt{d}}{2c(\Omega)})\|p_0^\perp\|^2_\nu + \alpha(1 - \frac{\sqrt{d}}{2c(\Omega)\delta})\|p_0\|^2.
\end{aligned}$$

The choice $\delta = \sqrt{d}\,c(\Omega)^{-1}$ and $\alpha_0 = \frac{2c(\Omega)^2}{c(\Omega)^2 + d}$ leads to

$$(\operatorname{div}(\tilde{\mathbf{u}} + \alpha_0\bar{\mathbf{u}}), p) \geq c_1(\|p_0^\perp\|^2_\nu + \|p_0\|^2) \quad \text{with } c_1 = \frac{c(\Omega)^2}{c(\Omega)^2 + d}.$$

Hence, for $\mathbf{u} = \tilde{\mathbf{u}} + \alpha_0\bar{\mathbf{u}}$ we get

$$c_1\|p\|^2_Q \leq (\operatorname{div}\mathbf{u}, p) \quad \text{and} \quad c\|\mathbf{u}\|^2_{\mathbf{V}} \leq \|p_0^\perp\|^2_\nu + \|p_0\|^2 = \|p\|^2_Q,$$

with $c > 0$ independent of ε. This implies the result of the theorem. □

From these results it follows that we have uniform (w.r.t. ε) well-posedness of the continuous variational interface Stokes problem in the spaces $\mathbf{V}$, Q with the norms $\|\cdot\|_{\mathbf{V}}$ and $\|\cdot\|_Q$, respectively. Using standard arguments (cf. [11]) it can be shown that the variational problem has a unique solution and that the a priori estimate

$$(\|\mathbf{u}\|_{\mathbf{V}}^2 + \|p\|_Q^2)^{\frac{1}{2}} \le c\|\mathbf{f}\|_{\mathbf{V}'} \tag{16}$$

holds with a constant c independent of $\mathbf{f}$ and of ε.

The dual norm $\|\mathbf{f}\|_{\mathbf{V}'}$ in (16) can replaced by a more trackable norm of $\mathbf{f}$. For this we need the Poincare type inequality

$$\|\nu^{\frac{1}{2}}\mathbf{v}\| \le C_P\|\mathbf{v}\|_{\mathbf{V}}, \quad \text{for all } \ \mathbf{v} \in \mathbf{V}. \tag{17}$$

The optimal constant C_P in (17) is uniformly bounded w.r.t. ε (cf. [12]). For $\mathbf{f} \in L_2(\Omega)^d$ the Cauchy inequality and (17) immediately yield the a-priori estimate

$$(\|\mathbf{u}\|_{\mathbf{V}}^2 + \|p\|_Q^2)^{\frac{1}{2}} \le c\, C_P\|\nu^{-\frac{1}{2}}\mathbf{f}\|,$$

with $c\, C_P$ independent of $\mathbf{f}$ and of ε.

3 Finite element discretization error analysis

In this section a finite element discretization of the Stokes interface problem using conforming finite element spaces is analyzed.

We consider a family of triangulations $\{\mathcal{T}_h\}$ in the sense of [10] and assume that each triangulation $\mathcal{T}_h$ is conforming w.r.t. the two subdomains Ω_1, Ω_2 in the following sense:

$$\exists\ \mathcal{T}_h^{(i)} \subset \mathcal{T}_h \ : \quad \cup\{\, T \mid T \in \mathcal{T}_h^{(i)}\,\} = \overline{\Omega}_i, \quad i = 1, 2 \tag{18}$$

This assumption is easily fulfilled if Ω_1 and Ω_2 are polyhedral subdomains.

Remark 1 In computational fluid dynamics for two-phase flow problems it is (more) realistic to assume that $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is smooth. Then the assumption (18) in general does not hold. However, in such applications it is common practice to approximate Γ by a polyhedral discrete interface Γ_h. In such a setting the assumption (18) may still make sense. As far as we know no rigorous analysis is available which for the (Navier)-Stokes equations shows the effect of approximating the smooth interface Γ by a piecewise smooth interface Γ_h. A theoretical analysis of this effect for a Poisson interface problem can be found in [9]. The results in [9], however, are not robust with respect to the jump in the diffusion coefficient.

We assume a pair of finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$ that is LBB stable with a constant $\hat{\beta}$ independent of h:

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\| \|q_h\|} \ge \hat{\beta} > 0\ .$$

For the analysis it is convenient to introduce the bilinear form

$$a(\mathbf{u}, p; \mathbf{v}, q) := (\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) + (\operatorname{div}\mathbf{u}, q)$$

on $(\mathbf{V}\times M)\times(\mathbf{V}\times M)$. The discrete problem is as follows: find $\{\mathbf{u}_h, p_h\}\in\mathbf{V}_h\times Q_h$ such that

$$a(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = \mathbf{f}(\mathbf{v}_h) \quad \text{for all} \quad \{\mathbf{v}_h, q_h\}\in\mathbf{V}_h\times Q_h\ . \tag{19}$$

In the proof of the discrete infsup condition below we will use a decomposition which is similar, but *not identical* to the one from the previous section. Let $\tilde{p}_h\in Q_h$ be the L^2-orthogonal projection of $\tilde{p}$ on Q_h,

$$(\tilde{p}-\tilde{p}_h, q_h) = 0 \quad \text{for all} \quad q_h\in Q_h \tag{20}$$

and define the one-dimensional subspace $Q_{0,h} := \operatorname{span}(\tilde{p}_h)$ of Q_h. This induces an L^2-orthogonal decomposition $Q = Q_{0,h}\oplus Q_{0,h}^{\perp}$, and for any $p\in Q$ we write

$$p = p_{0,h} + p_{0,h}^{\perp}, \qquad p_{0,h}\in Q_{0,h},\ p_{0,h}^{\perp}\in Q_{0,h}^{\perp}. \tag{21}$$

For this decomposition we have a discrete analog of the result in (9):

Lemma 2 *The following property holds:*

$$Q_h\cap Q_{0,h}^{\perp} = \{\, p_h\in Q_h \mid (p_h, 1)_{\Omega_1} = (p_h, 1)_{\Omega_2} = 0\,\}$$

Proof. For $p_h\in Q_h$ we define $c_i := (p_h, 1)_{\Omega_i},\ i=1,2$. Due to $Q_h\subset L_0^2(\Omega)$ the equality $c_1+c_2=0$ holds. By definition of $Q_{0,h}^{\perp}$ we have $p_h\in Q_{0,h}^{\perp}$ iff $(p_h,\tilde{p}_h)=0$. Hence, due to (20) we have $p_h\in Q_{0,h}^{\perp}$ iff $(p_h,\tilde{p})=0$. From the definition of $\tilde{p}$ and $c_1+c_2=0$ we get

$$(p_h,\tilde{p}) = 0 \ \Leftrightarrow\ \frac{1}{|\Omega_1|}c_1 - \frac{1}{|\Omega_2|}c_2 = 0 \ \Leftrightarrow\ (\frac{1}{|\Omega_1|}+\frac{1}{|\Omega_2|})c_1 = 0 \ \Leftrightarrow\ c_1 = c_2 = 0$$

and thus the result of the lemma holds. $\square$

Based on the decomposition (21) we introduce an *h-dependent* norm on Q:

$$\|p\|_{Q,h} := (\|p_{0,h}\|^2 + \|p_{0,h}^{\perp}\|_{\nu}^2)^{\frac{1}{2}} \qquad \text{for} \quad p\in Q$$

We define the quantity

$$\tilde{\mu}_h := \frac{\|\tilde{p}-\tilde{p}_h\|}{\|\tilde{p}\|}\ ,$$

which measures the error made by approximating $\tilde{p}$ in the finite element pressure space. In particular, $\tilde{\mu}_h = 0$ if Q_h contains piecewise constant finite elements. In general we have $\tilde{\mu}_h = \mathcal{O}(\tilde{h}^{\frac{1}{2}})$, where $\tilde{h}$ is the maximal diameter of the elements in $\mathcal{T}_h$ that have a nonempty intersection with Γ. We will assume $\tilde{\mu}_h\le\frac{1}{2}$.

The following continuity results hold:

Lemma 3 *There exists a constant C independent of h and ε such that for all $\mathbf{u}\in\mathbf{V}$, $p\in Q$ the following inequalities hold*

$$|(\operatorname{div}\mathbf{u},p)| \le C\|\mathbf{u}\|_{\mathbf{V}}(\|p\|^2_{Q,h}+\tfrac{\tilde\mu_h^2}{\varepsilon}\|p_{0,h}\|^2)^{\frac12} \tag{22}$$
$$|(\operatorname{div}\mathbf{u},p)| \le C(\|\mathbf{u}\|_{\mathbf{V}}+\tilde\mu_h\|\nabla\mathbf{u}\|)\|p\|_{Q,h}. \tag{23}$$

Proof. For $p\in Q$ we use the decomposition $p=p_{0,h}+p_{0,h}^\perp$. First we prove the estimate (22). For the component $p_{0,h}^\perp$ the Cauchy inequality gives for any $\mathbf{u}\in\mathbf{V}$:

$$(\operatorname{div}\mathbf{u},p_{0,h}^\perp)\le\|\nu^{\frac12}\operatorname{div}\mathbf{u}\|\|p_{0,h}^\perp\|_\nu\le\sqrt{d}\|\mathbf{u}\|_{\mathbf{V}}\|p_{0,h}^\perp\|_\nu. \tag{24}$$

For the component $p_{0,h}=\alpha\,\tilde p_h$ define $p_0:=\alpha\,\tilde p\in Q_0$. From $\|p_{0,h}-p_0\|=\tilde\mu_h\|p_0\|$ and $\tilde\mu_h\le\frac12$ it follows that $\|p_0\|\le(1-\tilde\mu_h)^{-1}\|p_{0,h}\|\le2\|p_{0,h}\|$. Now using the Stokes formula and a trace theorem, we have for any $\mathbf{u}\in\mathbf{V}$:

$$\begin{aligned}
|(\operatorname{div}\mathbf{u},p_{0,h})| &\le |(\operatorname{div}\mathbf{u},p_0)|+|(\operatorname{div}\mathbf{u},p_{0,h}-p_0)|\\
&\le 2|\int_\Gamma \mathbf{u}{\cdot}\mathbf{n}\,d\,s|\,\|p_0\|_{L_\infty(\Omega)}+\|\operatorname{div}\mathbf{u}\|\|p_{0,h}-p_0\|\\
&\le c_1\|\mathbf{u}|_\Gamma\|_{L_2(\Gamma)}\,\|p_0\|+\sqrt{d}\,\|\nabla\mathbf{u}\|\,\tilde\mu_h\|p_0\|\\
&\le c_2\,\|\mathbf{u}\|_{H^1(\Omega_1)}\|p_{0,h}\|+\sqrt{d}\,\varepsilon^{-\frac12}\|\mathbf{u}\|_{\mathbf{V}}\,2\,\tilde\mu_h\|p_{0,h}\|\\
&\le c_3\|\nabla\mathbf{u}\|_{\Omega_1}\|p_{0,h}\|+2\sqrt{d}\,\tilde\mu_h\varepsilon^{-\frac12}\|\mathbf{u}\|_{\mathbf{V}}\|p_{0,h}\|\\
&\le (c_3+2\sqrt{d}\,\tilde\mu_h\varepsilon^{-\frac12})\|\mathbf{u}\|_{\mathbf{V}}\|p_{0,h}\|.
\end{aligned}$$

Taking a sum of the latter inequality and (24) proves (22).

The estimate (23) is proved using similar arguments. For the component $p_{0,h}^\perp$ we already have the estimate (24). For the component $p_{0,h}$ we get using an estimate as in (12):

$$\begin{aligned}
|(\operatorname{div}\mathbf{u},p_{0,h})| &\le |(\operatorname{div}\mathbf{u},p_0)|+|(\operatorname{div}\mathbf{u},p_{0,h}-p_0)|\\
&\le c\|\mathbf{u}\|_{\mathbf{V}}\,\|p_{0,h}\|+\sqrt{d}\,\|\nabla\mathbf{u}\|\,2\,\tilde\mu_h\|p_{0,h}\|\\
&\le C(\|\mathbf{u}\|_{\mathbf{V}}+\tilde\mu_h\|\nabla\mathbf{u}\|)\|p_{0,h}\|.
\end{aligned}$$

Taking a sum of this inequality and (24) we obtain (23). □

We now prove a discrete infsup stability result.

Theorem 2 *There exists a constant $C>0$ independent of h and ε such that*

$$\sup_{\mathbf{u}_h\in\mathbf{V}_h}\frac{(\operatorname{div}\mathbf{u}_h,p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}}\ge C\|p_h\|_{Q,h}\quad \textit{for all}\ \ p_h\in Q_h$$

Proof. The proof uses arguments very similar to those used in the proof of theorem 1. Fix an arbitrary $p_h = p_{0,h} + p_{0,h}^{\perp} \in Q_h$. From lemma 2 we get $(p_{0,h}^{\perp}, 1)_{\Omega_i} = 0$, for $i = 1, 2$. Hence, we can apply the LBB stability result in each subdomain. This yields

$$\|p_{0,h}^{\perp}\|_{\nu}^2 = (\operatorname{div} \tilde{\mathbf{u}}_h, p_{0,h}^{\perp}) \quad \text{and} \quad c\|\tilde{\mathbf{u}}_h\|_{\mathbf{V}} \leq \|p_{0,h}^{\perp}\|_{\nu}, \quad \tilde{\mathbf{u}}_h = \mathbf{u}_1 + \mathbf{u}_2,$$

for suitable functions $\mathbf{u}_1 \in \mathbf{V}_h$ with $\mathbf{u}_1 = 0$ on Ω_2 and $\mathbf{u}_2 \in \mathbf{V}_h$ with $\mathbf{u}_2 = 0$ on Ω_1. For the component $p_{0,h}$ the LBB condition gives a function $\bar{\mathbf{u}}_h \in \mathbf{V}_h$ such that

$$\|p_{0,h}\|^2 = (\operatorname{div} \bar{\mathbf{u}}_h, p_{0,h}) \quad \text{and} \quad \hat{\beta}\,\|\nabla \bar{\mathbf{u}}_h\| \leq \|p_{0,h}\|.$$

Therefore we have for any $\alpha > 0$ and $\delta > 0$

$$(\operatorname{div}(\tilde{\mathbf{u}}_h + \alpha\bar{\mathbf{u}}_h), p_h) \geq (1 - \frac{\delta\alpha\sqrt{d}}{2\hat{\beta}})\|p_{0,h}^{\perp}\|_{\nu}^2 + \alpha(1 - \frac{\sqrt{d}}{2\hat{\beta}\delta})\|p_{0,h}\|^2.$$

Therefore setting $\mathbf{u} = \tilde{\mathbf{u}} + \alpha_0\bar{\mathbf{u}}$, $\delta = \sqrt{d}\,\hat{\beta}^{-1}$ and $\alpha_0 = \frac{2\hat{\beta}^2}{\hat{\beta}^2+d}$ we get

$$\|p_h\|_{Q,h}^2 \leq c\,(\operatorname{div} \mathbf{u}_h, p_h) \quad \text{and} \quad c\|\mathbf{u}_h\|_{\mathbf{V}}^2 \leq \|p_{0,h}^{\perp}\|_{\nu}^2 + \|p_{0,h}\|^2 = \|p_h\|_{Q,h}^2.$$

□

Now one can use standard arguments (cf. [11]) to derive continuity and stability results for the bilinear form $a(\cdot,\cdot)$. It is convenient to introduce the product norm

$$|||\mathbf{u}, p||| = (\|\mathbf{u}\|_{\mathbf{V}}^2 + \|p\|_{Q,h}^2)^{\frac{1}{2}} \quad \{\mathbf{u}, p\} \in \mathbf{V} \times Q$$

Lemma 3 and theorem 2 yield the following continuity and stability results:

Theorem 3 *There exist constants C and $c > 0$ independent of h and of ε such that*

$$a(\mathbf{u}, p; \mathbf{v}_h, q_h) \leq C(\|\mathbf{u}\|_{\mathbf{V}}^2 + \mu_h^2\|\nabla\mathbf{u}\|^2 + \|p\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\varepsilon}\|p_{0,h}\|^2)^{\frac{1}{2}}|||\mathbf{v}_h, q_h|||$$

for all $\{\mathbf{u}, p\} \in \mathbf{V} \times Q$, $\{\mathbf{v}_h, q_h\} \in \mathbf{V}_h \times Q_h$ and

$$\sup_{\{\mathbf{v}_h, q_h\} \in \mathbf{V}_h \times Q_h} \frac{a(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h)}{|||\mathbf{v}_h, q_h|||} \geq c\,|||\mathbf{u}_h, p_h||| \quad \textit{for all} \quad \{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times Q_h$$

□

As for the continuous problem we get as a direct corollary that the discrete problem (19) has a unique solution $\{\mathbf{u}_h, p_h\}$ and the inequality

$$|||\mathbf{u}_h, p_h||| \leq c^{-1}\|\mathbf{f}\|_{\mathbf{V}_h'}$$

holds, with the constant c from theorem 3. Moreover, if $\mathbf{f} \in L_2(\Omega)^d$, then using the Cauchy inequality and the Poincare inequality (17) we obtain the a-priori estimate:

$$|||\mathbf{u}_h, p_h||| \leq c^{-1}\,C_P\|\nu^{-\frac{1}{2}}\mathbf{f}\|.$$

Using the continuity and the infsup results in theorem 3 we can prove a discretization error bound.

Theorem 4 *Let $\{\mathbf{u}, p\}$ be the solution of the continuous problem (5) and $\{\mathbf{u}_h, p_h\}$ the solution of the discrete problem (19). The following holds with a constant C independent of h and of ε:*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}^2 + \|p-p_h\|_{Q,h}^2 \;\; &\leq \;\; C\Big(\min_{\mathbf{v}_h\in\mathbf{V}_h} (\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{V}}^2 + \tilde{\mu}_h^2 \|\nabla(\mathbf{u}-\mathbf{v}_h)\|^2) \\ &\qquad + \min_{q_h\in Q_h} (\|p-q_h\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\varepsilon}\|(p-q_h)_{0,h}\|^2)\Big)\end{aligned}$$

Proof. For arbitrary $\mathbf{v}_h \in \mathbf{V}_h$, $q_h \in Q_h$ define $\mathbf{e} := \mathbf{u}-\mathbf{v}_h$, $\mathbf{e}_h = \mathbf{u}_h-\mathbf{v}_h$, $g := p-q_h$, $g_h := p_h - q_h$. The Galerkin orthogonality property yields

$$a(\mathbf{e}_h, g_h; \mathbf{z}_h, r_h) = a(\mathbf{e}, g; \mathbf{z}_h, r_h) \quad \text{for all} \quad \{\mathbf{z}_h, r_h\} \in \mathbf{V}_h \times Q_h \; .$$

Using this in combination with the continuity and infsup results we obtain, for suitable $\{\mathbf{z}_h, r_h\} \in \mathbf{V}_h \times Q_h$:

$$\begin{aligned}|||\mathbf{e}_h, g_h||| \;\; &\leq \;\; c^{-1}\frac{a(\mathbf{e}_h, g_h; \mathbf{z}_h, r_h)}{|||\mathbf{z}_h, r_h|||} = c^{-1}\frac{a(\mathbf{e}, g; \mathbf{z}_h, r_h)}{|||\mathbf{z}_h, r_h|||} \\ &\leq \;\; c^{-1}C(\|\mathbf{e}\|_{\mathbf{V}}^2 + \tilde{\mu}_h^2\|\nabla\mathbf{e}\|^2 + \|g\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\varepsilon}\|g_{0,h}\|^2)^{\frac{1}{2}}.\end{aligned}$$

Now combine this with the triangle inequality $|||\mathbf{u}-\mathbf{u}_h, p-p_h||| \leq |||\mathbf{e}_h, g_h||| + |||\mathbf{e}, g|||$. □

Based on the result in theorem 4 and using approximation properties of the finite element spaces one can derive further bounds for the discretization error. For such an analysis one needs regularity results for the continuous Stokes interface problem. As far as we know, this regularity issue is largely unsolved.

Remark 2 In the bound in theorem 4 in addition to the norm $|||\mathbf{e}, g|||^2 = \|\mathbf{e}\|_{\mathbf{V}}^2 + \|g\|_{Q,h}^2$ two terms with $\tilde{\mu}_h$ occur. In remark 3 we give an indication that in the analysis as presented in this paper these terms can not be avoided. This, however, is not the case if instead of the space $\mathbf{V} \times Q$ one uses the (less standard) space $\mathbf{V} \times M$, where $M := \{\, p \in L^2(\Omega) \mid (p, 1)_\nu = 0 \,\}$ with scalar product $(\cdot,\cdot)_M = (\cdot,\cdot)_\nu$. An analysis for the space $\mathbf{V} \times M$ is presented in [12]. This analysis uses a splitting $M = M_0 \oplus M_0^{\perp_M}$, where M_0 is a one-dimensional space and $M_0^{\perp_M}$ is the orthogonal complement of this space w.r.t. $(\cdot,\cdot)_M$. It turns out that then perturbation terms like those with $\tilde{\mu}_h$ in theorem 4 can be avoided. For the discretization error we can prove a bound of the form (theorem 3.6 in [12]):

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}^2 + \|p-p_h\|_M^2 \leq C(\min_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{V}}^2 + \min_{q_h\in M_h} \|p-q_h\|_M^2)$$

This shows that for a theoretical analysis the nonstandard space M seems to be more natural than $Q = L_0^2(\Omega)$.

4 Preconditioners for the Schur complement

In this section we discuss several preconditioners for the Schur complement. For this we first introduce the standard matrix-vector formulation of the discrete problem (19).

In practice the discrete space Q_h for the pressure is constructed by taking a standard finite element space, which we denote by Q_h^+ (for example, continuous piecewise linear functions) and then adding an orthogonality condition:

$$Q_h = \{\, p_h \in Q_h^+ \mid (p_h, 1) = 0 \,\}$$

We assume standard (nodal) bases in $\mathbf{V}_h$ and Q_h^+. The bilinear forms $(\nu\nabla\cdot, \nabla\cdot)$, $(\operatorname{div}\cdot, \cdot)$ yield stiffness matrices $\mathrm{A} \in \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{m\times n}$, with $n := \dim(\mathbf{V}_h)$, $m := \dim(Q_h^+)$. Let $\mathrm{M} \in \mathbb{R}^{m\times m}$ be the mass matrix of Q_h^+ w.r.t. $(\cdot,\cdot)$ and $J_Q : \mathbb{R}^m \to Q_h^+$ the finite element isomorphism ($\mathrm{y} \in \mathbb{R}^m$ contains the nodal values of $J_Q\mathrm{y}$).

The matrix-vector formulation of the problem (19) is given by

$$\begin{pmatrix} \mathrm{A} & \mathrm{B}^T \\ \mathrm{B} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} \mathrm{f} \\ 0 \end{pmatrix} \tag{25}$$

The Schur complement is denoted by $\mathrm{S} := \mathrm{B}\mathrm{A}^{-1}\mathrm{B}^T$. Note that both S and the matrix in (25) are singular and have a one-dimensional kernel. Define the constant vector $\mathrm{e} := (1, \dots, 1)^T \in \mathbb{R}^m$. Then we have $\ker(\mathrm{S}) = \mathrm{span}\{\mathrm{e}\}$. To treat this singularity we need the space

$$(\mathrm{Me})^\perp := \{\, \mathrm{y} \in \mathbb{R}^m \mid \langle \mathrm{y}, \mathrm{Me}\rangle = 0 \,\}\ ,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product. Since

$$Q_h = \{\, J_Q\mathrm{y} \mid \mathrm{y} \in (\mathrm{Me})^\perp \,\}$$

we get the following matrix-vector representation of the discrete problem:

$$\text{Find}\quad (\mathrm{x}, \mathrm{y}) \in \mathbb{R}^n \times (\mathrm{Me})^\perp \quad \text{such that (25) holds} \tag{26}$$

In preconditioned MINRES and (inexact) Uzawa type of methods for solving (26) one needs good preconditioners Q_A of A and Q_S of S. It is known that if for Q_A we take a symmetric multigrid V-cycle then we have (cf. [5, 7, 20])

$$(1 - \sigma_A)\mathrm{Q}_A \le \mathrm{A} \le \mathrm{Q}_A,$$

with a constant $\sigma_A < 1$ independent of h and of ε.

Below we introduce preconditioners for S. Theorem 2 and estimate (22) imply

$$c_1\|p_h\|_{Q,h} \le \sup_{\mathbf{u}_h\in\mathbf{V}_h} \frac{(\operatorname{div}\mathbf{u}_h, p_h)}{\|\mathbf{u}_h\|_{\mathbf{V}}} \le c_2\,(\|p_h\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\varepsilon}\|p_{0,h}\|^2)^{\frac12} \quad \text{for } p_h \in Q_h \tag{27}$$

with $c_1 > 0$, c_2 independent of ε and h. From the definition of the Schur complement it follows that for arbitrary $\mathrm{y} \in \mathbb{R}^m$ we have

$$\langle \mathrm{Sy}, \mathrm{y} \rangle = \sup_{\mathbf{u}_h \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{u}_h, J_Q \mathrm{y})^2}{\|\mathbf{u}_h\|_{\mathbf{V}}^2} \tag{28}$$

Let G be the mass matrix of Q_h^+ w.r.t. the scalar product induced by the L^2-orthogonal decomposition in (21):

$$(p_h, q_h)_{Q,h} := (p_{0,h}, q_{0,h}) + (p_{0,h}^{\perp}, q_{0,h}^{\perp})_\nu \ , \quad p_h, q_h \in Q_h^+$$

The vector representation of $\tilde{p}_h \in Q_h$ from (20) is denoted by $\tilde{\mathrm{p}}$. i.e., $J_Q \tilde{\mathrm{p}} = \tilde{p}_h$.

Theorem 5 *The spectral equivalences*

$$c_1^2 \langle \mathrm{Gy}, \mathrm{y} \rangle \ \leq \langle \mathrm{Sy}, \mathrm{y} \rangle \ \leq c_2^2 (1 + \frac{\tilde{\mu}_h^2}{\varepsilon}) \langle \mathrm{Gy}, \mathrm{y} \rangle \qquad \forall \ \mathrm{y} \in (\mathrm{Me})^{\perp} \tag{29}$$

$$c_1^2 \langle \mathrm{Gy}, \mathrm{y} \rangle \ \leq \langle \mathrm{Sy}, \mathrm{y} \rangle \ \leq c_2^2 \langle \mathrm{Gy}, \mathrm{y} \rangle \qquad \forall \ \mathrm{y} \in (\mathrm{Me})^{\perp} \cap (\mathrm{M}\tilde{\mathrm{p}})^{\perp} \tag{30}$$

hold with c_1, c_2 as in (27).

Proof. From the definition of the mass matrix G we get

$$\langle \mathrm{Gy}, \mathrm{y} \rangle = \langle J_Q \mathrm{y}, J_Q \mathrm{y} \rangle_{Q,h} = \|J_Q \mathrm{y}\|_{Q,h}^2 \quad \text{for all} \ \ \mathrm{y} \in \mathbb{R}^m \tag{31}$$

We also have

$$\|p_h\|_{Q,h}^2 + \frac{\tilde{\mu}_h^2}{\varepsilon} \|p_{0,h}\|^2 \leq (1 + \frac{\tilde{\mu}_h^2}{\varepsilon}) \|p_h\|_{Q,h}^2 \tag{32}$$

Note that $J_Q \mathrm{y} = p_h \in Q_h$ iff $\mathrm{y} \in (\mathrm{Me})^{\perp}$. The result in (29) now follows from (27),(28),(31) and (32). For $\mathrm{y} \in \mathbb{R}^m$ let $J_Q \mathrm{y} =: p_h = p_{0,h} + p_{0,h}^{\perp}$ with $p_{0,h} \in \operatorname{span}(\tilde{p}_h)$ and $(p_{0,h}^{\perp}, \tilde{p}_h) = 0$. For $\mathrm{y} \in (\mathrm{M}\tilde{\mathrm{p}})^{\perp}$ we have $\langle \mathrm{y}, \mathrm{M}\tilde{\mathrm{p}} \rangle = 0$ and thus $(p_h, \tilde{p}_h) = 0$. This implies $p_{0,h} = 0$ and thus the term with $\tilde{\mu}_h$ in (27) vanishes. This yields the result in (30). $\square$

We note that the deterioration (for $\varepsilon \downarrow 0$) of the upper bound in (29) is not a serious problem, because it is caused by the *one*-dimensional subspace $\operatorname{span}(\mathrm{M}\tilde{\mathrm{p}})$ (cf. (30)). If we apply a Krylov subspace solver with a preconditioner G for S then already after a few iterations this one-dimensional subspace does not influence the effective spectral condition number anymore.

From

$$\langle \mathrm{Gy}, \mathrm{y} \rangle = (J_Q \mathrm{y}, J_Q \mathrm{y})_\nu \qquad \forall \ \mathrm{y} \in (\mathrm{Me})^{\perp} \cap (\mathrm{M}\tilde{\mathrm{p}})^{\perp}$$

we see that on the $(m-2)$-dimensional subspace $(\mathrm{Me})^{\perp} \cap (\mathrm{M}\tilde{\mathrm{p}})^{\perp}$ the mass matrix G coincides with the mass matrix corresponding to the scalar product $(\cdot, \cdot)_\nu$. The latter mass matrix, which is denoted by M_ν, is easier to construct than G. Hence, M_ν is a further candidate preconditioner for S. Furthermore, the next lemma shows that either the matrix M_ν can be replaced by a cheap diagonal preconditioner or a good approximation of $\mathrm{M}_\nu^{-1}\mathrm{y}$ can be obtained efficiently by applying a preconditioned CG method with a diagonal matrix as preconditioner .

Lemma 4 *Define the diagonal matrix* $\bar{\mathrm{M}}_\nu$ *by* $(\bar{\mathrm{M}}_\nu)_{ii} = \sum_{j=1}^m (\mathrm{M}_\nu)_{ij}$ *(diagonal lumping). Then for all* $\mathrm{y} \in \mathbb{R}^m$ *we have*

$$C_3 \langle \bar{\mathrm{M}}_\nu \mathrm{y}, \mathrm{y} \rangle \le \langle \mathrm{M}_\nu \mathrm{y}, \mathrm{y} \rangle \le C_4 \langle \bar{\mathrm{M}}_\nu \mathrm{y}, \mathrm{y} \rangle$$

with constants $C_3 > 0$ *and* C_4 *independent of* ε *and* h*.*

Proof. The proof follows from applying the analysis from [19]. The details are given in [12]. □

Summarizing, we discussed the following three preconditioners for the Schur complement S:

$$\mathrm{G} \;:\; \text{the mass matrix w.r.t. } (\cdot,\cdot)_{Q,h} \tag{33}$$

$$\mathrm{M}_\nu \;:\; \text{the mass matrix w.r.t. } (\cdot,\cdot)_\nu \tag{34}$$

$$\bar{\mathrm{M}}_\nu \;:\; \text{the diagonal matrix such that } \bar{\mathrm{M}}_\nu \mathrm{e} = \mathrm{M}_\nu \mathrm{e} \tag{35}$$

For the preconditioner G we have a spectral equivalence result on the whole pressure space $(\mathrm{Me})^\perp$ as in (29). For all three preconditioners we have uniform spectral bounds on the subspace $(\mathrm{Me})^\perp \cap (\mathrm{M}\tilde{\mathrm{p}})^\perp$, as in (30).

Remark 3 We performed a simple numerical experiment to show that the result in (29) is sharp, in the sense that the term $\frac{\tilde{\mu}_h^2}{\varepsilon}$ can not be avoided. We consider a 1D Stokes problem with $\Omega = (0,1)$, $\Omega_2 = (\frac{1}{4}, \frac{3}{4})$ and $P_2 iso P_1 - P_1$ finite elements on a uniform grid. In this case we have $\tilde{\mu}_h^2 \sim h$. In table 1 we show the values of $\langle \mathrm{Sy}, \mathrm{y} \rangle / \langle \mathrm{Gy}, \mathrm{y} \rangle$ for $\mathrm{y} = \tilde{\mathrm{p}} \in (\mathrm{Me})^\perp$.

	h		
ε	1/16	1/32	1/64
10^{-2}	5.0	3.5	2.7
10^{-4}	$3.1\,10^2$	$1.6\,10^2$	$8.1\,10^1$
10^{-6}	$3.1\,10^4$	$1.6\,10^4$	$7.9\,10^3$

Table 1: Estimates for upper bound in (29).

These results show a behaviour consistent with $\frac{\tilde{\mu}_h^2}{\varepsilon} \sim \frac{h}{\varepsilon}$. If instead of the spaces Q and Q_h we use slightly different spaces M and M_h as explained in remark 2 then it is possible to avoid the deterioration (for $\varepsilon \downarrow 0$) as in the upper bound in (29). In [12] (theorem 4.1) it is shown that on the space $M_h \cap (\mathrm{M}_\nu \mathrm{e})^\perp$ the mass matrix M_ν is uniformly spectrally equivalent to S.

5 Numerical experiments

In this section we present results of a few numerical experiments. We consider a Stokes interface problem as in (1)-(3) with

$$\Omega = (0,1)^3, \quad \Omega_2 = (0, \frac{1}{2})^3$$

For the discretization we start with a uniform tetrahedral grid with $h = \frac{1}{2}$ and we apply regular refinements to this starting triangulation. The resulting triangulations satisfy the conformity condition (18). For the finite element discretization we used the LBB stable pair of Hood-Taylor $P_2 - P_1$. We performed computations for the cases $h = 1/16$, $h = 1/32$ and with varying $\varepsilon \in (0, 1]$. Note that for $h = 1/32$ we have approximately $7.5{\cdot}10^5$ velocity unknowns and $3.3{\cdot}10^4$ pressure unknowns ($n \approx 7.5{\cdot}10^5,\ m \approx 3.3{\cdot}10^4$). We consider the problem in (26). In the iterative method we take a fixed arbitrary starting vector $(\mathrm{x}^0, \mathrm{y}^0)$, with $\mathrm{y}^0 \in (\mathrm{Me})^\perp$.

We use a preconditioned MINRES method. For this we consider a symmetric positive definite preconditioner

$$\tilde{\mathrm{K}} = \begin{pmatrix} \mathrm{Q}_A & 0 \\ 0 & \mathrm{Q}_S \end{pmatrix} \quad \text{for } \mathrm{K} := \begin{pmatrix} \mathrm{A} & \mathrm{B}^T \\ \mathrm{B} & 0 \end{pmatrix}$$

Define the norm $\|\mathrm{w}\|_{\tilde{K}} := \langle \tilde{\mathrm{K}}\mathrm{w}, \mathrm{w}\rangle^{\frac{1}{2}}$ for $\mathrm{w} \in \mathbb{R}^{n+m}$. Given a starting vector w^0 with corresponding error $\mathrm{e}^0 := \mathrm{w}^* - \mathrm{w}^0$, then in the preconditioned MINRES method one computes the vector $\mathrm{w}^k \in \mathrm{w}^0 + \mathrm{span}\{\tilde{\mathrm{K}}^{-1}\mathrm{K}\mathrm{e}^0, \ldots, (\tilde{\mathrm{K}}^{-1}\mathrm{K})^k\mathrm{e}^0\}$ which minimizes the preconditioned residual $\|\tilde{\mathrm{K}}^{-1}\mathrm{K}(\mathrm{w}^* - \mathrm{w})\|_{\tilde{\mathrm{K}}}$ over this subspace. For an efficient implementation of this method we refer to the literature. From the literature (cf. [18]) it is known that the convergence of the preconditioned MINRES method is fast if we have good preconditoners Q_A of A and Q_S of S. For the preconditioner Q_A we take *one* iteration of a standard multigrid V-cycle with one pre- and one post-smoothing iteration with a symmetric Gauss-Seidel method. We take $\mathrm{Q}_S \in \{\mathrm{M}, \mathrm{G}, \mathrm{M}_\nu, \bar{\mathrm{M}}_\nu\}$.

As a stopping criterion we use

$$\left\|\tilde{\mathrm{K}}^{-1}(\mathrm{K}\begin{pmatrix} \mathrm{x}^k \\ \mathrm{y}^k \end{pmatrix} - \begin{pmatrix} \mathrm{f} \\ 0 \end{pmatrix})\right\|_{\tilde{\mathrm{K}}} \le 10^{-6} \left\|\tilde{\mathrm{K}}^{-1}(\mathrm{K}\begin{pmatrix} \mathrm{x}^0 \\ \mathrm{y}^0 \end{pmatrix} - \begin{pmatrix} \mathrm{f} \\ 0 \end{pmatrix})\right\|_{\tilde{\mathrm{K}}} \tag{36}$$

Note that for $\mathrm{Q}_S = \mathrm{M}$ the iterands $\mathrm{w}^k = (\mathrm{x}^k, \mathrm{y}^k)$ of the preconditioned MINRES method satisfy $\mathrm{y}^k \in (\mathrm{Me})^\perp$ and thus y^k converges to the solution $\mathrm{y} \in (\mathrm{Me})^\perp$ as desired in (26). However, for $\mathrm{Q}_S \in \{\mathrm{G}, \mathrm{M}_\nu, \bar{\mathrm{M}}_\nu\}$, one in general has $\mathrm{y}^k \notin (\mathrm{Me})^\perp$. This inconsistency is easily repaired by using the projection $\mathrm{y}^k \to \mathrm{y}^k - \frac{\langle \mathrm{My}^k, \mathrm{e}\rangle}{\langle \mathrm{Me}, \mathrm{e}\rangle}\mathrm{e}$. Note that this projection does not influence the residual in (36), since $\mathrm{e} \in \ker(\mathrm{B}^T)$.

For $\mathrm{Q}_S \in \{\mathrm{M}, \mathrm{G}, \mathrm{M}_\nu\}$ the systems with matrix Q_S that occur in each MINRES iteration are solved using a CG method. This inner CG iteration is stopped if its starting residual has been reduced by a factor 10^3 (in the Euclidean norm).

A matrix-vector multiplication Gy can be computed as follows. Let $\tilde{p}$ be the piecewise constant function as in (7) and $\{\phi_i\}_{1\le i\le m}$ the standard nodal basis in Q_h^+. Define $\mathrm{p}_0, \mathrm{b}_0 \in \mathbb{R}^m$, $\alpha_0 \in \mathbb{R}$ by

$$\mathrm{b}_0 = ((\tilde{p},\phi_1),\ldots,(\tilde{p},\phi_m))^T, \quad \mathrm{M}\mathrm{p}_0 = \mathrm{b}_0, \quad \alpha_0 = \langle \mathrm{b}_0, \mathrm{p}_0\rangle^{-1}$$

A straightforward computation yields the representation

$$\mathrm{G} = \alpha_0 \mathrm{b}_0\mathrm{b}_0^T + (\mathrm{I} - \alpha_0\mathrm{b}_0\mathrm{p}_0^T)\mathrm{M}_\nu(\mathrm{I} - \alpha_0\mathrm{p}_0\mathrm{b}_0^T)$$

for the mass matrix G. Using this one can compute Gy efficiently.

In table 2, for different choices of the preconditioner Q_S, we show the number of iterations needed to satisfy the stopping criterion (36).

h	1/16				1/32			
ε	1	10^{-2}	10^{-4}	10^{-6}	1	10^{-2}	10^{-4}	10^{-6}
$\mathrm{Q}_S = \mathrm{M}$	48	370	1242	1500	42	340	1201	1488
$\mathrm{Q}_S = \mathrm{G}$	48	54	64	71	42	49	58	69
$\mathrm{Q}_S = \mathrm{M}_\nu$	48	53	63	67	42	49	58	66
$\mathrm{Q}_S = \bar{\mathrm{M}}_\nu$	62	68	98	157	50	58	85	116

Table 2: # preconditioned MINRES iterations.

As expected, the preconditioner $\mathrm{Q}_S = \mathrm{M}$ is not satisfactory for small ε values. The preconditiners $\mathrm{Q}_S = \mathrm{G}$ and $\mathrm{Q}_S = \mathrm{M}_\nu$ result in a very similar convergence behaviour of the preconditioned MINRES method. The latter, however, is easier to implement. Finally note that for $\mathrm{Q}_S = \bar{\mathrm{M}}_\nu$ there is some deterioration of the rate of convergence for small ε values.

Acknowledgements. The authors thank J. Peters for providing the results of the numerical experiments. M.A.Olshanskii was supported by the German Research Foundation through the guest program of SFB 540, RFBR grant 02-01-00592 and NWO-RFBR grant 047.016.008.

REFERENCES

[1] J.U. Brackbill, D.B. Kothe, and C. Zemach. A continuum method for modeling surface tension. *J. Comput. Phys.*, **100**, 335–354, 1992.

[2] I. Babuska.The finite element method for elliptic problems with discontinuous coefficients. *Computing*, **5**, 277–213, 1970.

[3] I. Babuska, C. Caloz and J. Osborn. Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM J. Numer. Anal.*, **31**, 945–981, 1994.

[4] C. Bernardi, R. Verfürth. Adaptive finite element methods for elliptic equations with non-smooth coefficients. *Numer. Math.*, **85**, 579–608, 2000.

[5] J.H. Bramble. *Multigrid Methods*. Pitman Research Notes in Mathematics, Vol. 294, John Wiley, New York, 1993.

[6] J.H. Bramble and J.T. King. A finite element method for interface problems in domains with smooth boundaries and interfaces. *Advances in Comp. Math.*, **6**, 109–138, 1996.

[7] J.H. Bramble, J.E. Pasciak, J. Wang and J. Xu. Convergence estimates for multigrid algorithms without regularity assumptions. *Math. Comp.*, **57**, 23–45, 1991.

[8] Y.C. Chang, T.Y. Hou, B. Merriman, and S. Osher. Eulerian capturing methods based on a level set formulation for incompressible fluid interfaces. *J. Comput. Phys.*, **124**, 449–464, 1996.

[9] Z. Chen and J. Zou. Finite element method and its convergence for elliptic and parabolic interface problems. *Numer. Math.*, **79**, 175–202, 1998.

[10] P.G. Ciarlet. Basic Error Estimates for Elliptic Problems. in *Handbook of Numerical Analysis*, Vol. II, P.G. Ciarlet and J.L. Lions, eds., Elsevier Science, 1991.

[11] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer, Berlin, 1986.

[12] M.A. Olshanskii, A. Reusken. Analysis of a Stokes interface problem. *IGPM report*, **237**, RWTH-Aachen, 2004. Submitted.

[13] S. Osher and R.P. Fedkiw. Level set methods: An overview and some recent results. *J. Comput. Phys.*, **169**, 463–502, 2001.

[14] M. Petzoldt. A posteriori error estimators for elliptic equations with discontinuous diffusion coefficients. *Advances in Computational Mathematics*, **16**, 47–75, 2002.

[15] C. Powell and D. Silvester. Black-box preconditioning for mixed formulation of self-adjoint elliptic PDEs. In: *Challenges in Scientific Computing - CISC 2002*, ed. E. Bänsch. Lecture Notes in Computational Science and Engineering, **35**, 268–285, 2003.

[16] A.-K. Tornberg and B. Engquist. A finite element based level-set method for multiphase flow applications. *Comput. Visual. Sci.*, **3**, 93–101, 2000.

[17] M. Plum and C. Wieners. Optimal a priori estimates for interface problems. *Numer. Math.*, **95**, 735–759, 2003.

[18] D. Silvester, A. Wathen. Fast iterative solution of stabilised stokes systems. part II: using general block preconditioners. *SIAM J. Numer. Anal.*, **31**, 1352–1367, 1994.

[19] A. Wathen. Realistic eigenvalues bounds for the Galerkin mass matrix. *IMA J. Numer. Anal.*, **7**, 449–457, 1987.

[20] H. Yserentant. Two preconditioners based on the multi-level splitting of finite element spaces. *Numer. Math.*, **58**, 163–184, 1990.