# Uniform preconditioners for a parameter dependent saddle point problem with application to generalized Stokes interface equations 

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#### Abstract

We consider an abstract parameter dependent saddle-point problem and present a general framework for analyzing robust Schur complement preconditioners. The abstract analysis is applied to a generalized Stokes problem, which yields robustness of the Cahouet-Chabard preconditioner. Motivated by models for two-phase incompressible flows we consider a generalized Stokes interface problem. Application of the general theory results in a new Schur complement preconditioner for this class of problems. The robustness of this preconditioner with respect to several parameters is treated. Results of numerical experiments are given that illustrate robustness properties of the preconditioner.


Keywords Generalized Stokes equations • Interface problem • Two-phase flow • Preconditioning • Schur complement

AMS Subject Classifications $65 \mathrm{~N} 15 \cdot 65 \mathrm{~N} 22 \cdot 65 \mathrm{~N} 30 \cdot 65 \mathrm{~F} 10$

## 1 Introduction

Let $H_{1} \subset H_{2}$ and $M$ be Hilbert spaces such that the identity $I: H_{1} \rightarrow H_{2}$ is a dense embedding. Let there be given continuous symmetric elliptic bilinear

[^0]forms $a: H_{1} \times H_{1} \rightarrow \mathbb{R}, c: H_{2} \times H_{2} \rightarrow \mathbb{R}$ and a continuous bilinear form $b: H_{1} \times M \rightarrow \mathbb{R}$ that satisfies a standard infsup condition. Operators corresponding to these bilinear forms are denoted by $A: H_{1} \rightarrow H_{1}^{\prime}, C: H_{2} \rightarrow H_{2}^{\prime}$ and $B: M \rightarrow H_{1}^{\prime}$, respectively. In this paper we consider the following saddlepoint system: Find $(u, p) \in H_{1} \times M$ such that
\[

\left\{$$
\begin{align*}
A u+\tau C u+B p & =f  \tag{1.1}\\
B^{\prime} u & =0
\end{align*}
$$\right.
\]

with $f \in H_{1}^{\prime}$ and a parameter $\tau \geq 0$. Similar abstract saddle point problems are thoroughly analyzed in the literature, e.g. [6,11]. Important examples that fit in this general setting are the stationary Stokes equation (then $\tau=0$ ) and the so-called generalized Stokes problem, which results from an implicit time integration applied to a nonstationary Stokes equation (then $\tau$ is proportional to the inverse of the time step). Another (less standard) example, which motivated the research that led to the results presented in this paper, is the following generalized Stokes interface problem. Assume bounded Lipschitz subdomains $\Omega_{1}$ and $\Omega_{2}$ of $\Omega \subset \mathbb{R}^{d}$ such that $\bar{\Omega}=\bar{\Omega}_{1} \cup \bar{\Omega}_{2}, \Omega_{1} \cap \Omega_{2}=\emptyset$. The boundary between the subdomains is denoted by $\Gamma=\partial \Omega_{1} \cap \partial \Omega_{2}$. Consider a problem of the following form: Find $\mathbf{u}$ and $p$ such that

$$
\begin{align*}
-\operatorname{div}(\nu(\mathbf{x}) \mathrm{Du})+\tau \rho(\mathbf{x}) \mathbf{u}+\nabla p & =\mathbf{f} & & \text { in } \Omega_{k}, \\
\operatorname{div} \mathbf{u} & =0 & & \text { in } \Omega_{k}, \quad k=1,2 . \\
{[\mathbf{u}]=0,[\sigma(\mathbf{u}, p) \mathbf{n}] } & =\mathbf{g} & & \text { on } \Gamma,  \tag{1.2}\\
\mathbf{u} & =0 & & \text { on } \partial \Omega .
\end{align*}
$$

In this formulation we use standard notations: $\sigma(\mathbf{u}, p)=-p I+2 v \mathrm{Du}$ is the stress tensor, $\mathbf{D u}=\frac{1}{2}\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{T}\right)$ the rate of deformation tensor, $\mathbf{n}$ is a unit normal vector to $\Gamma,\left.[a]\right|_{\Gamma}=\left.\left(\left.a\right|_{\Omega_{1}}-\left.a\right|_{\Omega_{2}}\right)\right|_{\Gamma}$. We assume piecewise constant viscosity ( $v_{k}>0$ in $\Omega_{k}$ ) and density ( $\rho_{k}>0$ in $\Omega_{k}$ ). An important motivation for considering this type of generalized Stokes interface equations comes from twophase incompressible flows. Often such two-phase problems can be modeled by time-dependent Navier-Stokes equations with discontinuous density and viscosity coefficients, [4,9,12,21,22]. A localized force $\mathbf{g}$ on $\Gamma$ can be used to describe the effect of surface tension. If in such a setting one has highly viscous flows then the nonstationary Stokes equations are a reasonable model problem. After implicit time integration one obtains a problem of the form (1.2). A variational formulation of this problem results in a saddle point problem of the form (1.1).

A Galerkin discretization appoach applied to (1.1) results in a finite dimensional saddle point problem. In the examples mentioned above one applies iterative methods for solving the matrix-vector representation of such discrete problems, cf. [1] for a recent overview. Most of these iterative solvers use block preconditioners [8,10,14,19,23]. For such methods a good preconditioner of the Schur complement is crucial for the efficiency of the iterative solver. There
is an extensive literature on this issue of preconditioning the Schur complement. We mention results that are related to those presented in this paper. For the generalized Stokes problem one is interested in a preconditioner that is robust with respect to variation in both $h$ (mesh size parameter) and $\tau$. Such a preconditioner was introduced by Cahouet and Chabard in [8]. A proof of robustness of this preconditioner (w.r.t. $\tau$ and $h$ ) in a finite element setting is given in [5]. An analysis of robustness (w.r.t. $\tau$ ) of this method in a continuous setting can be found in [13]. In a recent paper [15] an analysis is presented which shows how the construction of this Cahouet-Chabard preconditioner is related to certain mapping properties of the gradient operator. This results in a unifying framework in which robust preconditioning of both the continuous and the discrete Schur complement can be analyzed. In [16] it is noted that an important assumption about the regularity of the stationary Stokes problem is implicitly used in the proof of Lemma 1 in [15] and not stated explicitly.

We do not know any literature in which Schur complement preconditioners for the generalized Stokes interface problem (1.2) are treated. A preconditioner for the stationary Stokes interface problem, i.e., $\tau=0$ in (1.2), that is robust with respect to the size of the jump in $v$ across the interface is introduced and analyzed in [17,18].

The two main topics of this paper are the following. Firstly, we extend the analysis that is presented in [15] for the generalized Stokes problem to the general abstract saddle point problem (1.1), resulting in an abstract framework for analyzing the Schur complement $S=B^{\prime}(A+\tau C)^{-1} B: M \rightarrow M^{\prime}$. In this framework we obtain a natural preconditioner $\tilde{S}$ for this Schur complement. We show that a spectral inequality $S \lesssim \tilde{S}$ that is uniform with respect to $\tau$ is easy to derive. For a uniform spectral inequality $\tilde{S} \lesssim S$, however, we need a certain boundedness property for the orthogonal projection $P: H_{1}^{\prime} \rightarrow B(M) \subset H_{1}^{\prime}$ (Assumption 1). We apply the abstract theory to the continuous and to the discrete generalized Stokes problem. The preconditioner $\tilde{S}$ then coincides with the Cahouet-Chabard preconditioner. To prove the robustness with respect to $\tau$ we have to verify the boundedness property, which turns out to be a regularity property for the stationary Stokes problem that is very closely related to the "hidden" assumption in Lemma 1 in [15] (cf. [16]). The preconditioner $\tilde{S}=\tilde{S}_{h}$ is of the form $\tilde{S}_{h}^{-1}=I_{h}+\tau\left(B_{h}^{\prime} C_{h}^{-1} B_{h}\right)^{-1}$, where $I_{h}$ is the identity on the pressure finite element space $M_{h}$, and $B_{h}, C_{h}$ are discrete analogons of the operators $B, C$ in (1.1). We show that if $M_{h} \subset H^{1}(\Omega)$ holds, the operator $\tilde{S}_{h}^{-1}$ is uniformly spectrally equivalent to the simpler operator $\hat{S}_{h}^{-1}=I_{h}+\tau N_{h}^{-1}$, with $N_{h}^{-1}$ a solution operator of a discrete Neumann problem in the space $M_{h}$.

Secondly, we introduce and analyze a Schur complement preconditioner for the generalized Stokes interface problem (1.2). This preconditioner is new and is obtained by applying the general abstract analysis to the variational formulation of the generalized Stokes interface problem. In this interface problem it is interesting (for two-phase flows with large differences in viscosity and density of the two phases) to have a preconditioner that is robust not only with respect to variation in $\tau$ but also with respect to the jumps in $\nu$ and $\rho$ across
the interface. From our general analysis applied to the continuous generalized Stokes interface problem it follows that a spectral inequality $S \lesssim \widetilde{S}$ holds uniformly with respect to $\tau$ and the jumps in $v, \rho$. For the spectral inequality $\tilde{S} \lesssim S$ we can only show uniformity with respect to $\tau$. An equality $\tilde{S} \lesssim S$ that is uniform w.r.t. the jumps in $v$ and $\rho$, too, would hold if we could verify the boundedness assumption formulated in the abstract theory. It turns out, however, that this requires certain regularity results for the stationary Stokes interface problem that are not known in the literature. This issue of the dependence of the constant in the spectral inequality $\tilde{S} \leq c S$ on the jumps in $v$ and $\rho$ is an open problem.

The preconditioner for the continuous generalized Stokes interface problem has an obvious discrete analogon. For a standard finite element discretization ( $P_{2}-P_{1}$ Hood-Taylor) we present results of numerical experiments that illustrate robustness properties of this preconditioner for the discrete Schur complement.

The remainder of the paper is organized as follows. In Sect. 2 we present a general analysis for the abstract problem (1.1). We introduce a preconditioner $\tilde{S}$ for the Schur complement $S$ and derive spectral inequalities $\tilde{S} \lesssim S \lesssim \tilde{S}$. A crucial assumption to obtain the lower spectral inequality uniformly in $\tau$ is introduced in Sect. 2.5. In Sect. 3 we apply the general theory to the continuous generalized Stokes problem and we show that this crucial assumption corresponds to a regularity assumption for the stationary Stokes equations. In Sect. 4 we consider a finite element discretization of the generalized Stokes problem with an LBB stable pair of spaces and show how the general theory can be used to prove robustness of the Cahouet-Chabard preconditioner. In Sect. 5 we apply the abstract analysis to the continuous generalized Stokes interface problem (1.2) and derive a robust preconditioner for the Schur complement. This preconditioner has an obvious discrete analogon. In Sect. 6 results of numerical experiments are presented that illustrate certain robustness properties of this discrete Schur complement preconditioner.

## 2 General analysis

Consider Hilbert spaces $H_{1}$ and $M$. In Sect. 2.1 we describe a parameter dependent saddle point problem in the pair of spaces $H_{1} \times M$. We are interested in a uniform (w.r.t. variation in the parameter) preconditioner for the selfadjoint and positive definite Schur complement operator $S: M \rightarrow M^{\prime}$. In Sect. 2.2 we collect some results that will be used in our analysis. In Sect. 2.3 a Schur complement preconditioner $\tilde{S}$ is introduced. In Sects. 2.4 and 2.5 we prove uniform spectral inequalities $S \lesssim \tilde{S}$ and $\tilde{S} \lesssim S$, respectively.

Apart from $H_{1}$ we will also use a Hilbert space $H_{2}$ such that $H_{1} \subset H_{2}$ and the identity $I: H_{1} \rightarrow H_{2}$ is a continuous dense embedding. We use $(\cdot, \cdot)_{H}$ to denote a scalar product in a Hilbert space $H$ and $\langle\cdot, \cdot\rangle_{H^{\prime} \times H}$ for the duality pairing. The subscripts are omitted when these are obvious from the context.

### 2.1 A parameter dependent saddle point problem and its Schur complement

In this section we introduce a parameter dependent saddle point problem. Assume bilinear forms $a: H_{1} \times H_{1} \rightarrow \mathbb{R}, c: H_{2} \times H_{2} \rightarrow \mathbb{R}$ and $b: H_{1} \times M$. Related to these bilinear forms we make the following assumptions. $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are symmetric and the following ellipticity, continuity and infsup conditions hold with strictly positive constants $\gamma_{a}, \gamma_{b}, \gamma_{c}$ :

$$
\begin{align*}
\gamma_{a}\|u\|_{H_{1}}^{2} \leq a(u, u), \quad a(u, v) \leq \Gamma_{a}\|u\|_{H_{1}}\|v\|_{H_{1}} & \text { for all } u, v \in H_{1},  \tag{2.1}\\
\gamma_{c}\|u\|_{H_{2}}^{2} \leq c(u, u), \quad c(u, v) \leq \Gamma_{c}\|u\|_{H_{2}}\|v\|_{H_{2}} & \text { for all } u, v \in H_{2},  \tag{2.2}\\
\gamma_{b}\|p\|_{M} \leq \sup _{v \in H_{1}} \frac{b(v, p)}{\|v\|_{H_{1}}}, \quad b(v, p) \leq \Gamma_{b}\|v\|_{H_{1}}\|p\|_{M} & \text { for all } v \in H_{1}, p \in M . \tag{2.3}
\end{align*}
$$

For the analysis given below it is convenient to introduce corresponding linear mappings

$$
\begin{aligned}
& A: H_{1} \rightarrow H_{1}^{\prime},\langle A u, v\rangle=a(u, v) \\
& C: H_{2} \rightarrow H_{2}^{\prime},\langle C u, v\rangle=c(u, v) \\
& \text { for all } u, v \in H_{1}, \\
& B: M \rightarrow H_{1}^{\prime},\langle B p, v\rangle=b(v, p)
\end{aligned} \quad \text { for all } p \in M, v \in H_{2}, ~ . ~ . ~ f o H_{1} .
$$

The assumptions on the bilinear forms imply that

$$
\begin{align*}
\gamma_{a}\|u\|_{H_{1}} \leq\|A u\|_{H_{1}^{\prime}} \leq \Gamma_{a}\|u\|_{H_{1}} & \text { for all } u \in H_{1},  \tag{2.4}\\
\gamma_{c}\|u\|_{H_{2}} \leq\|C u\|_{H_{2}^{\prime}} \leq \Gamma_{c}\|u\|_{H_{2}} & \text { for all } u \in H_{2},  \tag{2.5}\\
\gamma_{b}\|p\|_{M} \leq\|B p\|_{H_{1}^{\prime}} \leq \Gamma_{b}\|p\|_{M} & \text { for all } p \in M . \tag{2.6}
\end{align*}
$$

Note that the operator $A: H_{1} \rightarrow H_{1}^{\prime}$ is selfadjoint: $\langle A u, v\rangle_{H_{1}^{\prime} \times H_{1}}=\langle A v, u\rangle_{H_{1}^{\prime} \times H_{1}}$. The operator $C: H_{2} \rightarrow H_{2}^{\prime}$ is selfadjoint, too.

Consider the following general saddle point problem: Given $\tau \geq 0$ and $f \in H_{1}^{\prime}$, find $(u, p) \in H_{1} \times M$ such that

$$
\begin{equation*}
a(u, v)+\tau c(u, v)+b(v, p)+b(u, q)=\langle f, v\rangle \quad \text { for all } v \in H_{1}, q \in M \tag{2.7}
\end{equation*}
$$

The problem (2.7) can be rewritten in operator formulation: Find $(u, p) \in$ $H_{1} \times M$ such that

$$
\left\{\begin{align*}
A u+\tau C u+B p & =f,  \tag{2.8}\\
B^{\prime} u & =0 .
\end{align*}\right.
$$

Standard analyses of saddle point problems (e.g., [6]) yield that this problem has a unique solution. The Schur complement

$$
S:=B^{\prime}(A+\tau C)^{-1} B
$$

of the system is a selfadjoint positive definite operator $S: M \rightarrow M^{\prime}$. It defines a scalar product (and corresponding norm) on $M$ :

$$
\begin{equation*}
\|p\|_{S}:=\langle S p, p\rangle^{\frac{1}{2}}=\sup _{v \in H_{1}} \frac{\langle B p, v\rangle}{\langle(A+\tau C) v, v\rangle^{\frac{1}{2}}}, \quad p \in M \tag{2.9}
\end{equation*}
$$

Example 1 For the (nonstationary) Stokes system on a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^{d}$ we take the spaces

$$
H_{1}:=\mathbf{H}_{0}^{1}(\Omega), \quad H_{2}:=\mathbf{L}^{2}(\Omega), \quad M=L_{0}^{2}(\Omega)
$$

with scalar products

$$
(u, v)_{H_{1}}:=(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^{2}}, \quad(u, v)_{H_{2}}:=(\mathbf{u}, \mathbf{v})_{L^{2}}, \quad(p, q)_{M}:=(p, q)_{L^{2}} .
$$

The bilinear forms are

$$
a(\mathbf{u}, \mathbf{v}):=(\nabla \mathbf{u}, \nabla \mathbf{v})_{L^{2}}, \quad c(\mathbf{u}, \mathbf{v}):=(\mathbf{u}, \mathbf{v})_{L^{2}}, \quad b(\mathbf{v}, p):=-(p, \operatorname{div} \mathbf{v})_{L^{2}}
$$

and the problem is as follows: Find $(\mathbf{u}, p) \in H_{1} \times M$ such that

$$
\left\{\begin{align*}
a(\mathbf{u}, \mathbf{v})+\tau c(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p) & =\langle f, \mathbf{v}\rangle & & \text { for all } \mathbf{v} \in H_{1}  \tag{2.10}\\
b(\mathbf{u}, q) & =0 & & \text { for all } q \in M .
\end{align*}\right.
$$

Recall the infsup inequality (Nečas inequality):

$$
\sup _{\mathbf{v} \in H_{1}} \frac{(\operatorname{div} \mathbf{v}, p)_{L^{2}}}{\|\nabla \mathbf{v}\|_{L_{2}}} \geq \gamma_{b}\|p\|_{L_{2}} \quad \text { for all } p \in M
$$

with $\gamma_{b}>0$. Using this one easily verifies that the conditions in (2.1)-(2.3) are satisfied with $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=\Gamma_{b}=1, \gamma_{b}>0$ the constant from the infsup inequality.

### 2.2 Preliminaries

In this section we derive some properties of the saddle point problem (2.8) that will be used in the analysis of the Schur complement preconditioner. We use the concept of sums and intersections of vector spaces (cf. [3]). The idea of applying this concept in the analysis of Schur complement preconditioners is introduced in [15].

Let $X, Y$ be compatible normed spaces, i.e., both $X$ and $Y$ are subspaces of some larger topological vector space $Z$. Then we can form their sum $X+Y$ and intersection $X \cap Y$. The sum $X+Y$ consists of all $z \in Z$ such that $z=x+y$
with $x \in X, y \in Y$. The spaces $X \cap Y$ and $X+Y$ are normed vector spaces with norms

$$
\begin{aligned}
\|x\|_{X \cap Y} & =\left(\|x\|_{X}^{2}+\|x\|_{Y}^{2}\right)^{\frac{1}{2}} \quad(x \in X \cap Y) \\
\|z\|_{X+Y} & =\inf _{z=x+y}\left(\|x\|_{X}^{2}+\|y\|_{Y}^{2}\right)^{\frac{1}{2}} \quad(x \in X, y \in Y)
\end{aligned}
$$

If $X$ and $Y$ are complete then both $X \cap Y$ and $X+Y$ are complete (Lemma 2.3.1 in [3]). A few properties that we will need further on are given in the following lemma. The space of bounded linear mappings $X \rightarrow Y$ is denoted by $\mathcal{L}(X, Y)$.

Lemma 2.1 Let $X_{1}, X_{2}$ and $Y_{1}, Y_{2}$ be pairs of compatible normed vector spaces and let $T$ be a linear mapping on $X_{1}+X_{2}$ such that $T \in \mathcal{L}\left(X_{1}, Y_{1}\right) \cap \mathcal{L}\left(X_{2}, Y_{2}\right)$. Then $T: X_{1}+X_{2} \rightarrow Y_{1}+Y_{2}$ is bounded and

$$
\begin{equation*}
\|T\|_{X_{1}+X_{2} \rightarrow Y_{1}+Y_{2}} \leq\left(\|T\|_{X_{1} \rightarrow Y_{1}}^{2}+\|T\|_{X_{2} \rightarrow Y_{2}}^{2}\right)^{\frac{1}{2}} \tag{2.11}
\end{equation*}
$$

holds. If $X_{1}$ and $X_{2}$ are Hilbert spaces such that $X_{1} \cap X_{2}$ is dense in both $X_{1}$ and $X_{2}$, then $\left(X_{1} \cap X_{2}\right)^{\prime}=X_{1}^{\prime}+X_{2}^{\prime}$ holds and

$$
\begin{equation*}
\|g\|_{\left(X_{1} \cap X_{2}\right)^{\prime}}=\|g\|_{X_{1}^{\prime}+X_{2}^{\prime}} \quad \text { for all } g \in\left(X_{1} \cap X_{2}\right)^{\prime} \tag{2.12}
\end{equation*}
$$

Proof Proofs are given in [3]. Since these results are fundamental for our further considerations, we present an elementary proof to make the paper selfcontained.

Consider $x \in X_{1}+X_{2}$ and an arbitrary decomposition $x=x_{1}+x_{2}, x_{1} \in X_{1}$, $x_{2} \in X_{2}$. For $T x=T x_{1}+T x_{2}$ we have

$$
\begin{aligned}
\|T x\|_{Y_{1}+Y_{2}} & =\inf _{T x=y_{1}+y_{2}}\left(\left\|y_{1}\right\|_{Y_{1}}^{2}+\left\|y_{2}\right\|_{Y_{2}}^{2}\right)^{\frac{1}{2}} \leq\left(\left\|T x_{1}\right\|_{Y_{1}}^{2}+\left\|T x_{2}\right\|_{Y_{2}}^{2}\right)^{\frac{1}{2}} \\
& \leq\left(\|T\|_{X_{1} \rightarrow Y_{1}}^{2}+\|T\|_{X_{2} \rightarrow Y_{2}}^{2}\right)^{\frac{1}{2}}\left(\left\|x_{1}\right\|_{X_{1}}^{2}+\left\|x_{2}\right\|_{X_{2}}^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

Since the decomposition $x=x_{1}+x_{2}$ is arbitrary we obtain

$$
\|T x\|_{Y_{1}+Y_{2}} \leq\left(\|T\|_{X_{1} \rightarrow Y_{1}}^{2}+\|T\|_{X_{2} \rightarrow Y_{2}}^{2}\right)^{\frac{1}{2}}\|x\|_{X_{1}+X_{2}}
$$

Thus (2.11) is proved.
To prove the second part of the lemma consider arbitrary $g \in X_{1}^{\prime}+X_{2}^{\prime}$ and take $x \in X_{1} \cap X_{2}$. For an arbitrary decomposition $g=g_{1}+g_{2}$ with $g_{1} \in X_{1}^{\prime}$, $g_{2} \in X_{2}^{\prime}$ we have $\langle g, x\rangle_{\left(X_{1} \cap X_{2}\right)^{\prime} \times\left(X_{1} \cap X_{2}\right)}=\left\langle g_{1}, x\right\rangle_{X_{1}^{\prime} \times X_{1}}+\left\langle g_{2}, x\right\rangle_{X_{2}^{\prime} \times X_{2}}$ and thus

$$
\left|\langle g, x\rangle_{\left(X_{1} \cap X_{2}\right)^{\prime} \times\left(X_{1} \cap X_{2}\right)}\right| \leq\left(\left\|g_{1}\right\|_{X_{1}^{\prime}}^{2}+\left\|g_{2}\right\|_{X_{2}^{\prime}}^{2}\right)^{\frac{1}{2}}\left(\|x\|_{X_{1}}^{2}+\|x\|_{X_{2}}^{2}\right)^{\frac{1}{2}}
$$

This yields

$$
\begin{aligned}
\left|\langle g, x\rangle_{\left(X_{1} \cap X_{2}\right)^{\prime} \times\left(X_{1} \cap X_{2}\right)}\right| & \leq \inf _{g=g_{1}+g_{2}}\left(\left\|g_{1}\right\|_{X_{1}^{\prime}}^{2}+\left\|g_{2}\right\|_{X_{2}^{\prime}}^{2}{ }^{\frac{1}{2}}\|x\|_{X_{1} \cap X_{2}}\right. \\
& =\|g\|_{X_{2}^{\prime}+X_{2}^{\prime}}\|x\|_{X_{1} \cap X_{2}} .
\end{aligned}
$$

Therefore $\left(X_{1} \cap X_{2}\right)^{\prime} \supset X_{1}^{\prime}+X_{2}^{\prime}$ and $\|g\|_{\left(X_{1} \cap X_{2}\right)^{\prime}} \leq\|g\|_{X_{2}^{\prime}+X_{2}^{\prime}}$.
Now take $g \in\left(X_{1} \cap X_{2}\right)^{\prime}$. Since $X_{1} \cap X_{2}$ with scalar product $(\cdot, \cdot)_{X_{1}}+$ $(\cdot, \cdot)_{X_{2}}$ is a Hilbert space, there exists an element $G \in X_{1} \cap X_{2}$ such that $\langle g, x\rangle_{\left(X_{1} \cap X_{2}\right)^{\prime} \times\left(X_{1} \cap X_{2}\right)}=(G, x)_{X_{1}}+(G, x)_{X_{2}}$ and

$$
\|g\|_{\left(X_{1} \cap X_{2}\right)^{\prime}}=\|G\|_{X_{1} \cap X_{2}}=\left(\|G\|_{X_{1}}^{2}+\|G\|_{X_{2}}^{2}\right)^{\frac{1}{2}}
$$

For $i=1,2$ define $\hat{g}_{i}: x \rightarrow(G, x)_{X_{i}}$ for all $x \in X_{i}$. Then $\hat{g}_{i} \in X_{i}^{\prime},\left\|\hat{g}_{i}\right\|_{X_{i}^{\prime}}=\|G\|_{X_{i}}$ and $\langle g, x\rangle_{\left(X_{1} \cap X_{2}\right)^{\prime} \times\left(X_{1} \cap X_{2}\right)}=\left\langle\hat{g}_{1}, x\right\rangle_{X_{1}^{\prime} \times X_{1}}+\left\langle\hat{g}_{2}, x\right\rangle_{X_{2}^{\prime} \times X_{2}}$. Because $X_{1} \cap X_{2}$ is dense in $X_{1}$ and in $X_{2}$, both $\hat{g}_{1}$ and $\hat{g}_{2}$ are uniquely defined by their values at $x \in X_{1} \cap X_{2}$. Hence we get $g=\hat{g}_{1}+\hat{g}_{2} \in X_{1}^{\prime}+X_{2}^{\prime}$ and

$$
\begin{aligned}
\|g\|_{X_{1}^{\prime}+X_{2}^{\prime}} & =\inf _{g=g_{1}+g_{2}}\left(\left\|g_{1}\right\|_{X_{1}^{\prime}}^{2}+\left\|g_{2}\right\|_{X_{2}^{\prime}}^{2}\right)^{\frac{1}{2}} \\
& \leq\left(\left\|\hat{g}_{1}\right\|_{X_{1}^{\prime}}^{2}+\left\|\hat{g}_{2}\right\|_{X_{2}^{\prime}}^{2}\right)^{\frac{1}{2}}=\left(\|G\|_{X_{1}}^{2}+\|G\|_{X_{2}}^{2}\right)^{\frac{1}{2}}=\|g\|_{\left(X_{1} \cap X_{2}\right)^{\prime}}
\end{aligned}
$$

which completes the proof.
In the remainder we assume $\tau>0$, unless stated otherwise. By $\tau \mathrm{H}_{2}$ we denote the space $H_{2}$ with the scaled scalar product $\tau(\cdot, \cdot)_{H_{2}}$. Using the previous lemma we obtain the following equivalence result for the Schur complement norm in (2.9).

Theorem 2.2 For all $p \in M$ we have

$$
\begin{equation*}
\min \left\{\gamma_{a}, \gamma_{c}\right\}\|p\|_{S}^{2} \leq\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}}^{2} \leq \max \left\{\Gamma_{a}, \Gamma_{c}\right\}\|p\|_{S}^{2} \tag{2.13}
\end{equation*}
$$

Proof For $p \in M$ we have

$$
\begin{equation*}
\|B p\|_{\left(H_{1} \cap \tau H_{2}\right)^{\prime}}=\sup _{v \in H_{1}} \frac{\langle B p, v\rangle}{\left(\|v\|_{H_{1}}^{2}+\tau\|v\|_{H_{2}}^{2}\right)^{\frac{1}{2}}} \tag{2.14}
\end{equation*}
$$

Due to the properties of $A$ and $C$ and the definition of $\|\cdot\| S$ we get

$$
\min \left\{\gamma_{a}, \gamma_{c}\right\}\|p\|_{S}^{2} \leq\|B p\|_{\left(H_{1} \cap \tau H_{2}\right)^{\prime}}^{2} \leq \max \left\{\Gamma_{a}, \Gamma_{c}\right\}\|p\|_{S}^{2} \quad \text { for all } p \in M
$$

Now we apply the result in (2.12) to the case $X_{1}=H_{1}, X_{2}=\tau H_{2}$. Note that $H_{1} \cap \tau H_{2}=H_{1}$ (this should be understood as equality of sets) and that the intersection is dense in $\tau \mathrm{H}_{2}$. Hence, we get

$$
\|B p\|_{\left(H_{1} \cap \tau H_{2}\right)^{\prime}}=\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}}
$$

and thus the result is proved.
Remark 1 For the nonstationary Stokes problem described in example 1 we obtain

$$
(S p, p)_{L^{2}}=\|\nabla p\|_{\mathbf{H}^{-1}+\tau^{-1} \mathbf{L}^{2}}^{2} \quad \text { for all } p \in L_{0}^{2}(\Omega)
$$

We introduce a subspace $W$ of $M$ :

$$
\begin{equation*}
W=\left\{p \in M \left\lvert\, \sup _{v \in H_{1}} \frac{\langle B p, v\rangle}{\|v\|_{H_{2}}}<\infty\right.\right\}=\left\{p \in M \mid B p \in H_{2}^{\prime}\right\} . \tag{2.15}
\end{equation*}
$$

(Recall that $H_{1}$ is dense in $H_{2}$ ). We define the following functional on $W$ :

$$
\begin{equation*}
\|p\|_{W}:=\sup _{v \in H_{2}} \frac{\langle B p, v\rangle}{\langle C v, v\rangle^{\frac{1}{2}}} . \tag{2.16}
\end{equation*}
$$

The lemma below summarizes several useful properties of $W$.
Lemma 2.3 The following holds:

> The identity $I: W \rightarrow M$ is a continuous embedding.
> $B(W)$ is a closed subspace of $H_{2}^{\prime}$.
> $\|\cdot\|_{W}$ defines a norm and $\left(W,\|\cdot\|_{W}\right)$ is a Hilbert space.
> If $\operatorname{dim}\left(H_{2}\right)<\infty$ then $W=M$ (as sets) holds.

Proof Note that for all $p \in W$ we have

$$
\begin{align*}
& \Gamma_{c}^{-\frac{1}{2}}\|B p\|_{H_{2}^{\prime}} \leq\|p\|_{W} \leq \gamma_{c}^{-\frac{1}{2}}\|B p\|_{H_{2}^{\prime}}  \tag{2.21}\\
& \|p\|_{M} \leq \gamma_{b}^{-1}\|B p\|_{H_{1}^{\prime}} \leq c\|B p\|_{H_{2}^{\prime}} \leq c \Gamma_{c}^{\frac{1}{2}}\|p\|_{W} \tag{2.22}
\end{align*}
$$

with $c$ independent of $p$.
Hence, $\|\cdot\|_{W}$ indeed defines a norm on $W$ and $I: W \rightarrow M$ is a continuous embedding. Let $\left(B p_{n}\right)_{n \geq 1}$ be a Cauchy-sequence in $B(W)=B(M) \cap H_{2}^{\prime}$ w.r.t. $\|\cdot\|_{H_{2}^{\prime}}$. Since $H_{2}^{\prime}$ is complete there exists a $w \in H_{2}^{\prime}$ such that
$\lim _{n \rightarrow \infty}\left\|B p_{n}-w\right\|_{H_{2}^{\prime}}=0$. We have assumed that $H_{1}$ is continuously embedded in $H_{2}$, therefore convergence in $H_{2}^{\prime}$ implies convergence in $H_{1}^{\prime}$ and thus $\lim _{n \rightarrow \infty}\left\|B p_{n}-w\right\|_{H_{1}^{\prime}}=0$. Due to (2.6) $B(M)$ is a closed subspace of $H_{1}^{\prime}$ and thus $w \in B(M)$ holds. We conclude that $w \in B(M) \cap H_{2}^{\prime}=B(W)$ and thus $B(W)$ is a closed subspace of $H_{2}^{\prime}$.

Let $\left(p_{n}\right)_{n \geq 1}$ be a Cauchy-sequence in $\left(W,\|\cdot\|_{W}\right)$. From (2.21) it follows that $\left(B p_{n}\right)_{n \geq 1}$ is a Cauchy-sequence in $\left(B(W),\|\cdot\|_{H_{2}^{\prime}}\right)$. This space is closed and thus there exists $p \in W$ such that $\lim _{n \rightarrow \infty}\left\|B\left(p-p_{n}\right)\right\|_{H_{2}^{\prime}}=0$. Using (2.21) we obtain $\lim _{n \rightarrow \infty}\left\|p-p_{n}\right\|_{W}=0$ and thus $\left(W,\|\cdot\|_{W}\right)$ is a Banach space. It remains to define a scalar product on $W$ that induces $\|\cdot\|_{W}$. For this we need the adjoint of $B: W \rightarrow H_{2}^{\prime}$. Recall that $B^{\prime}: H_{1} \rightarrow M^{\prime}$ is the adjoint of $B: M \rightarrow H_{1}^{\prime}$, i.e., $\left\langle B^{\prime} v, p\right\rangle_{M^{\prime} \times M}=\langle B p, v\rangle_{H_{1}^{\prime} \times H_{1}}$. To distinguish from this adjoint we use the notation $B_{2}^{\prime}$ for the adjoint of $B: W \rightarrow H_{2}^{\prime}$. Hence, $\left\langle B_{2}^{\prime} v, p\right\rangle_{W^{\prime} \times W}=\langle B p, v\rangle_{H_{2}^{\prime} \times H_{2}}$ for all $v \in H_{2}, p \in W$. Using this we define $S_{2}: W \rightarrow W^{\prime}$ by

$$
\begin{equation*}
S_{2}:=B_{2}^{\prime} C^{-1} B \tag{2.23}
\end{equation*}
$$

and we get

$$
\begin{equation*}
\|p\|_{W}^{2}=\left\langle S_{2} p, p\right\rangle_{W^{\prime} \times W} \quad \text { for all } p \in W \tag{2.24}
\end{equation*}
$$

Thus the scalar product on $W$ that corresponds to $\|\cdot\|_{W}$ is given by $(p, q)_{W}=$ $\left\langle S_{2} p, q\right\rangle_{W^{\prime} \times W}$.

From $\operatorname{dim}\left(H_{2}\right)<\infty$ and the assumption that the embedding $H_{1} \hookrightarrow H_{2}$ is dense it follows that $H_{1}=H_{2}$ (with possibly different norms) and that $g \in H_{1}^{\prime}$ iff $g \in H_{2}^{\prime}$. Using that $B p \in H_{1}^{\prime}$ for all $p \in M$ we conclude that $W=M$ holds.

In our analysis we will need the orthogonal projection on $B(M)$ in $H_{1}^{\prime}$. This projection, which is well-defined since $B(M)$ is a closed subspace of $H_{1}^{\prime}$, is denoted by $P$. The following lemma gives another characterization of this projection $P$.

Lemma 2.4 Let $I_{1}: H_{1} \rightarrow H_{1}^{\prime}$ be the Riesz isomorphism, i.e., $\left\langle I_{1} u, v\right\rangle=(u, v)_{H_{1}}$ for all $u, v \in H_{1}$. For $f \in H_{1}^{\prime}$ let $(u, p) \in H_{1} \times M$ be the unique solution of

$$
\begin{array}{r}
I_{1} u+B p=f, \\
B^{\prime} u=0 .
\end{array}
$$

Define the solution operator $S_{1}: H_{1}^{\prime} \rightarrow M$ by $f \rightarrow p$. Then $P=B S_{1}$ holds.
Proof For arbitrary $f \in H_{1}^{\prime}$ we have $B S_{1} f=B p \in B(M)$ and for any $q \in M$ :
$(f-B p, B q)_{H_{1}^{\prime}}=\left\langle I_{1}^{-1}(f-B p), B q\right\rangle_{H_{1} \times H_{1}^{\prime}}=\langle u, B q\rangle_{H_{1} \times H_{1}^{\prime}}=\left\langle B^{\prime} u, q\right\rangle_{M^{\prime} \times M}=0$
and thus the result holds.

### 2.3 Schur complement preconditioner

We introduce the norm

$$
\begin{equation*}
\|p\|_{M+\tau^{-1} W}=\inf _{q \in W}\left(\|p-q\|_{M}^{2}+\tau^{-1}\|q\|_{W}^{2}\right)^{\frac{1}{2}} \tag{2.25}
\end{equation*}
$$

From the analysis below (Sects. 2.4 and 2.5) it follows that (under a certain assumption) this norm is uniformly (w.r.t. $\tau$ ) equivalent to $\|p\|_{S}=\langle S p, p\rangle^{\frac{1}{2}}$. It is not obvious how to use $\|p\|_{M+\tau^{-1} W}$ to construct a feasible preconditioner for the Schur complement $S$. In this section we address this issue.

Let $I_{M}: M \rightarrow M^{\prime}$ be the Riesz isomorphism. Because the identity $I: W \rightarrow M$ is a continuous embedding we have $I_{M}(W) \subset W^{\prime}$. The mapping $I_{M}: W \rightarrow W^{\prime}$ is denoted by $I_{W}$ (note that in general this is not the Riesz-isomorphism in $W$ ).

Theorem 2.5 Define $\tilde{S}: M \rightarrow M^{\prime}$ by $\tilde{S}=I_{M}-I_{M}\left(I_{W}+\tau^{-1} S_{2}\right)^{-1} I_{M}$ with $S_{2}$ defined in (2.23). Then $\tilde{S}$ is selfadjoint and positive definite and

$$
\begin{equation*}
\|p\|_{M+\tau^{-1} W}^{2}=\langle\tilde{S} p, p\rangle \quad \text { for all } p \in M \tag{2.26}
\end{equation*}
$$

Proof By assumption the operator $C^{-1}: H_{2}^{\prime} \rightarrow H_{2}^{\prime}$ is selfadjoint, therefore $\tilde{S}$ is selfadjoint as well.

With the help of elementary variational analysis we see that the infimum on the right handside in (2.25) is attained for $\tilde{q} \in W$ that satisfies

$$
(\tilde{q}-p, \xi)_{M}+\tau^{-1}(\tilde{q}, \xi)_{W}=0 \quad \text { for all } \xi \in W
$$

This can be reformulated in operator notation, using the definition of the $W$ scalar product:

$$
\begin{equation*}
\left\langle I_{M}(\tilde{q}-p)+\tau^{-1} S_{2} \tilde{q}, \xi\right\rangle_{W^{\prime} \times W}=0 \quad \text { for all } \xi \in W \tag{2.27}
\end{equation*}
$$

Note that $I_{M} p \in M^{\prime} \subset W^{\prime}$ holds. The solution $\tilde{q} \in W$ of (2.27) is given by

$$
\left(I_{W}+\tau^{-1} S_{2}\right) \tilde{q}=I_{M p}
$$

and thus $\tilde{q}=\left(I_{W}+\tau^{-1} S_{2}\right)^{-1} I_{M} p$. A straightforward computation yields

$$
\|p\|_{M+\tau^{-1} W}^{2}=\|p-\tilde{q}\|_{M}^{2}+\tau^{-1}\|\tilde{q}\|_{W}^{2}=(p-\tilde{q}, p)_{M}=\left\langle I_{M}(p-\tilde{q}), p\right\rangle
$$

Substituting $\tilde{q}=\left(I_{W}+\tau^{-1} S_{2}\right)^{-1} I_{M} p$, we obtain (2.26). From (2.26) it follows that $\tilde{S}$ is positive definite.

In the setting of preconditioning one is interested in the inverse of the preconditioner. By a straightforward computation one can check that the inverse
$\tilde{S}^{-1}: M^{\prime} \rightarrow M$ of $\tilde{S}$ is given by

$$
\begin{equation*}
\tilde{S}^{-1}:=I_{M}^{-1}+\tau S_{2}^{-1} . \tag{2.28}
\end{equation*}
$$

2.4 Uniform spectral bound $S \lesssim \tilde{S}$

The proof of a spectral bound $S \leq c \tilde{S}$ with a constant $c$ independent of $\tau$ is very simple.

Theorem 2.6 Define $\Gamma_{s}=\frac{\Gamma_{b}^{2}+\Gamma_{c}}{\min \left\{\gamma_{a}, \gamma_{c}\right\}}$. For all $p \in M$ we have

$$
\langle S p, p\rangle \leq \Gamma_{s}\langle\tilde{S} p, p\rangle
$$

Proof From Theorem 2.2 we get

$$
\langle S p, p\rangle \leq \frac{1}{\min \left\{\gamma_{a}, \gamma_{c}\right\}}\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}}^{2}
$$

From (2.5), (2.6) and the definition of $\|\cdot\|_{W}$ we have

$$
\|B\|_{M \rightarrow H_{1}^{\prime}} \leq \Gamma_{b}, \quad\|B\|_{W \rightarrow H_{2}^{\prime}} \leq \Gamma_{c}^{\frac{1}{2}}
$$

and thus from (2.11) we obtain

$$
\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}} \leq\left(\Gamma_{b}^{2}+\Gamma_{c}\right)^{\frac{1}{2}}\|p\|_{M+\tau^{-1} W}
$$

Hence, using theorem 2.5, we obtain

$$
\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}}^{2} \leq\left(\Gamma_{b}^{2}+\Gamma_{c}\right)\langle\tilde{S} p, p\rangle
$$

### 2.5 Uniform spectral bound $\tilde{S} \lesssim S$

The derivation of a spectral inequality $\tilde{S} \leq \hat{c} S$ with a constant $\hat{c}>0$ independent of $\tau$ turns out to be more delicate than the bound $S \leq c \tilde{S}$ that is shown in Theorem 2.6. We present an analysis which requires an assumption on the orthogonal projection $P: H_{1}^{\prime} \rightarrow B(M)$ (cf. Sect. 2.2).
This crucial assumption is as follows.
Assumption 1 Assume that $P: H_{2}^{\prime} \rightarrow H_{2}^{\prime}$ and that there exist constants $c_{P} \geq 1$, $d_{P} \geq 0$ such that

$$
\begin{equation*}
\|P f\|_{H_{2}^{\prime}}^{2} \leq c_{P}^{2}\left(\|f\|_{H_{2}^{\prime}}^{2}+d_{P}^{2}\|(I-P) f\|_{H_{1}^{\prime}}^{2}\right) \quad \text { for all } f \in H_{2}^{\prime} \tag{2.29}
\end{equation*}
$$

Lemma 2.7 If Assumption 1 holds, then we have

$$
\begin{equation*}
B(W)=P\left(H_{2}^{\prime}\right) \tag{2.30}
\end{equation*}
$$

Proof Take $p \in W$. Then $B p \in H_{2}^{\prime} \subset H_{1}^{\prime}$ and with the solution operator $S$ as in Lemma 2.4 we get $S_{1} B p=p$. This yields $P B p=B S_{1} B p=B p$ and thus $B p \in P\left(H_{2}^{\prime}\right)$, which proves $B(W) \subset P\left(H_{2}^{\prime}\right)$. Take $P f \in P\left(H_{2}^{\prime}\right)$. Then $P f \in H_{2}^{\prime}$ and $P f=B S_{1} f=B p$ with $p:=S_{1} f \in M$. Thus $P f \in B(W)$, i.e., $P\left(H_{2}^{\prime}\right) \subset B(W)$. Hence, the result (2.30) holds.

Below we use the Hilbert spaces $\left(B(M),\|\cdot\|_{H_{1}^{\prime}}\right)$ and $\left(B(W),\|\cdot\|_{H_{2}^{\prime}}\right)$.
Lemma 2.8 Let Assumption 1 hold. Then for all $p \in M$ we have

$$
\|B p\|_{B(M)+\tau^{-1} B(W)} \leq c_{P}\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}} \quad \text { for all } \tau \geq d_{P}^{2}
$$

Proof We use the notation $f:=B p$. Note that

$$
\|f\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}}=\inf _{w \in H_{2}^{\prime}}\left(\|f-w\|_{H_{1}^{\prime}}^{2}+\tau^{-1}\|w\|_{H_{2}^{\prime}}^{2}\right)^{\frac{1}{2}}
$$

Take an arbitrary $w \in H_{2}^{\prime}$. Using $f \in B(M)$ we get

$$
\begin{aligned}
\|f-w\|_{H_{1}^{\prime}}^{2} & =\|P(f-w)+(P-I) w\|_{H_{1}^{\prime}}^{2}=\|P f-P w\|_{H_{1}^{\prime}}^{2}+\|(P-I) w\|_{H_{1}^{\prime}}^{2} \\
& =\|f-P w\|_{H_{1}^{\prime}}^{2}+\|(I-P) w\|_{H_{1}^{\prime}}^{2} .
\end{aligned}
$$

From $\|P w\|_{H_{2}^{\prime}}^{2} \leq c_{P}^{2}\left(\|w\|_{H_{2}^{\prime}}^{2}+d_{P}^{2}\|(I-P) w\|_{H_{1}^{\prime}}^{2}\right)$ we get $\|w\|_{H_{2}^{\prime}}^{2} \geq c_{P}^{-2}\|P w\|_{H_{2}^{\prime}}^{2}-$ $d_{P}^{2}\|(I-P) w\|_{H_{1}^{\prime}}^{2}$. Hence we obtain, using $\tau \geq d_{P}^{2}$ and $c_{P} \geq 1$,

$$
\begin{aligned}
\inf _{w \in H_{2}^{\prime}}\left(\|f-w\|_{H_{1}^{\prime}}^{2}+\tau^{-1}\|w\|_{H_{2}^{\prime}}^{2}\right)^{\frac{1}{2}} \geq & \inf _{w \in H_{2}^{\prime}}\left(\|f-P w\|_{H_{1}^{\prime}}^{2}+\tau^{-1} c_{P}^{-2}\|P w\|_{H_{2}^{\prime}}^{2}\right. \\
& \left.+\left(1-\tau^{-1} d_{P}^{2}\right)\|(I-P) w\|_{H_{1}^{\prime}}^{2}\right)^{\frac{1}{2}} \\
\geq & \inf _{w \in H_{2}^{\prime}}\left(\|f-P w\|_{H_{1}^{\prime}}^{2}+\tau^{-1} c_{P}^{-2}\|P w\|_{H_{2}^{\prime}}^{2}\right)^{\frac{1}{2}} \\
\geq & c_{P}^{-1} \inf _{w \in H_{2}^{\prime}}\left(\|f-P w\|_{H_{1}^{\prime}}^{2}+\tau^{-1}\|P w\|_{H_{2}^{\prime}}^{2}\right)^{\frac{1}{2}} \\
\geq & c_{P}^{-1} \inf _{\substack{f=f_{1}+f_{2} \\
f_{1} \in B(M), f_{2} \in B(W)}}\left(\left\|f_{1}\right\|_{H_{1}^{\prime}}^{2}+\tau^{-1}\left\|f_{2}\right\|_{H_{2}^{\prime}}^{2}\right)^{\frac{1}{2}} \\
= & c_{P}^{-1}\|f\|_{B(M)+\tau^{-1} B(W)}
\end{aligned}
$$

and thus the result is proved.

Remark 2 Consider the finite dimensional case $\operatorname{dim}\left(H_{2}\right)<\infty$. We then have $H_{1}=H_{2}, H_{1}^{\prime}=H_{2}^{\prime}$ and $W=M$ (where " $=$ " allows different norms in the spaces). We can apply a symmetry argument involving an alternative to the assumption 1. Let $\hat{P}: H_{2}^{\prime} \rightarrow B(W)$ be the orthogonal projection on $B(W)$ in $H_{2}^{\prime}$. Assume that $c_{P} \geq 1, \hat{d}_{P} \geq 0$ are such that

$$
\begin{equation*}
\|\hat{P} f\|_{H_{1}^{\prime}}^{2} \leq c_{P}^{2}\left(\|f\|_{H_{1}^{\prime}}^{2}+\hat{d}_{P}^{2}\|(I-\hat{P}) f\|_{H_{2}^{\prime}}^{2}\right) \quad \text { for all } f \in H_{2}^{\prime} \tag{2.31}
\end{equation*}
$$

Lemma 2.8 then yields, for all $p \in M$ :

$$
\|B p\|_{B(W)+\tau^{-1} B(M)} \leq c_{P}\|B p\|_{H_{2}^{\prime}+\tau^{-1} H_{1}^{\prime}} \quad \text { for all } \tau \geq \hat{d}_{P}^{2}
$$

which is equivalent to

$$
\begin{equation*}
\|B p\|_{B(M)+\tau^{-1} B(W)} \leq c_{P}\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}^{\prime}} \quad \text { for all } \tau \leq \hat{d}_{P}^{-2} \tag{2.32}
\end{equation*}
$$

This will be used in the analysis of the finite element discretization in Sect. 4.
Theorem 2.9 Let assumption 1 hold. Define $\gamma_{s}:=\frac{\gamma_{b}^{2} \gamma_{c}}{c_{P}^{2}\left(\gamma_{b}^{2}+\gamma_{c}\right) \max \left\{\Gamma_{a}, \Gamma_{c}\right\}}$. For all $p \in M$ we have

$$
\begin{equation*}
\gamma_{s}\langle\tilde{S} p, p\rangle \leq\langle S p, p\rangle \quad \text { for all } \tau \geq d_{P}^{2} \tag{2.33}
\end{equation*}
$$

Proof From (2.5), (2.6) and the definition of $\|\cdot\|_{W}$ we also have

$$
\left\|B^{-1}\right\|_{B(M) \rightarrow M} \leq \gamma_{b}^{-1}, \quad\left\|B^{-1}\right\|_{B(W) \rightarrow W} \leq \gamma_{c}^{-\frac{1}{2}}
$$

and thus

$$
\left\|B^{-1} g\right\|_{M+\tau^{-1} W} \leq\left(\gamma_{b}^{-2}+\gamma_{c}^{-1}\right)^{\frac{1}{2}}\|g\|_{B(M)+\tau^{-1} B(W)} \quad \text { for all } g \in B(M)
$$

Hence,

$$
\frac{\gamma_{b}^{2} \gamma_{c}}{\gamma_{b}^{2}+\gamma_{c}}\|p\|_{M+\tau^{-1} W}^{2} \leq\|B p\|_{B(M)+\tau^{-1} B(W)}^{2} \quad \text { for all } p \in M
$$

Using Lemma 2.8 we obtain

$$
\frac{\gamma_{b}^{2} \gamma_{c}}{c_{P}^{2}\left(\gamma_{b}^{2}+\gamma_{c}\right)}\|p\|_{M+\tau^{-1} W}^{2} \leq\|B p\|_{H_{1}^{\prime}+\tau^{-1} H_{2}}^{2} \quad \text { for all } \tau \geq d_{P}^{2}
$$

and combining this with Theorem 2.5 and Theorem 2.2 proves the inequality in (2.33).

Remark 3 Consider the finite dimensional setting as in Remark 2 and assume that besides Assumption 1 also (2.31) holds. Then (2.32) holds, and a slight modification of the last step in the proof of Theorem 2.9 then yields, for all $p \in M$ :

$$
\begin{equation*}
\gamma_{s}\langle\tilde{S} p, p\rangle \leq\langle S p, p\rangle \quad \forall \tau \in\left(0, \hat{d}_{P}^{-2}\right] \cup\left[d_{P}^{2}, \infty\right) . \tag{2.34}
\end{equation*}
$$

The main result of the general analysis is the following.
Corollary 2.10 Suppose Assumption 1 holds. The following inequalities hold for any $p \in M$ :

$$
\begin{equation*}
\gamma_{s}\langle\tilde{S} p, p\rangle \leq\langle S p, p\rangle \leq \Gamma_{s}\langle\tilde{S} p, p\rangle \quad \text { for all } \tau \geq d_{P}^{2} \tag{2.35}
\end{equation*}
$$

Proof Direct consequence of Theorems 2.6 and 2.9.
Remark 4 Consider the finite dimensional case $\operatorname{dim}\left(H_{2}\right)<\infty$. From remark 3 it follows that if the assumptions 1 and (2.31) hold, then the equivalence result in (2.35) holds for all $\tau \in\left(0, \hat{d}_{P}^{-2}\right] \cup\left[d_{P}^{2}, \infty\right)$.

As a final result in this section we give a simple criterion that will be used in the applications in the next sections to show that Assumption 1 holds.

Lemma 2.11 Let $S_{1}: H_{1}^{\prime} \rightarrow M$ be the solution operator from Lemma 2.4.
Assume that there is a subspace $\tilde{W} \subset M$ with norm $\|\cdot\|_{\tilde{W}}$ such that both $S_{1}: H_{2}^{\prime} \rightarrow \tilde{W}$ and $B: \tilde{W} \rightarrow H_{2}^{\prime}$ are bounded, i.e.,

$$
\left\|S_{1} f\right\|_{\tilde{W}} \leq c_{1}\|f\|_{H_{2}^{\prime}} \quad \forall f \in H_{2}^{\prime}, \quad\|B p\|_{H_{2}^{\prime}} \leq c_{2}\|p\|_{\tilde{W}} \quad \forall p \in \tilde{W}
$$

then Assumption 1 is fulfilled with $c_{P}=c_{1} c_{2}, d_{P}=0$.
Proof The proof immediately follows from $P=B S_{1}$ and

$$
\|P f\|_{H_{2}^{\prime}}=\left\|B S_{1} f\right\|_{H_{2}^{\prime}} \leq c_{2}\left\|S_{1} f\right\|_{\tilde{W}} \leq c_{2} c_{1}\|f\|_{H_{2}^{\prime}} \quad \text { for all } f \in H_{2}^{\prime}
$$

## 3 Application to the continuous generalized Stokes problem

In this section we apply the above abstract analysis to the generalized Stokes problem. The spaces and bilinear forms used in the variational problem are as in example 1. It was noted that we have the properties (2.1)-(2.3), with $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=\Gamma_{b}=1, \gamma_{b}>0$ the constant from the infsup inequality. For the operators $A, B, B^{\prime}, C$ corresponding to the bilinear forms we use the (usual) notation

$$
A=:-\Delta, B:=\nabla, \quad B^{\prime}=:-\operatorname{div}, \quad C=: \mathrm{I} .
$$

We now consider Assumption 1. We use the criterion given in Lemma 2.11. Note that $-\Delta$ is the Riesz isomorphism $\mathbf{H}_{0}^{1}(\Omega) \rightarrow \mathbf{H}_{0}^{1}(\Omega)^{\prime}=: \mathbf{H}^{-1}$. Thus for $\mathbf{f} \in \mathbf{H}^{-1}$ the solution $p=S_{1} \mathbf{f}$, with $S_{1}$ from lemma 2.4, satisfies the weak formulation of the stationary Stokes problem:

$$
\begin{align*}
-\Delta \mathbf{u}+\nabla p & =\mathbf{f} \\
\operatorname{div} \mathbf{u} & =0  \tag{3.1}\\
\left.\mathbf{u}\right|_{\partial \Omega} & =0 .
\end{align*}
$$

In the following lemma it is shown that $H^{2}$-regularity of the Stokes problem implies that Assumption 1 holds.

Lemma 3.1 Assume that the domain $\Omega$ is such that the Stokes problem (3.1) is $H^{2}$-regular, i.e., there is a constant $c_{R}$ such that for any $\mathbf{f} \in \mathbf{L}^{2}(\Omega)$ the solution $(\mathbf{u}, p)$ is an element of $H^{2}(\Omega)^{d} \times H^{1}(\Omega)$ and satisfies

$$
\begin{equation*}
\|\mathbf{u}\|_{H^{2}(\Omega)}+\|\nabla p\|_{L^{2}} \leq c_{R}\|\mathbf{f}\|_{L^{2}} . \tag{3.2}
\end{equation*}
$$

Then Assumption 1 is satisfied with $c_{P}=c_{R}$ and $d_{P}=0$. Furthermore, we have $W=H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ and $\|p\|_{W}=\|\nabla p\|_{L^{2}}$.

Proof We apply Lemma 2.11 with $\tilde{W}:=H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ and norm $\|p\|_{\tilde{W}}^{2}=$ $(\nabla p, \nabla p)_{L^{2}}$. Due to the regularity assumption we have $\left\|S_{1} \mathbf{f}\right\|_{\tilde{W}}=\|\nabla p\|_{L^{2}} \leq$ $c_{R}\|\mathbf{f}\|_{L^{2}}$. Furthermore, for $p \in \tilde{W}$ we have $\|B p\|_{H_{2}^{\prime}}=\|\nabla p\|_{L^{2}}=\|p\|_{\tilde{W}}$. Thus the assumptions in Lemma 2.11 hold with $c_{1}=c_{R}, c_{2}=1$. It follows that Assumption 1 is fulfilled.

Definition (2.15) of $W$ takes the form $W:=\left\{p \in L_{0}^{2} \mid \nabla p \in \mathbf{L}^{2}\right\}$. Thus $W=H^{1}(\Omega) \cap L_{0}^{2}(\Omega)=\tilde{W}$. Finally by the definition of the $W$-norm we have for $p \in \tilde{W}:$

$$
\|p\|_{W}:=\sup _{v \in H_{2}} \frac{\langle B p, v\rangle}{\langle C v, v\rangle^{\frac{1}{2}}}=\sup _{\mathbf{v} \in \mathbf{L}^{2}} \frac{(\nabla p, \mathbf{v})_{L^{2}}}{\|\mathbf{v}\|_{L^{2}}}=\|\nabla p\|_{L^{2}} .
$$

Now consider the Schur complement of the generalized Stokes problem:

$$
\begin{equation*}
S:=-\operatorname{div}(\tau \mathrm{I}-\Delta)^{-1} \nabla \tag{3.3}
\end{equation*}
$$

We identify $L_{0}^{2}(\Omega)$ with its dual. Then $S: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$ and $\langle\cdot, \cdot\rangle_{M^{\prime} \times M}=$ $(\cdot, \cdot)_{L^{2}}$.

If the stationary Stokes problem is $H^{2}$-regular our abstract theory can be applied, with $d_{P}=0$ in assumption 1 , and we have a uniform equivalence result given in Corollary 2.10. This yields the following main result of this section.

Theorem 3.2 Assume that the domain $\Omega \subset \mathbb{R}^{d}$ is such that the Stokes problem (3.1) is $H^{2}$-regular. Denote by $-\Delta_{N}^{-1}: L_{0}^{2}(\Omega) \rightarrow H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ the solution
operator of the following Neumann pressure problem: Given $f \in L_{0}^{2}(\Omega)$ find $p \in H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ such that

$$
(\nabla p, \nabla q)_{L^{2}}=(f, q)_{L^{2}}, \quad \forall q \in H^{1}(\Omega) \cap L_{0}^{2}(\Omega)
$$

Define $\tilde{S}^{-1}=I-\tau \Delta_{N}^{-1}$. Then $\tilde{S}^{-1}: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$ is selfadjoint and positive definite, and for all $p \in L_{0}^{2}(\Omega)$ and all $\tau \geq 0$ the following holds:

$$
\gamma_{s}(\tilde{S} p, p)_{L^{2}} \leq(S p, p)_{L^{2}} \leq \Gamma_{S}(\tilde{S} p, p)_{L^{2}}
$$

with $\gamma_{s}=\frac{\gamma_{b}^{2}}{c_{R}^{2}\left(\gamma_{b}^{2}+1\right)}, \Gamma_{s}=2$.
Proof We apply Corollary 2.10. In the setting here we have $W=H_{0}^{1}(\Omega) \cap L_{0}^{2}(\Omega)$, $M=L_{0}^{2}(\Omega)=M^{\prime}$. The mapping $\tilde{S}: M \rightarrow M$ is defined by, cf. (2.28), $\tilde{S}^{-1}=$ $I_{L^{2}}^{-1}+\tau S_{2}^{-1}=I+\tau S_{2}^{-1}$ with $S_{2}=B_{2}^{\prime} C^{-1} B$. For $f \in M$ we have $w=S_{2}^{-1} f \in W$ iff

$$
\begin{aligned}
& \left\langle B_{2}^{\prime} C^{-1} B w, q\right\rangle_{W^{\prime} \times W}=(f, q)_{L^{2}} \quad \forall q \in W \\
\Leftrightarrow & \left\langle B q, C^{-1} B w\right\rangle_{L^{2} \times L^{2}}=(f, q)_{L^{2}} \quad \forall q \in W \\
\Leftrightarrow & \left\langle\nabla q, I_{L^{2}}^{-1} \nabla w\right\rangle_{L^{2} \times L^{2}}=(f, q)_{L^{2}} \quad \forall q \in W \\
\Leftrightarrow & (\nabla w, \nabla q)_{L^{2}}=(f, q)_{L^{2}} \quad \forall q \in W
\end{aligned}
$$

and thus $S_{2}^{-1}$ is equal to the Neumann solution operator $-\Delta_{N}^{-1}$. Hence $\tilde{S}^{-1}=I+$ $\tau S_{2}^{-1}=I-\tau \Delta_{N}^{-1}$. The values for the spectral bounds follow from Corollary 2.10 and from $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=\Gamma_{b}=1$, and $c_{P}=c_{R}$.

## 4 Application to finite element discretization of the generalized Stokes problem

In this section we apply the abstract analysis of Sect. 2 to a finite element discretization of the generalized Stokes problem (2.10).

Let $\mathbf{V}_{h} \times M_{h} \subset \mathbf{H}_{0}^{1}(\Omega) \times L_{0}^{2}(\Omega)$ be a pair of conforming finite element spaces. We assume the LBB stability condition:

$$
\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{\left(\operatorname{div} \mathbf{v}_{h}, p_{h}\right)_{L^{2}}}{\left\|\nabla \mathbf{v}_{h}\right\|_{L_{2}}} \geq \gamma_{b}\left\|p_{h}\right\|_{L_{2}} \quad \text { for all } p_{h} \in M_{h}
$$

with a constant $\gamma_{b}>0$ independent of $h$. We also assume a global inverse inequality and an approximation property:

$$
\begin{aligned}
& \left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}} \leq c_{\text {inv }} h^{-1}\left\|\mathbf{v}_{h}\right\|_{L^{2}} \quad \forall \mathbf{v}_{h} \in \mathbf{V}_{h} \\
& \inf _{\mathbf{v}_{h} \in \mathbf{V}_{h}}\left\|\nabla\left(\mathbf{u}-\mathbf{v}_{h}\right)\right\|_{L^{2}}+\inf _{q_{h} \in M_{h}}\left\|p-q_{h}\right\|_{L^{2}} \leq \operatorname{Ch}\left(\|\mathbf{u}\|_{H^{2}}+\|p\|_{H^{1}}\right)
\end{aligned}
$$

for all $\mathbf{u} \in H^{2}(\Omega)^{d} \cap \mathbf{H}_{0}^{1}(\Omega), p \in H^{1}(\Omega)$. In the setting of the general analysis we take the spaces

$$
H_{1}=\left(\mathbf{V}_{h},(\nabla \cdot, \nabla \cdot)_{L^{2}}\right), \quad H_{2}=\left(\mathbf{V}_{h},(\cdot, \cdot)_{L^{2}}\right), \quad M=\left(M_{h},(\cdot, \cdot)_{L^{2}}\right) .
$$

In this finite dimensional case we have $W=M$ as sets (note, however, that in general $\|\cdot\|_{W} \neq\|\cdot\|_{M}$ ). The bilinear forms are the same as in Sect. 3. The operators corresponding to these bilinear forms are denoted by $A_{h}, C_{h}, B_{h}$. As in the continuous case we identify $M_{h}$ with its dual $M_{h}^{\prime}$. Thus we have $\langle\cdot, \cdot\rangle_{M^{\prime} \times M}=\langle\cdot, \cdot\rangle_{W^{\prime} \times W}=(\cdot, \cdot)_{L^{2}}$ and

$$
\begin{array}{cll}
A_{h}: \mathbf{V}_{h} \rightarrow \mathbf{V}_{h}^{\prime}, & \left\langle A_{h} \mathbf{u}_{h}, \mathbf{v}_{h}\right\rangle=a\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right) & \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
C_{h}: \mathbf{V}_{h} \rightarrow \mathbf{V}_{h}^{\prime}, & \left\langle C_{h} \mathbf{u}_{h}, \mathbf{v}_{h}\right\rangle=c\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right) & \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
B_{h}: M_{h} \rightarrow \mathbf{V}_{h}^{\prime}, & \left\langle B_{h} p_{h}, \mathbf{v}_{h}\right\rangle=b\left(\mathbf{v}_{h}, p_{h}\right) & \text { for all } p_{h} \in M_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
B_{h}^{\prime}: \mathbf{V}_{h} \rightarrow M_{h}, & \left(B_{h}^{\prime} \mathbf{v}_{h}, p_{h}\right)_{L^{2}}=b\left(\mathbf{v}_{h}, p_{h}\right) & \text { for all } p_{h} \in M_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h} .
\end{array}
$$

The discrete generalized Stokes problem is as follows: given $\mathbf{f}_{h} \in \mathbf{V}_{h}^{\prime}$ find $\left(\mathbf{u}_{h}, p_{h}\right) \in \mathbf{V}_{h} \times M_{h}$ such that

$$
\begin{align*}
A_{h} \mathbf{u}_{h}+\tau C_{h} \mathbf{u}_{h}+B_{h} p_{h} & =\mathbf{f}_{h},  \tag{4.1}\\
B_{h}^{\prime} \mathbf{u}_{h} & =0 .
\end{align*}
$$

The corresponding Schur complement is given by $S_{h}=B_{h}^{\prime}\left(A_{h}+\tau C_{h}\right)^{-1} B_{h}$ : $M_{h} \rightarrow M_{h}$. Application of the general analysis yields the following main result of this section.

Theorem 4.1 Assume that $\Omega \subset \mathbb{R}^{d}$ is such that the continuous stationary Stokes problem (3.1) is $H^{2}$-regular. With $I_{h}$ the identity operator on $M_{h}$ define $\tilde{S}_{h}^{-1}=$ $I_{h}+\tau\left(B_{h}^{\prime} C_{h}^{-1} B_{h}\right)^{-1}$. Then the following inequalities hold for any $p_{h} \in M_{h}$ with $c_{d}>0$ independent of $\tau$ and $h$ :

$$
c_{d}\left(\tilde{S}_{h} p_{h}, p_{h}\right)_{L^{2}} \leq\left(S_{h} p_{h}, p_{h}\right)_{L^{2}} \leq 2\left(\tilde{S}_{h} p_{h}, p_{h}\right)_{L^{2}}
$$

Proof The properties (2.1)-(2.3) hold with $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=\Gamma_{b}=1$, and $\gamma_{b}>0$ the constant from the LBB condition.

We now treat Assumption 1. Let $P_{h}: H_{1}^{\prime} \rightarrow B_{h}\left(M_{h}\right)$ be the orthogonal projection on $B_{h}\left(M_{h}\right)$ in $H_{1}^{\prime}$. From Lemma 2.4 we have $P_{h}=B_{h} S_{1, h}$, where (for $\left.\mathbf{f}_{h} \in H_{2}^{\prime}\right) S_{1, h} \mathbf{f}_{h}=p_{h}$ is the solution operator corresponding to the discrete stationary Stokes problem

$$
\begin{align*}
A_{h} \mathbf{u}_{h}+B_{h} p_{h} & =\mathbf{f}_{h}, \\
B_{h}^{\prime} \mathbf{u}_{h} & =0 . \tag{4.2}
\end{align*}
$$

The functional $\mathbf{f}_{h}$ can be extended to $\mathbf{f} \in \mathbf{L}^{2}(\Omega)^{\prime}$ with $\left\langle f, v_{h}\right\rangle=\left(f_{h}, v_{h}\right)$ for all $\mathbf{v}_{h} \in \mathbf{V}_{h}$ and $\|\mathbf{f}\|_{L^{2}(\Omega)^{\prime}}=\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}\left(\right.$ recall, $H_{2}=\left(\mathbf{V}_{h},\|\cdot\|_{L^{2}}\right)$ ). Consider the continuous stationary Stokes problem with right hand side $\mathbf{f}$ :

$$
\begin{align*}
A \mathbf{u}+B p & =\mathbf{f},  \tag{4.3}\\
B^{\prime} \mathbf{u} & =0 .
\end{align*}
$$

Comparison of (4.2) and (4.3) yields

$$
a\left(\mathbf{u}-\mathbf{u}_{h}, \mathbf{v}_{h}\right)+b\left(\mathbf{v}_{h}, p-p_{h}\right)=0 \quad \text { for all } \mathbf{v}_{h} \in \mathbf{V}_{h}
$$

Using the $H^{2}$-regularity, the inverse inequality and the approximation property of the finite element spaces we obtain

$$
\begin{aligned}
\left\|P_{h} \mathbf{f}_{h}\right\|_{H_{2}^{\prime}} & =\left\|B_{h} S_{1, h} \mathbf{f}_{h}\right\|_{H_{2}^{\prime}}=\left\|B_{h} p_{h}\right\|_{H_{2}^{\prime}}=\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{b\left(\mathbf{v}_{h}, p_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}} \\
& \leq \sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{b\left(\mathbf{v}_{h}, p\right)}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}}+\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{a\left(\mathbf{u}-\mathbf{u}_{h}, \mathbf{v}_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}} \\
& \leq\|\nabla p\|_{L^{2}}+\left\|\nabla\left(\mathbf{u}-\mathbf{u}_{h}\right)\right\|_{L^{2}} \sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}}}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}} \\
& \leq\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}+c\left\|\nabla\left(\mathbf{u}-\mathbf{u}_{h}\right)\right\|_{L^{2}} h^{-1} \\
& \leq\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}+c\|\mathbf{u}\|_{H^{2}(\Omega)} \leq c\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}} .
\end{aligned}
$$

Hence, Assumption 1 holds with $d_{P}=0$ and a constant $c_{P}$ independent of $h$. Application of Corollary 2.10 proves the result.

Both in the analysis of the continuous generalized Stokes problem (Theorem 3.2) and of its finite element discretization (Theorem 4.1) we need a $H^{2}$-regularity assumption. We now show, that for a certain range of $\tau$ values a regularity assumption can be avoided.

Theorem 4.2 Let $\tilde{S}_{h}^{-1}=I_{h}+\tau\left(B_{h}^{\prime} C_{h}^{-1} B_{h}\right)^{-1}$ be as in Theorem 4.1. There exist positive constants $c_{1}, c_{2}$, independent of $h$ and $\tau$, such for all $p_{h} \in M_{h}$ the following holds:
$\gamma_{s}\left(\tilde{S}_{h} p_{h}, p_{h}\right)_{L^{2}} \leq\left(S_{h} p_{h}, p_{h}\right)_{L^{2}} \leq \Gamma_{S}\left(\tilde{S}_{h} p_{h}, p_{h}\right)_{L^{2}} \quad$ for all $\tau \in\left[0, c_{1}\right] \cup\left[c_{2} h^{-2}, \infty\right)$, with $\gamma_{s}=\frac{\gamma_{b}^{2}}{2\left(\gamma_{b}^{2}+1\right)}, \Gamma_{s}=2$.

Proof We use the result given in Remark 4. The properties (2.1)-(2.3) hold with $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=\Gamma_{b}=1$, and $\gamma_{b}$ the constant from the LBB condition.

Let $P_{h}: H_{1}^{\prime} \rightarrow B_{h}\left(M_{h}\right)$ be the orthogonal projection on $B_{h}\left(M_{h}\right)$ in $H_{1}^{\prime}$. Using the inverse inequality we get, for $\mathbf{f}_{h} \in H_{2}^{\prime}$ :

$$
\begin{aligned}
\left\|P_{h} \mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2} & \leq 2\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}+2\left\|\left(I-P_{h}\right) \mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}=2\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}+2 \sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{\left(\left(I-P_{h}\right) \mathbf{f}_{h}, \mathbf{v}_{h}\right)_{L^{2}}^{2}}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}^{2}} \\
& \leq 2\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}+2 c_{\mathrm{inv}}^{2} h^{-2} \sup _{\mathbf{v}_{h} \in \mathbf{v}_{h}} \frac{\left(\left(I-P_{h}\right) \mathbf{f}_{h}, \mathbf{v}_{h}\right)_{L^{2}}^{2}}{\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}}^{2}} \\
& =2\left(\left\|\mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}+c_{\mathrm{inv}}^{2} h^{-2}\left\|\left(I-P_{h}\right) \mathbf{f}_{h}\right\|_{H_{1}^{\prime}}^{2}\right) .
\end{aligned}
$$

Thus Assumption 1 holds with $c_{P}=\sqrt{2}$ and $d_{P}=c_{\text {inv }} h^{-1}$.
Let $\hat{P}_{h}: H_{2}^{\prime} \rightarrow B_{h}\left(M_{h}\right)$ be the orthogonal projection on $B_{h}\left(M_{h}\right)$ in $H_{2}^{\prime}$. Using the Friedrichs inequality, $\left\|\mathbf{v}_{h}\right\|_{L^{2}} \leq c_{F}\left\|\nabla \mathbf{v}_{h}\right\|_{L^{2}}$, we obtain, for $\mathbf{f}_{h} \in H_{1}^{\prime}$ :

$$
\left\|\hat{P}_{h}\right\|_{H_{1}^{\prime}}^{2} \leq 2\left\|\mathbf{f}_{h}\right\|_{H_{1}^{\prime}}^{2}+2\left\|\left(I-\hat{P}_{h}\right) \mathbf{f}_{h}\right\|_{H_{1}^{\prime}}^{2} \leq 2\left(\left\|\mathbf{f}_{h}\right\|_{H_{1}^{\prime}}^{2}+c_{F}^{2}\left\|\left(I-\hat{P}_{h}\right) \mathbf{f}_{h}\right\|_{H_{2}^{\prime}}^{2}\right)
$$

Thus (2.31) holds with $c_{P}=\sqrt{2}, d_{P}=c_{F}$. Using the result in Remark 4 we obtain the equivalence result with spectral constants $\gamma_{s}=\frac{\gamma_{b}^{2}}{2\left(\gamma_{b}^{2}+1\right)}, \Gamma_{s}=2$.

Remark 5 The equivalence result for the Schur complement operator $S_{h}$ : $M_{h} \rightarrow M_{h}$ has an obvious analogon if we use matrix representations. Assume that we have chosen (nodal) bases in $\mathbf{V}_{h}$ and $M_{h}$. The coefficient vectors of $\mathbf{u}_{h}$ and $p_{h}$ in these bases are denoted by $\overline{\mathbf{u}}_{h}, \bar{p}_{h}$, respectively. The Euclidean scalar product in $\mathbb{R}^{n}$ is denoted by $\langle\cdot, \cdot\rangle_{2}$. Let $\mathbf{Q}_{h}$ be the mass matrix in $M_{h}$. The matrix representations of $A_{h}, C_{h}, B_{h}$ are defined by

$$
\begin{aligned}
\left\langle\mathbf{A}_{h} \overline{\mathbf{u}}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2} & =\left(\nabla \mathbf{u}_{h}, \nabla \mathbf{v}_{h}\right)_{L^{2}} \quad \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
\left\langle\mathbf{C}_{h} \overline{\mathbf{u}}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2} & =\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)_{L^{2}} \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
\left\langle\mathbf{B}_{h} \bar{p}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2} & =\left(\operatorname{div} \mathbf{v}_{h}, p_{h}\right)_{L^{2}} \quad \text { for all } p_{h} \in M_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}
\end{aligned}
$$

The discrete generalized Stokes problem has a matrix-vector formulation with matrix

$$
\left(\begin{array}{cc}
\mathbf{A}_{h}+\tau \mathbf{C}_{h} \mathbf{B}_{h} \\
\mathbf{B}_{h}^{T} & 0
\end{array}\right)
$$

and thus the Schur complement matrix is $\mathbf{S}_{h}=\mathbf{B}_{h}^{T}\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right)^{-1} \mathbf{B}_{h}$. Using

$$
\begin{aligned}
\left(B_{h}^{\prime} C_{h}^{-1} B_{h} p_{h}, p_{h}\right)_{L^{2}} & =\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{b\left(\mathbf{v}_{h}, p_{h}\right)^{2}}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}^{2}}=\sup _{\overline{\mathbf{v}}_{h} \in \mathbb{R}^{n}} \frac{\left\langle\mathbf{B}_{h} \bar{p}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2}^{2}}{\left\langle\mathbf{C}_{h} \overline{\mathbf{v}}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2}} \\
& =\left\langle\mathbf{B}_{h}^{T} \mathbf{C}_{h}^{-1} \mathbf{B}_{h} \bar{p}_{h}, \bar{p}_{h}\right\rangle_{2} \text { for all } p_{h} \in M_{h},
\end{aligned}
$$

it follows that the Schur complement preconditioner $\tilde{S}_{h}$ given in Theorems 4.1 and 4.2 has the matrix representation $\tilde{\mathbf{S}}^{-1}=\mathbf{Q}_{h}^{-1}+\tau\left(\mathbf{B}_{h}^{T} \mathbf{C}_{h}^{-1} \mathbf{B}_{h}\right)^{-1}$.

The operator $B_{h}^{\prime} C_{h}^{-1} B_{h}$ in the definition of $\tilde{S}_{h}$ corresponds to a mixed discretization of the saddle point formulation of a Neumann problem: Find $\mathbf{u} \in \mathbf{H}_{0}($ div $), p \in L_{0}^{2}(\Omega)$ such that

$$
\begin{aligned}
\mathbf{u}+\nabla p & =0 \\
\operatorname{div} \mathbf{u} & =g \\
\left.\mathbf{u} \cdot \mathbf{n}\right|_{\partial \Omega} & =0
\end{aligned}
$$

This mixed discretization is convenient when the discrete pressure is not continuous, i.e. $M_{h} \nsubseteq H^{1}(\Omega)$. On the other hand, if $M_{h} \subset H^{1}(\Omega)$, then one may wish to use a conforming finite element discretization of the Neumann problem and thus obtain a discrete analogon of the preconditioner given for the continuous case in Theorem 3.2. This is treated in Sect. 4.1.
4.1 Schur complement preconditioner for the case $M_{h} \subset H^{1}(\Omega)$

Assume $M_{h} \subset H^{1}(\Omega)$. Let $N_{h}^{-1}: M_{h} \rightarrow M_{h}, N_{h}^{-1} g_{h}=p_{h}$ be the solution operator of the discrete Neumann problem in $M_{h}$ :

$$
\left(\nabla p_{h}, \nabla q_{h}\right)_{L^{2}}=\left(g_{h}, q_{h}\right)_{L^{2}} \quad \text { for all } q_{h} \in M_{h}
$$

Note that $\left(N_{h} p_{h}, p_{h}\right)=\left\|\nabla p_{h}\right\|_{L^{2}}^{2}$ for all $p_{h} \in M_{h}$. We define $\tilde{S}_{h, N}: M_{h} \rightarrow M_{h}$ by

$$
\tilde{S}_{h, N}^{-1}:=I_{h}^{-1}+\tau N_{h}^{-1} .
$$

This preconditioner has been proposed in [8] and analyzed in [5]. The preconditioner $\tilde{S}_{h, N}$ is uniformly (w.r.t. $h$ and $\tau$ ) spectrally equivalent to $\tilde{S}_{h}$ from Theorem 4.1 iff $N_{h}$ is, uniformly in $h$, spectrally equivalent to $B_{h}^{\prime} C_{h}^{-1} B_{h}$. Note that, for all $p_{h} \in M_{h}$,

$$
\left(B_{h}^{\prime} C_{h}^{-1} B_{h} p_{h}, p_{h}\right)_{L^{2}}=\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{\left(p_{h}, \operatorname{div} \mathbf{v}_{h}\right)_{L^{2}}^{2}}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}^{2}} \leq\left\|\nabla p_{h}\right\|_{L^{2}}^{2}=\left(N_{h} p_{h}, p_{h}\right)_{L^{2}}
$$

Hence, $\tilde{S}_{h, N}$ is uniformly spectrally equivalent to $\tilde{S}_{h}$ iff

$$
\begin{equation*}
\sup _{\mathbf{v}_{h} \in \mathbf{V}_{h}} \frac{\left(p_{h}, \operatorname{div} \mathbf{v}_{h}\right)_{L^{2}}}{\left\|\mathbf{v}_{h}\right\|_{L^{2}}} \geq \gamma_{w}\left\|\nabla p_{h}\right\|_{L^{2}} \quad \text { for all } p_{h} \in M_{h} \tag{4.4}
\end{equation*}
$$

holds with $\gamma_{w}>0$, independent of $h$. This modified stability condition (also called weak inf-sup condition) can be found at several places in the literature,
e.g., [2, 7,24]. In [2] a proof of this result for $P_{1}$ iso $P_{2}-P_{1}$ and for the Hood-Taylor $P_{2}-P_{1}$ pair is given for the two-dimensional case. The approach in [2] can probably be extended to the three-dimensional case. Because the weak inf-sup condition in (4.4) is essential for the analysis in this paper we decided to include an elementary proof for $P_{2}-P_{1}$ Hood-Taylor finite elements in a $d$-dimensional domain, with $d=2,3$. We assume that the family of triangulations $\left\{\mathcal{T}_{h}\right\}$ is regular but not necessarily quasi-uniform.

Lemma 4.3 For $\Omega \subset \mathbb{R}^{d}$, $d=2,3$, let $\left\{\mathcal{T}_{h}\right\}$ be a regular family of triangulations consisting of d-simplices. Assume that every simplex has at least one vertex which is not on $\partial \Omega$. Then the Hood-Taylor $P_{2}-P_{1}$ pair of finite element spaces satisfies (4.4).

Proof The Hood-Taylor $P_{2}-P_{1}$ pair is denoted by $\left(\mathbf{V}_{h}, M_{h}\right)$. Take $q_{h} \in M_{h}$, $q_{h} \neq 0$. The constants used below are independent of $\mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}$ and of $q_{h}$. The set of edges in $\mathcal{T}_{h}$ is denoted by $\mathcal{E}$. This set is partitioned in edges which are in the interior of $\Omega$ and edges which are part of $\partial \Omega: \mathcal{E}=\mathcal{E}_{\text {int }} \cup \mathcal{E}_{\text {bnd }}$. For every $E \in \mathcal{E}, m_{E}$ denotes the midpoint of the edge $E$. Every $E \in \mathcal{E}$ int with endpoints $a_{1}, a_{2} \in \mathbb{R}^{d}$ is assigned a vector $\mathbf{t}_{E}:=a_{1}-a_{2}$. For $E \in \mathcal{E}_{\text {bnd }}$ we define $\mathbf{t}_{E}:=0$. Since $q_{h}$ is continuous piecewise linear the function $x \rightarrow \mathbf{t}_{E} \cdot \nabla q_{h}(x)$ is continuous across $E$, for $E \in \mathcal{E}_{\text {int }}$. We define

$$
\begin{aligned}
& \hat{\mathbf{t}}_{E}:=\left\|\mathbf{t}_{E}\right\|_{2}^{-1} \mathbf{t}_{E} \quad\left(\hat{\mathbf{t}}_{E}:=0 \text { if } E \in \mathcal{E}_{\mathrm{bnd}}\right), \\
& \mathbf{w}_{E}:=\left(\hat{\mathbf{t}}_{E} \cdot \nabla q_{h}\left(m_{E}\right)\right) \hat{\mathbf{t}}_{E} \quad \text { for } E \in \mathcal{E} .
\end{aligned}
$$

A unique $\mathbf{w}_{h} \in \mathbf{V}_{h}$ is defined by

$$
\mathbf{w}_{h}\left(x_{i}\right)=\left\{\begin{array}{cl}
0 & \text { if } x_{i} \text { is a vertex of } T \in \mathcal{T}_{h} \\
\mathbf{w}_{E} & \text { if } x_{i}=m_{E} \text { for } E \in \mathcal{E}
\end{array}\right.
$$

The set of edges of $T \in \mathcal{T}_{h}$ is denoted by $E_{T}$. By using quadrature we see that for any $p \in P_{2}$ which is zero at the vertices of $T$ we have

$$
\int_{T} p(x) d x=\frac{|T|}{2 d-1} \sum_{E \in E_{T}} p\left(m_{E}\right) .
$$

We obtain

$$
\begin{align*}
-\int_{\Omega} q_{h} \operatorname{div} \mathbf{w}_{h} d x & =\int_{\Omega} \nabla q_{h} \cdot \mathbf{w}_{h} d x=\sum_{T \in \mathcal{T}_{h}}\left(\nabla q_{h}\right)_{\mid T} \cdot \int_{T} \mathbf{w}_{h} d x \\
& =\sum_{T \in \mathcal{T}_{h}} \frac{|T|}{2 d-1}\left(\nabla q_{h}\right)_{\mid T} \cdot \sum_{E \in E_{T}} \mathbf{w}_{h}\left(m_{E}\right)  \tag{4.5}\\
& =\sum_{T \in \mathcal{T}_{h}} \frac{|T|}{2 d-1} \sum_{E \in E_{T}}\left(\hat{\mathbf{t}}_{E} \cdot \nabla q_{h}\left(m_{E}\right)\right)^{2}
\end{align*}
$$

Using the fact that $\left(\nabla q_{h}\right)_{\mid T}$ is constant and for each $T$ at least two independent nonzero vectors $\hat{\mathbf{t}}_{E}$ exist, one easily checks that

$$
c\left\|\nabla q_{h}\right\|_{L^{2}(T)}^{2} \leq|T| \sum_{E \in E_{T}}\left(\hat{\mathbf{t}}_{E} \cdot \nabla q_{h}\left(m_{E}\right)\right)^{2} \leq \tilde{c}\left\|\nabla q_{h}\right\|_{L^{2}(T)}^{2}, \quad c>0 .
$$

Combining this with (4.5) we get

$$
\begin{equation*}
-\int_{\Omega} q_{h} \operatorname{div} \mathbf{w}_{h} d x \geq C \sum_{T \in \mathcal{T}_{h}}\left\|\nabla q_{h}\right\|_{L^{2}(T)}^{2}=C\left\|\nabla q_{h}\right\|_{L^{2}}^{2} \tag{4.6}
\end{equation*}
$$

Let $E_{\hat{T}}$ be the set of edges of the unit $d$-simplex. In the space $\left\{\hat{v} \in P_{2} \mid \hat{v}\right.$ is zero at the vertices of $\hat{T}\}$ the norms $\|\hat{v}\|_{L^{2}(\hat{T})}$ and $\left(\sum_{E \in E_{\hat{T}}} \hat{v}\left(m_{E}\right)^{2}\right)^{\frac{1}{2}}$ are equivalent. Using this componentwise for the vector-function $\hat{\mathbf{w}}_{h}:=\mathbf{w}_{h} \circ F$, with $F$ the affine mapping such that $F(\hat{T})=T$, we get

$$
\begin{aligned}
\left\|\mathbf{w}_{h}\right\|_{L^{2}(T)}^{2} & \leq C|T|\left\|\hat{\mathbf{w}}_{h}\right\|_{L^{2}(\hat{T})}^{2} \\
& \leq C|T| \sum_{E \in E_{\hat{T}}}\left\|\hat{\mathbf{w}}_{h}\left(m_{E}\right)\right\|_{2}^{2}=C|T| \sum_{E \in E_{T}}\left\|\mathbf{w}_{E}\right\|_{2}^{2} .
\end{aligned}
$$

Summation over all simplices $T$ yields

$$
\begin{align*}
\left\|\mathbf{w}_{h}\right\|_{L^{2}}^{2} & \leq C \sum_{T \in \mathcal{T}_{h}}|T| \sum_{E \in E_{T}}\left\|\mathbf{w}_{E}\right\|_{2}^{2}=C \sum_{T \in \mathcal{T}_{h}}|T| \sum_{E \in E_{T}}\left(\hat{\mathbf{t}}_{E} \cdot \nabla q_{h}\left(m_{E}\right)\right)^{2} \\
& \leq C \sum_{T \in \mathcal{T}_{h}}\left\|\nabla q_{h}\right\|_{L^{2}(T)}^{2}=C\left\|\nabla q_{h}\right\|_{L^{2}}^{2} . \tag{4.7}
\end{align*}
$$

From (4.6) and (4.7) we obtain

$$
\frac{\left(q_{h}, \operatorname{div}\left(-\mathbf{w}_{h}\right)\right)_{L^{2}}}{\left\|\mathbf{w}_{h}\right\|_{L^{2}}} \geq C\left\|\nabla q_{h}\right\|_{L^{2}}
$$

with a constant $C>0$ independent of $q_{h}$ and of $\mathcal{T}_{h} \in\left\{\mathcal{T}_{h}\right\}$.

## 5 A Stokes interface problem

In this section we consider a generalized Stokes interface problem. Assume bounded Lipschitz subdomains $\Omega_{1}$ and $\Omega_{2}$ of $\Omega$ such that $\bar{\Omega}=\bar{\Omega}_{1} \cup \bar{\Omega}_{2}$, $\Omega_{1} \cap \Omega_{2}=\emptyset$. The interface between the subdomains is denoted by $\Gamma=\partial \Omega_{1} \cap$ $\partial \Omega_{2}$. The problem we consider is as follows: Find $\mathbf{u}$ and $p$ such that

$$
\begin{align*}
-\operatorname{div}(v(\mathbf{x}) \mathrm{Du})+\tau \rho(\mathbf{x}) \mathbf{u}+\nabla p & =\mathbf{f} & & \text { in } \Omega_{k},  \tag{5.1}\\
\operatorname{div} \mathbf{u} & =0 & & \text { in } \Omega_{k}, \quad k=1,2 .  \tag{5.2}\\
{[\mathbf{u}]=0,[\sigma(\mathbf{u}, p) \mathbf{n}] } & =\mathbf{g} & & \text { on } \Gamma,  \tag{5.3}\\
\mathbf{u} & =0 & & \text { on } \partial \Omega . \tag{5.4}
\end{align*}
$$

In this formulation we use standard notations: $\sigma(\mathbf{u}, p)=-p I+2 \nu \mathrm{Du}$ is the stress tensor, $\mathrm{Du}=\frac{1}{2}\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{T}\right)$ the rate of deformation tensor, $\mathbf{n}$ is a unit normal vector to $\Gamma,\left.[a]\right|_{\Gamma}=\left.\left(\left.a\right|_{\Omega_{1}}-\left.a\right|_{\Omega_{2}}\right)\right|_{\Gamma}$.

We assume piecewise constant viscosity and density. A localized force term $g$ occurs, for example, in models that take surface tension effects into account, cf. [12,22]. Suitable scaling can be used to ensure that viscosity and density are equal to one in $\Omega_{1}$. Hence, we assume

$$
v=\left\{\begin{array}{ll}
1 & \text { in } \Omega_{1}  \tag{5.5}\\
v_{2}>0 & \text { in } \Omega_{2}
\end{array}, \quad \rho=\left\{\begin{array}{ll}
1 & \text { in } \Omega_{1} \\
\rho_{2}>0 & \text { in } \Omega_{2}
\end{array} .\right.\right.
$$

The weak formulation leads to a saddle point problem as in (2.7), (2.8): find $\mathbf{u}, p \in \mathbf{H}_{0}^{1}(\Omega) \times L_{0}^{2}(\Omega)$ such that

$$
\begin{equation*}
a_{v}(\mathbf{u}, \mathbf{v})+\tau c_{\rho}(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p)+b(\mathbf{u}, q)=f(\mathbf{v}) \quad \text { for all } \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega), q \in L_{0}^{2}(\Omega), \tag{5.6}
\end{equation*}
$$

with

$$
\begin{aligned}
& a_{v}(\mathbf{u}, \mathbf{v}):=\int_{\Omega} v \operatorname{tr}(\mathrm{DuDv}) \mathrm{d} x, \quad c_{\rho}(\mathbf{u}, \mathbf{v}):=(\rho \mathbf{u}, \mathbf{v})_{L^{2}}, \quad b(\mathbf{v}, p):=-(p, \operatorname{div} \mathbf{v})_{L^{2}}, \\
& \langle f, v\rangle:=(\mathbf{f}, \mathbf{v})_{L^{2}}+\int_{\Gamma} \mathbf{g} \cdot \mathbf{v} \mathrm{d} s .
\end{aligned}
$$

In view of the general analysis it is natural to introduce the following Hilbert spaces:

$$
\begin{aligned}
& H_{1}=\left\{\mathbf{H}_{0}^{1}(\Omega), \text { with }\|\mathbf{v}\|_{H_{1}}^{2}:=\int_{\Omega} v \operatorname{tr}\left((\mathrm{D} \mathbf{v})^{2}\right) \mathrm{d} x\right\} \\
& H_{2}=\left\{\mathbf{L}^{2}(\Omega), \text { with }\|\mathbf{v}\|_{H_{2}}:=\left\|\rho^{\frac{1}{2}} \mathbf{v}\right\|_{L^{2}}\right\} .
\end{aligned}
$$

Due to Korn's inequality $\|\cdot\|_{H_{1}}$ defines a norm on $\mathbf{H}_{0}^{1}(\Omega)$. Related to this norm we need a uniform (w.r.t. v) equivalence result that is proved in [17], Lemma 6.1.

This result is as follows. Assume that one of the following conditions is satisfied:

$$
\begin{array}{ll}
\operatorname{meas}\left(\partial \Omega_{k} \cap \partial \Omega\right)>0 & \text { for } k=1,2 \\
\operatorname{meas}\left(\partial \Omega_{1} \cap \partial \Omega\right)>0 & \text { and } \nu_{2} \leq C . \tag{5.8}
\end{array}
$$

Then there exists a constant $\tilde{c}>0$ independent of $v$ such that

$$
\begin{equation*}
\tilde{c}\left\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L^{2}} \leq\|\mathbf{v}\|_{H_{1}} \leq\left\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L^{2}} \quad \text { for all } \mathbf{v} \in H_{1} \tag{5.9}
\end{equation*}
$$

Before we introduce the (pressure) space $M$ we recall a result from [18] that we need in the analysis below. Let $\tilde{p}$ be the piecewise constant function

$$
\tilde{p}=\left\{\begin{align*}
\left|\Omega_{1}\right|^{-1} & \text { on } \Omega_{1},  \tag{5.10}\\
-\left|\Omega_{2}\right|^{-1} & \text { on } \Omega_{2} .
\end{align*}\right.
$$

Since $(\tilde{p}, 1)_{L^{2}}=0$, we have $\tilde{p} \in L_{0}^{2}(\Omega)$. Consider the one-dimensional subspace $P_{0}:=\operatorname{span}\{\tilde{p}\}$ of $L_{0}^{2}(\Omega)$ and an $L^{2}$-orthogonal decomposition $L_{0}^{2}(\Omega)=P_{0} \oplus P_{0}^{\perp}$. For $p \in L_{0}^{2}(\Omega)$ we use the notation

$$
\begin{equation*}
p=p_{0}+p_{0}^{\perp}, \quad p_{0} \in P_{0}, p_{0}^{\perp} \in P_{0}^{\perp} . \tag{5.11}
\end{equation*}
$$

One easily checks that

$$
\begin{equation*}
P_{0}^{\perp}=\left\{p \in L_{0}^{2}(\Omega) \mid(p, 1)_{L^{2}\left(\Omega_{1}\right)}=(p, 1)_{L^{2}\left(\Omega_{2}\right)}=0\right\} \tag{5.12}
\end{equation*}
$$

Using this splitting we can define an appropriate norm on $L_{0}^{2}(\Omega)$ :

$$
M=\left\{L_{0}^{2}(\Omega), \text { with }\|p\|_{M}:=\left(\left\|p_{0}\right\|_{L^{2}}^{2}+\left\|\nu^{-\frac{1}{2}} p_{0}^{\perp}\right\|_{L^{2}}^{2}\right)^{\frac{1}{2}}\right\} .
$$

The scalar product corresponding to $\|\cdot\|_{M}$ is denoted by $(\cdot, \cdot)_{M}$. In [18], Lemma 1 and Theorem 1, the following results are proved. There exist constants $\hat{C}$ and $\hat{c}>0$ independent of $v$ such that

$$
\begin{align*}
\left|(\operatorname{div} \mathbf{v}, p)_{L^{2}}\right| & \leq \hat{C}\left\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L^{2}}\|p\|_{M} \quad \text { for all } \mathbf{v} \in H_{1}, p \in M,  \tag{5.13}\\
\sup _{\mathbf{v} \in \mathbf{H}_{0}^{1}} \frac{(\operatorname{div} \mathbf{v}, p)_{L^{2}}}{\left\|v^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L_{2}}} & \geq \hat{c}\|p\|_{M} \quad \text { for all } p \in M . \tag{5.14}
\end{align*}
$$

We identify $L^{2}(\Omega)$ with its dual and then have

$$
\begin{aligned}
& H_{1}^{\prime}=\left\{\mathbf{H}^{-1}, \text { with }\|\mathbf{f}\|_{H_{1}^{\prime}}=\sup _{\mathbf{v} \in \mathbf{H}_{0}^{1}} \frac{\langle\mathbf{f}, \mathbf{v}\rangle}{\|\mathbf{v}\|_{H_{1}}}\right\} \\
& H_{2}^{\prime}=\left\{\mathbf{L}^{2}(\Omega), \text { with }\|\mathbf{v}\|_{H_{2}^{\prime}}:=\left\|\rho^{-\frac{1}{2}} \mathbf{v}\right\|_{L^{2}}\right\} \\
& M^{\prime}=\left\{L_{0}^{2}(\Omega), \text { with }\|p\|_{M^{\prime}}:=\left(\left\|p_{0}\right\|_{L^{2}}^{2}+\left\|v^{\frac{1}{2}} p_{0}^{\perp}\right\|_{L^{2}}^{2}\right)^{\frac{1}{2}}\right\} .
\end{aligned}
$$

With these spaces $H_{1}$ and $M$ the properties (2.1)-(2.3) can be shown to hold.
Lemma 5.1 Assume that (5.7) or (5.8) is satisfied. The properties (2.1)-(2.3) hold with $\gamma_{a}=\Gamma_{a}=\gamma_{c}=\Gamma_{c}=1$ and constants $\Gamma_{b}, \gamma_{b}>0$ independent of $v$ and $\rho$.
Proof From $a_{v}(\mathbf{u}, \mathbf{u})=\|\mathbf{u}\|_{H_{1}}^{2}$ we get property (2.1) with $\gamma_{a}=\Gamma_{a}=1$. Due to $c_{\rho}(\mathbf{u}, \mathbf{u})=\|\mathbf{u}\|_{H_{2}}^{2}$ property (2.2) holds. Using (5.9) and (5.13), we get

$$
(\operatorname{div} \mathbf{v}, p)_{L^{2}} \leq \hat{C}\left\|\nu^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L^{2}}\|p\|_{M} \leq \hat{C} \tilde{c}^{-1}\|\mathbf{v}\|_{H_{1}}\|p\|_{M}
$$

and thus the upper bound in (2.3) with $\Gamma_{b}=\hat{C} \tilde{c}^{-1}$. Using (5.9) and (5.14), we obtain

$$
\sup _{\mathbf{v} \in H_{1}} \frac{(\operatorname{div} \mathbf{v}, p)_{L^{2}}}{\|\mathbf{v}\|_{H_{1}}} \geq \sup _{\mathbf{v} \in H_{1}} \frac{(\operatorname{div} \mathbf{v}, p)_{L^{2}}}{\left\|v^{\frac{1}{2}} \nabla \mathbf{v}\right\|_{L_{2}}} \geq \hat{c}\|p\|_{M}
$$

and thus the lower bound in (2.3). Because $\rho$ is not used in the definitions of $H_{1}$ and $M$ the constants $\Gamma_{b}$ and $\gamma_{b}$ are independent of $\rho$.

The norm $\|\cdot\|_{H_{2}}$ is equivalent to the standard $L^{2}$-norm. Hence, the space $W=\left\{p \in M \mid B p \in H_{2}^{\prime}\right\}$ is the same as the one for the generalized Stokes problem in Sect. 3:

$$
\begin{equation*}
W=H^{1}(\Omega) \cap L_{0}^{2}(\Omega), \quad \text { with norm }\|p\|_{W}=\sup _{\mathbf{v} \in H_{2}} \frac{\langle B p, \mathbf{v}\rangle}{\|\mathbf{v}\|_{H_{2}}}=\left\|\rho^{-\frac{1}{2}} \nabla p\right\|_{L^{2}} \tag{5.15}
\end{equation*}
$$

The Schur complement $S_{v, \rho}: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$ corresponding to this Stokes interface problem is characterized by

$$
\begin{equation*}
\left(S_{v, \rho} p, p\right)_{L^{2}}^{\frac{1}{2}}=\sup _{\mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})_{L^{2}}}{\left(a_{v}(\mathbf{v}, \mathbf{v})+\tau c_{\rho}(\mathbf{v}, \mathbf{v})\right)^{\frac{1}{2}}} \tag{5.16}
\end{equation*}
$$

We take the preconditioner from Theorem 2.5:

$$
\begin{equation*}
\left(\tilde{S}_{v, \rho} p, p\right)_{L^{2}}^{\frac{1}{2}}=\|p\|_{M+\tau^{-1} W}=\inf _{q \in W}\left(\|p-q\|_{M}+\tau^{-1}\left\|\rho^{-\frac{1}{2}} \nabla q\right\|_{L^{2}}^{2}\right)^{\frac{1}{2}} \tag{5.17}
\end{equation*}
$$

This preconditioner can be characterized using a Neumann solution operator by applying a similar approach as in Theorem 3.2. We can apply the general analysis of Sect. 2.4 to derive a uniform spectral bound $S \lesssim S$. This is summarized in the following theorem.
Theorem 5.2 Assume that one of the conditions (5.7) or (5.8) is satisfied. Denote by $-\Delta_{\rho}^{-1}: L_{0}^{2}(\Omega) \rightarrow H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ the solution operator of the following Neumann interface problem: Given $f \in L_{0}^{2}(\Omega)$ find $p \in H^{1}(\Omega) \cap L_{0}^{2}(\Omega)$ such that

$$
\left(\rho^{-1} \nabla p, \nabla q\right)_{L^{2}}=(f, q)_{L^{2}} \quad \forall q \in H^{1}(\Omega) \cap L_{0}^{2}(\Omega)
$$

$I_{\nu}: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$ is defined by $\left(I_{\nu} p, q\right)_{L^{2}}=(p, q)_{M}$ for all $p, q \in L_{0}^{2}(\Omega)$. Then

$$
\tilde{S}_{v, \rho}^{-1}=I_{v}^{-1}-\tau \Delta_{\rho}^{-1}
$$

holds and or all $p \in L_{0}^{2}(\Omega)$

$$
\left(S_{v, \rho} p, p\right)_{L^{2}} \leq c\left(\tilde{S}_{v, \rho} p, p\right)_{L^{2}}
$$

with a constant $c$ independent of $\tau, \nu$ and $\rho$.
Proof For $W$ as in (5.15) define $\tilde{S}$ as in (5.17). Using the same arguments as in the proof of Theorem 3.2 it can be shown that $\tilde{S}^{-1}=I_{\nu}^{-1}-\tau \Delta_{\rho}^{-1}$ holds. The spectral inequality $S \leq c \tilde{S}$ follows from Theorem 2.6. The constant $c$ is uniform not only in $\tau$ but also w.r.t. $v$ and $\rho$, due to the fact that $\Gamma_{a}, \Gamma_{c}, \gamma_{a}$ and $\gamma_{c}$ are independent of $\nu$ and $\rho$.
Remark 6 We comment on a discrete version of the preconditioner $\tilde{S}_{v, \rho}$ in a pair of finite element spaces $\mathbf{V}_{h} \times M_{h}$ as considered in Sect. 4.1. In the operator $I_{v}$ the scalar product $(\cdot, \cdot)_{M}$ and thus the orthogonal projection on the onedimensional subspace $P_{0}=\operatorname{span}(\tilde{p})$ is used. This projection is avoided in the following operator $\hat{I}_{\nu}: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$

$$
\left(\hat{I}_{\nu} p, q\right)_{L^{2}}=\left(v^{-1} p, q\right)_{L^{2}} \quad \text { for all } q \in L_{0}^{2}(\Omega)
$$

Note that $\hat{I}_{\nu} p=I_{\nu} p$ for all $p \in L_{0}^{2}(\Omega) \cap P_{0}^{\perp}$. In general a poor behaviour of a preconditioner on a one-dimensional subspace is harmless if the preconditioner is combined with a CG method. Therefore we base our discrete preconditioner on the simpler operator $\hat{I}_{v}$ instead of on $I_{v}$. Let $\hat{I}_{v, h}^{-1}: M_{h} \rightarrow M_{h}, \hat{I}_{v, h}^{-1} g_{h}=p_{h}$ be such that:

$$
\left(v^{-1} p_{h}, q_{h}\right)_{L^{2}}=\left(g_{h}, q_{h}\right)_{L^{2}} \quad \text { for all } q_{h} \in M_{h}
$$

Let $N_{\rho, h}^{-1}: M_{h} \rightarrow M_{h}, N_{\rho, h}^{-1} g_{h}=p_{h}$ be the solution operator of the following discrete Neumann problem:

$$
\left(\rho^{-1} \nabla p_{h}, \nabla q_{h}\right)_{L^{2}}=\left(g_{h}, q_{h}\right)_{L^{2}} \quad \text { for all } q_{h} \in M_{h}
$$

We define $\tilde{S}_{v, \rho, h}: M_{h} \rightarrow M_{h}$ by

$$
\tilde{S}_{v, \rho, h}^{-1}:=\hat{I}_{v, h}^{-1}+\tau N_{\rho, h}^{-1} .
$$

This preconditioner is used in our numerical experiments. To evaluate $\tilde{S}_{v, \rho, h}^{-1} g_{h}$ one has to solve a system with a pressure mass matrix, w.r.t. the scalar product $\left(v^{-1} \cdot, \cdot\right)_{L^{2}}$, and a discrete Neumann interface problem.

Note that in Theorem 5.2 we have a spectral inequality $S_{v, \rho} \lesssim \tilde{S}_{v, \rho}$ that is uniform with respect to both the parameter $\tau$ and the jumps in the coefficients $\underset{\sim}{\nu}, \rho$ without using any regularity assumptions. To derive a spectral inequality $\tilde{S}_{v, \rho} \lesssim S_{v, \rho}$ we need (at least in our analysis) regularity results for a stationary Stokes interface problem of the form

$$
\begin{align*}
-\operatorname{div}(v(\mathbf{x}) \mathrm{Du})+\nabla p & =\mathbf{f} & & \text { in } \Omega_{k},  \tag{5.18}\\
\operatorname{div} \mathbf{u} & =0 & & \text { in } \Omega_{k}, \quad k=1,2 .  \tag{5.19}\\
{[\mathbf{u}]=0,[\sigma(\mathbf{u}, p) \cdot \mathbf{n}] } & =\mathbf{g} & & \text { on } \Gamma,  \tag{5.20}\\
\mathbf{u} & =0 & & \text { on } \Omega . \tag{5.21}
\end{align*}
$$

Similarly to the Stokes case in Sect. 3, verifying Assumption 1 is based on regularity properties of the solution of this problem. This important issue is largely unsolved. The following result is found in the literature (see, [20]): If the interface $\Gamma=\partial \Omega_{1} \cap \partial \Omega_{2}$ is sufficiently smooth and has no common points with $\partial \Omega$ and $\mathbf{f} \in \mathbf{L}^{2}$ then a solution $\mathbf{u}, p$ of (5.18)-(5.21) belongs to $H^{2}\left(\Omega_{k}\right)^{d} \times H^{1}\left(\Omega_{k}\right)$, $k=1,2$. However, in these results and in other analyses known in the literature the dependence of constants in a priori estimates on $v$ is not known. Due to this we are not able to prove a result $\tilde{S}_{v, \rho} \lesssim S_{v, \rho}$ that is uniform both with respect to $\tau$ and the jumps in $v, \rho$. Below we present an analysis where the spectral inequality is uniform with respect to $\tau$ only.

Theorem 5.3 Assume that one of the conditions (5.7) or (5.8) is satisfied and that the domain $\Omega \subset \mathbb{R}^{d}$ is such that the Stokes problem (3.1) is $H^{2}$-regular. Let $\tilde{S}_{v, \rho}$ be the preconditioner from (5.17). There exists a constant $c>0$ independent of $\tau$ such that for all $p \in L_{0}^{2}(\Omega)$

$$
c\left(\tilde{S}_{v, \rho} p, p\right)_{L^{2}} \leq\left(S_{v, \rho} p, p\right)_{L^{2}}
$$

holds.
Proof Let $S: L_{0}^{2}(\Omega) \rightarrow L_{0}^{2}(\Omega)$ as in (3.3) be the Schur complement for the generalized Stokes problem and Let $\tilde{S}$ be the preconditioner from theorem 3.2. For this preconditioner we have

$$
(\tilde{S} p, p)_{L^{2}}^{\frac{1}{2}}=\inf _{q \in H^{1}(\Omega) \cap L_{0}^{2}(\Omega)}\left(\|p-q\|_{L^{2}}^{2}+\tau^{-1}\|\nabla q\|_{L^{2}}^{2}\right)^{\frac{1}{2}}
$$

$\operatorname{Using}$ (5.17) and the norm equivalences $\|\cdot\|_{M} \sim\|\cdot\|_{L^{2}}\left(\right.$ on $\left.L_{0}^{2}(\Omega)\right),\left\|\rho^{-\frac{1}{2}} \nabla \cdot\right\|_{L^{2}} \sim$ $\|\nabla \cdot\|_{L^{2}}\left(\right.$ on $\left.H^{1}(\Omega) \cap L_{0}^{2}(\Omega)\right)$ it follows that there exists a constant $c$ independent of $\tau$ such that

$$
\begin{equation*}
\left(\tilde{S}_{v, \rho} p, p\right)_{L^{2}} \leq c(\tilde{S} p, p)_{L^{2}} \quad \text { for all } p \in L_{0}^{2}(\Omega) \tag{5.22}
\end{equation*}
$$

From Theorem 3.2 it follows that

$$
\begin{equation*}
(\tilde{S} p, p)_{L^{2}} \leq c(S p, p)_{L^{2}} \quad \text { for all } p \in L_{0}^{2}(\Omega) \tag{5.23}
\end{equation*}
$$

holds with a constant $c$ independent of $\tau$. From

$$
(S p, p)_{L^{2}}=\sup _{\mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})_{L^{2}}}{\left(\|\nabla \mathbf{v}\|_{L^{2}}^{2}+\tau\|\mathbf{v}\|_{L^{2}}^{2}\right)^{\frac{1}{2}}}
$$

the representation for $S_{v, \rho}$ in (5.16) and the equivalences $a_{v}(\cdot, \cdot) \sim\|\nabla \cdot\|_{L^{2}}^{2}$, $c_{\rho}(\cdot, \cdot) \sim\|\mathbf{v}\|_{L^{2}}^{2}\left(\right.$ on $\left.\mathbf{H}_{0}^{1}(\Omega)\right)$ it follows that

$$
\begin{equation*}
(S p, p)_{L^{2}} \leq c\left(S_{v, \rho} p, p\right)_{L^{2}} \quad \text { for all } p \in L_{0}^{2}(\Omega) \tag{5.24}
\end{equation*}
$$

Combination of the results in (5.22), (5.23) and (5.24) completes the proof.

## 6 Numerical experiments

We present results for a model generalized Stokes interface problem. Numerical results for a stationary $(\tau=0)$ Stokes interface problem can be found in [17].

We take $\Omega=(0,1)^{3}$ with subdomains $\Omega_{2}=\left(0, \frac{1}{2}\right)^{3}, \Omega_{1}=\Omega \backslash \bar{\Omega}_{2}$. The model problem reads: Find $(\mathbf{u}, p) \in \mathbf{H}_{0}^{1}(\Omega) \times L_{0}^{2}(\Omega)$ such that

$$
\begin{aligned}
& \hat{a}_{v}(\mathbf{u}, \mathbf{v})+\tau c_{\rho}(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p)=0 \text { for all } \mathbf{v} \in \mathbf{H}_{0}^{1}(\Omega) \\
& b(\mathbf{u}, q)=0 \\
& \text { for all } q \in L_{0}^{2}(\Omega) .
\end{aligned}
$$

The bilinear forms $b(\cdot, \cdot)$ and $c_{\rho}(\cdot, \cdot)$ are as in Sect. 5, $\hat{a}_{v}(\mathbf{u}, \mathbf{v}):=(\nu \nabla \mathbf{u}, \nabla \mathbf{v})_{L^{2}}$ with $\nu$ and $\rho$ piecewise constants as in (5.5).

For the discretization we start with a uniform tetrahedral grid with $h=\frac{1}{2}$ and apply regular (red) refinement to this triangulation. The resulting grids $\mathcal{T}_{h}$ satisfy a conformity condition:

$$
\exists \mathcal{T}_{h}^{(i)} \subset \mathcal{T}_{h}: \cup\left\{T \mid T \in \mathcal{T}_{h}^{(i)}\right\}=\bar{\Omega}_{i}, \quad i=1,2
$$

We use the LBB-stable $P_{2}-P_{1}$ Hood-Taylor finite element pair, denoted by $\mathbf{V}_{h} \times M_{h}$, and perform computations for $h=\frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and various values of $\nu_{2}$,
$\rho_{2}$ and $\tau$. For $h=\frac{1}{32}$ we have approximately $7.5 \cdot 10^{5}$ velocity unknowns and $3.3 \times 10^{4}$ pressure unknowns.

The following matrices are introduced (notation as in remark 5):

$$
\begin{aligned}
&\left\langle\mathbf{A}_{h} \overline{\mathbf{u}}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2}=\left(v \nabla \mathbf{u}_{h}, \nabla \mathbf{v}_{h}\right)_{L^{2}} \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
&\left\langle\mathbf{C}_{h} \overline{\mathbf{u}}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2}=\left(\rho \mathbf{u}_{h}, \mathbf{v}_{h}\right)_{L^{2}} \text { for all } \mathbf{u}_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
&\left\langle\mathbf{B}_{h} \bar{p}_{h}, \overline{\mathbf{v}}_{h}\right\rangle_{2}=-\left(\operatorname{div} \mathbf{v}_{h}, p_{h}\right)_{L^{2}} \text { for all } p_{h} \in M_{h}, \mathbf{v}_{h} \in \mathbf{V}_{h}, \\
&\left\langle\mathbf{A}_{h}^{\left.\mathrm{N} \bar{p}_{h}, \bar{q}_{h}\right\rangle_{2}}=\left(\rho^{-1} \nabla p_{h}, \nabla q_{h}\right)_{L^{2}} \text { for all } p_{h}, q_{h} \in M_{h},\right. \\
&\left\langle\mathbf{Q}_{h} \bar{p}_{h}, \bar{q}_{h}\right\rangle_{2}=\left(v^{-1} p_{h}, q_{h}\right)_{L^{2}} \quad \text { for all } p_{h}, q_{h} \in M_{h} .
\end{aligned}
$$

The discrete model problem has the following matrix-vector formulation: Find $\overline{\mathbf{u}}_{h} \in \mathbb{R}^{n}, \bar{p}_{h} \in \mathbb{R}^{m}$, such that

$$
\left(\begin{array}{cc}
\mathbf{A}_{h}+\tau \mathbf{C}_{h} \mathbf{B}_{h} \\
\mathbf{B}_{h}^{T} & 0
\end{array}\right)\binom{\overline{\mathbf{u}}_{h}}{\bar{p}_{h}}=\binom{\mathbf{f}_{h}}{0} .
$$

In the experiments we use $\mathbf{f}_{h}=0$ and a fixed starting vector $\left(\overline{\mathbf{u}}_{h}^{(0)}, \bar{p}_{h}^{(0)}\right) \neq(0,0)$. The Schur complement is

$$
\mathbf{S}_{h}=\mathbf{B}_{h}^{T}\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right)^{-1} \mathbf{B}_{h} .
$$

The linear system of equations is solved with the Uzawa method:
(1) Solve $\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right) \overline{\mathbf{z}}=\mathbf{f}_{h}$.
(2) Solve $\mathbf{S}_{h} \bar{p}_{h}=\mathbf{B}_{h}^{T} \overline{\mathbf{z}}$.
(3) Solve $\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right) \overline{\mathbf{u}}_{h}=\mathbf{f}_{h}-\mathbf{B}_{h} \bar{p}_{h}$.

In steps (1)-(3) the equations of the form $\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right) \mathbf{x}=\mathbf{r}$ are all solved with a standard multigrid V-cycle with one pre- and one post-smoothing iteration with a symmetric Gauss-Seidel method. The iteration is stopped as soon as the relative scaled residual satisfies

$$
\begin{equation*}
\frac{\left\|\mathbf{D}^{-1}\left(\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right) \mathbf{x}^{(k)}-\mathbf{r}\right)\right\|}{\left\|\mathbf{D}^{-1}\left(\left(\mathbf{A}+\tau \mathbf{C}_{h}\right) \overline{\mathbf{u}}_{h}^{(0)}-\mathbf{r}\right)\right\|} \leq 10^{-10}, \quad \mathbf{D}:=\operatorname{diag}\left(\mathbf{A}_{h}+\tau \mathbf{C}_{h}\right) . \tag{6.2}
\end{equation*}
$$

Here $\|\cdot\|$ denotes the Euclidean norm. The system in step (2) is solved with a preconditioned conjugate gradient method. The iteration is stopped as soon as the relative preconditioned residual satisfies

$$
\begin{equation*}
\frac{\left\|\mathbf{Q}_{S}^{-1}\left(\mathbf{S}_{h} \bar{p}_{h}^{k}-\mathbf{B}_{h}^{T} \overline{\mathbf{z}}\right)\right\|}{\left\|\mathbf{Q}_{S}^{-1}\left(\mathbf{S}_{h} \bar{p}_{h}^{(0)}-\mathbf{B}_{h}^{T} \overline{\mathbf{z}}\right)\right\|} \leq 10^{-6} \tag{6.3}
\end{equation*}
$$

The preconditioner $\mathbf{Q}_{S}$ is derived from our theoretical analysis as explained in remark 6. We compute $\mathbf{Q}_{S}^{-1} \mathbf{r}$ as follows:
(a) solve $\mathbf{A}_{h}^{\mathrm{N}} \mathbf{a}=\mathbf{r}$,
(b) solve $\mathbf{Q}_{h} \mathbf{b}=\mathbf{r}$,
(c) compute $\mathbf{Q}_{S}^{-1} \mathbf{r}:=\tau \mathbf{a}+\mathbf{b}$.

The linear systems in (a) and (b) are solved up to machine accuracy (using an SSOR-preconditioned conjugate gradient method).

Of course, in practice the Uzawa method is not very attractive because one has to solve the systems with $\mathbf{A}_{h}+\tau \mathbf{C}_{h}$ accurately. In this paper we take the Uzawa method to illustrate the robustness of the multigrid solver and of the preconditioner $\mathbf{Q}_{S}$ for the Schur complement. In practice one would use a blockpreconditioner combined with a MINRES method or an inexact version of the Uzawa method, cf. [19]. The efficiency of such methods is mainly determined by the efficiency of the preconditioners. The quality of the MG preconditioner (for the $\mathbf{A}_{h}+\tau \mathbf{C}_{h}$ systems) and of the $\mathbf{Q}_{s}$ preconditioner for the Schur complement is illustrated in the numerical experiments below.

In the first experiment (Table 1) we take $h=\frac{1}{16}, \tau=h^{-1}$ and vary $\nu_{2}=\nu_{\mid \Omega_{2}}$ and $\rho_{2}=\rho_{\mid \Omega_{2}}$ (recall that $\nu_{\mid \Omega_{1}}=\rho_{\mid \Omega_{1}}=1$ ). We present the average iteration numbers of the solvers in the Uzawa method (6.1). The first row (\#-MG) gives the average number of V-cycle steps for solving the systems with $\mathbf{A}_{h}+\tau \mathbf{C}_{h}$. In the second row (\#-PCG) we give the average iteration number of the preconditioned conjugate gradient solver in step (2) of (6.1).
We repeat the experiment from Table 1 for the case $\nu_{2}=10 \rho_{2}$, but now with $h=\frac{1}{32}$. The results are given in Table 2.

Table 1 Iteration counts for MG and PCG in Uzawa method, with $h=1 / 16$, $\tau=h^{-1}$

| $\nu_{2}=\rho_{2}$ | 1 e 4 | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ | $1 \mathrm{e}-4$ |
| :--- | ---: | ---: | ---: | ---: | ---: |
| \#-MG | 13 | 13 | 13 | 13 | 13 |
| \#-PCG | 23 | 23 | 20 | 24 | 22 |
| $\nu_{2}=\frac{1}{10} \rho_{2}, \rho_{2}$ | 1 e 4 | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ | $1 \mathrm{e}-4$ |
| \#-MG | 13 | 13 | 13 | 13 | 13 |
| \#-PCG | 22 | 22 | 21 | 22 | 22 |
| $\nu_{2}=10 \rho_{2}, \rho_{2}$ | 1 e 4 | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ | $1 \mathrm{e}-4$ |
| \#-MG | 13 | 13 | 13 | 13 | 13 |
| \#-PCG | 23 | 24 | 23 | 22 | 23 |

Table 2 Iteration counts for MG and PCG in Uzawa method, with $h=1 / 32$, $\tau=h^{-1}$

| $\nu_{2}=10 \rho_{2}, \rho_{2}$ | 1 e 4 | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ | $1 \mathrm{e}-4$ |
| :--- | ---: | ---: | ---: | ---: | ---: |
| \#-MG | 13 | 13 | 13 | 13 | 13 |
| \#-PCG | 24 | 24 | 23 | 21 | 22 |

Table 3 Iteration counts for MG and PCG in Uzawa method, with $h=1 / 16$, $\nu_{2}=\frac{1}{10} \rho_{2}$

| $\rho_{2}=10, \tau$ | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ |
| :--- | ---: | ---: | ---: |
| \#-MG | 12 | 13 | 13 |
| \#-PCG | 20 | 20 | 20 |
| $\rho_{2}=\frac{1}{10}, \tau$ | 1 e 2 | 1 e 0 | $1 \mathrm{e}-2$ |
| \#-MG | 12 | 13 | 13 |
| \#-PCG | 23 | 24 | 24 |

Table 4 Iteration counts for MG and PCG in Uzawa method, with $\nu_{2}=1 \mathrm{e}-6$, $\rho_{2}=1 \mathrm{e}+4, \tau=10$

| $h$ | $1 / 8$ | $1 / 16$ | $1 / 32$ |
| :--- | ---: | ---: | ---: |
| \#-MG | 14 | 14 | 14 |
| \#-PCG | 125 | 211 | 324 |

In Table 3 we present results for different $\tau$ values, with $h=\frac{1}{16}, \nu_{2}=\frac{1}{10} \rho_{2}$.
In all these experiments we observe a clear robustness both of the MG and the PCG method in large parameter ranges. We observe robustness of the PCG method with respect to the jumps in $v$ and $\rho$ across the interface, too. Note, however, that the analysis in Sect. 5 does not yield such a robustness result. We observed in numerical experiments, that if we take very large jumps in opposite directions in $\nu$ and $\rho$ (which is not likely to occur in realistic two-phase problems) then the Schur complement preconditioner turns out to become (much) less efficient, whereas the muligrid method remains robust. Results of one such an experiment are given in Table 4.
These results motivate a further theoretical analysis of the Schur complement preconditioner with respect to jumps in the coefficients.

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