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On the Stokes problem with model boundary conditions

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Abstract. For a wide class of two- and three-dimensional domains, boundary conditions on the velocity vector field for the Stokes problem are indicated ensuring that the corresponding Schur complement is the identity operator. These boundary conditions make it possible to 'decouple' the Stokes problem into two separate problems, for pressure and for velocity. The solubility of the problem and the regularity of its solutions are studied and the connections between the results obtained and certain aspects of numerical methods in hydrodynamics (such as the LBB condition and the numerical solution of the generalized Stokes problem) are considered.

Bibliography: 23 titles.

Introduction

We consider the following system of partial differential equations with respect to the vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ and the function p in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$:

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \\ \operatorname{div}\mathbf{u} &= g. \end{aligned} \tag{1}$$

The system (1) is called the Stokes system, and its numerical solution is one of the central problems in computational hydrodynamics. To obtain a closed system, equations (1) must be supplemented with certain boundary conditions. Most common in theoretical and applied studies is the Dirichlet boundary condition for the velocity function $\mathbf{u}$, which we assume to be homogeneous:

$$\mathbf{u}\big|_{\partial\Omega} = 0. \tag{2}$$

A detailed analysis of the problems concerning the solubility of the system (1), (2) and the regularity of its solutions can be found, for instance, in [1].

By the *Schur complement* (*Schur operator*) of the system (1), (2) we mean the operator

$$\mathbf{A}_0 \equiv \operatorname{div}\Delta_0^{-1}\nabla.$$

(Here we denote by $\Delta_0^{-1}\mathbf{f}$ the function $\mathbf{u}$ such that $\Delta\mathbf{u} = \mathbf{f}$, $\mathbf{u}|_{\partial\Omega} = 0$.) Under fairly general assumptions on Ω it is a linear bounded positive self-adjoint operator in $L_2(\Omega)/\mathbb{R}$.

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The pressure function p in (1) is defined up to a constant and satisfies the equation

$$\mathbf{A}_0 p = g + \operatorname{div}\Delta_0^{-1}\mathbf{f}. \tag{3}$$

It is remarkable that equation (3) *does not require* additional boundary conditions or smoothness assumptions on the pressure function, unlike the Poisson equation for p, which can be obtained by a formal application of the div operator to the first equation in (1) (see, for instance, [2] and [3]). Equation (3) can be used as a basis in the development of numerical methods for the solution of (1), (2); a knowledge of the spectral (and other) properties of $\mathbf{A}_0$ and (or) its discrete analogues is crucial for the analysis of various finite-difference schemes for the Stokes problem and certain iterative methods of their numerical solution [4]–[6].

It is known from the properties of the Cossera spectrum of the operator of elasticity theory [7] and Lemma 1 in [5] that the spectrum of $\mathbf{A}_0$ lies in a certain interval $[c,1]$, $0<c\leqslant\frac{1}{2}$, and 1 is an isolated eigenvalue of infinite multiplicity. The closeness of $\mathbf{A}_0$ to the identity operator in the finite-difference case has been investigated in [8]. In a certain sense, the reasons for the fact that $\mathbf{A}_0$ is not equal to the identity 'lie at the boundary', as suggested by the following example. We consider a rectangular domain $\Omega=(0,l_1)\times(0,l_2)$ and boundary conditions of the third kind for $\mathbf{u}$:

$$\mathbf{u}\cdot\nu\big|_{\partial\Omega}=\frac{\partial\mathbf{u}\cdot\tau}{\partial\nu}\bigg|_{\partial\Omega}=0, \tag{4}$$

where ν and τ are the normal and the tangent vectors to $\partial\Omega$. The Schur complement of the system (1), (4) is the identity operator on $L_2(\Omega)/\mathbb{R}$. In fact, this can be verified in a straightforward way for an arbitrary (finite) trigonometric polynomial

$$\sum_{\substack{k,j=0\\k+j>0}}^{M} a_{k,j}\cos k\pi l_1^{-1}x\,\cos j\pi l_2^{-1}y$$

in $L_2(\Omega)/\mathbb{R}$. For the entire space $L_2(\Omega)/\mathbb{R}$ this follows from the density of trigonometric polynomials in $L_2(\Omega)/\mathbb{R}$, since the Schur complement of the system (1), (4) is linear and bounded.

It is well known that the Schur complement of system (1) with boundary conditions (4) in a rectangular domain is equal to the identity operator in the finite-difference case, provided that the finite difference scheme is suitably chosen (see [8]). Because of this and since the problem can be explicitly solved in terms of Fourier series, the boundary conditions (4) have important applications in the analysis of finite-difference schemes and the construction of efficient iterative methods for problems with Dirichlet boundary conditions [8]–[14].

The following problem has long remained unsolved: is it possible to generalize boundary conditions (4) to a wider class of domains so that the Schur complement remains equal to the identity? In this paper we give an affirmative answer to this question for arbitrary bounded simply connected Lipschitz domains. The corresponding boundary conditions are as follows:

$$\begin{aligned}
&\mathbf{u}\cdot\nu\big|_{\partial\Omega}=0, \quad \operatorname{curl}\mathbf{u}\big|_{\partial\Omega}=0 && \text{for } n=2;\\
&\mathbf{u}\cdot\nu\big|_{\partial\Omega}=0, \quad \operatorname{curl}\mathbf{u}\times\nu\big|_{\partial\Omega}=0 && \text{for } n=3.
\end{aligned}$$

Here and in what follows, $\operatorname{curl}\mathbf{u} = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ for $n = 2$; we denote by $a \cdot b$ and $a \times b$ the scalar and the vector products of vectors a and b, respectively.

The boundary conditions under consideration are discussed in greater detail in § 2. Meanwhile, we note that vanishing of the normal component of the velocity is an essential boundary condition, which we build into the definition of the function space that must contain the generalized solution. By contrast, the conditions on the curl of the velocity are natural conditions in the generalized formulation of the problem considered in § 2. However, under additional assumptions on the regularity of the data the generalized solution turns out to be more regular and the conditions on $\operatorname{curl}\mathbf{u}$ are satisfied in the usual sense.

In our work we impose fairly weak conditions on the smoothness of the boundary of the domain $\partial\Omega$ and we pay for it by seeking the generalized solution $\mathbf{u}$ in a wider space than $W_2^1(\Omega)^n$, namely, in the space of vector-valued functions in $L_2(\Omega)^n$ with divergence and curl in $L_2(\Omega)^r$, $r = r(n)$. Such functions belong to $W_2^1(\Omega)^n$ only locally. However, certain additional conditions on the smoothness of the boundary enable us to consider a generalized formulation of the problem in the space of vector-valued functions in $W_2^1(\Omega)^n$ with zero normal component on $\partial\Omega$ and ensure that the generalized solution belongs to this space. In the present paper these conditions are as follows: either $\partial\Omega \in C^2$, or Ω is a convex polygon ($n = 2$), or it is a convex polytope ($n = 3$).

We present the relevant definitions and auxiliary statements in § 1. In § 2 we introduce certain model boundary conditions and prove the solubility in the generalized sense of the corresponding system with vector Laplace operator and of the Stokes system; we also prove the main theorem on the Schur complement. In § 3 we prove certain regularity results for the solution of the Stokes problem with model boundary conditions. Several consequences of the theorem on the Schur complement are presented in § 4. Generalizations for the Stokes problem with a parameter are given in § 5.

§ 1. Main definitions and auxiliary results

We consider a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary. In certain cases we shall assume that Ω satisfies the following condition:

(I) *either the boundary $\partial\Omega$ belongs to the class C^2, or the domain is a convex polygon ($n = 2$), or it is a convex polytope ($n = 3$).*

The outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ can be defined almost everywhere on $\partial\Omega$. If $n = 2$, then we denote by $\tau = (-\nu_2, \nu_1)$ the tangent vector to $\partial\Omega$. If $n = 3$, then at those points $P \in \partial\Omega$ where ν is defined we consider the tangent hyperplane with linearly independent system $\{\tau_i(P)\}$, $i = 1, 2$, of vectors tangent to $\partial\Omega$. We denote by τ_i, $i = 1, 2$, the vector field on $\partial\Omega$ such that $\tau_i = \tau_i(P)$ for all points $P \in \partial\Omega$ at which ν is defined.

We shall use the following function spaces:

$$
\begin{aligned}
L_2/\mathbb{R} &\equiv \left\{ q \in L_2(\Omega) : \int_\Omega q \, d\Omega = 0 \right\}, \\
\mathbf{H}_0 &\equiv W^1_{2,0}(\Omega)^n, \\
\mathbf{H}_\nu &\equiv \left\{ \mathbf{u} = (u_1, \dots, u_n) \in W^1_2(\Omega)^n : \mathbf{u} \cdot \nu = 0 \ \partial\Omega \right\}, \\
\mathbf{R}_\nu &\equiv \left\{ \mathbf{u} \in \mathbf{H}_\nu : \operatorname{curl} \mathbf{u} = 0 \right\}.
\end{aligned}
$$

We set

$$
(\mathbf{u}, \mathbf{v})_1 = (\nabla \mathbf{u}, \nabla \mathbf{v})_{L_2}, \quad \|\mathbf{u}\|_1 = (\mathbf{u}, \mathbf{u})_1^{1/2}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0 \cup \mathbf{H}_\nu \cup \mathbf{R}_\nu,
$$

and also use the space

$$
\mathbf{H}(\operatorname{div}) \equiv \left\{ \mathbf{u} \in L_2(\Omega)^n, \ \operatorname{div} \mathbf{u} \in L_2(\Omega) \right\}
$$

with norm

$$
\|\mathbf{u}\|_{\mathbf{H}(\operatorname{div})} = \left(\|\mathbf{u}\|_0^2 + \| \operatorname{div} \mathbf{u}\|_0^2 \right)^{1/2}.
$$

(Here and in what follows $\| \cdot \|_0$ denotes the L_2-norm.) Finally, we shall consider the space

$$
\mathbf{H}(\operatorname{curl}) \equiv \left\{ \mathbf{u} \in L_2(\Omega)^n, \ \operatorname{curl} \mathbf{u} \in L_2(\Omega)^r \right\}
$$

(where $r = 1$ for $n = 2$ and $r = 3$ for $n = 3$) with norm

$$
\|\mathbf{u}\|_{\mathbf{H}(\operatorname{curl})} = \left(\|\mathbf{u}\|_0^2 + \| \operatorname{curl} \mathbf{u}\|_0^2 \right)^{1/2},
$$

and the space $\mathbf{H}_0(\operatorname{div}) \equiv \overline{(C_0^\infty(\Omega)^n)}^{\mathbf{H}(\operatorname{div})}$, which is the closure with respect to the norm in $\mathbf{H}(\operatorname{div})$ of the space $C_0^\infty(\Omega)^n$ of smooth vector-valued functions in Ω with compact support.

We denote by $\mathbf{H}^{1/2}(\partial\Omega)$ the space of traces on $\partial\Omega$ of functions in $W_2^1(\Omega)^n$ with norm

$$
\|\mu\|_{1/2} = \inf_{\substack{\mathbf{v} \in W_2^1(\Omega)^n \\ \mathbf{v} = \mu \text{ on } \partial\Omega}} \|\mathbf{v}\|_{W_2^1}, \qquad \mu \in \mathbf{H}^{1/2}(\partial\Omega);
$$

$\mathbf{H}^{-1/2}(\partial\Omega)$ is the space dual to $\mathbf{H}^{1/2}(\partial\Omega)$.

We use the following statements concerning the spaces defined above.

Lemma 1 [15]. *Let γ_ν be the map $\mathbf{u} \mapsto \mathbf{u} \cdot \nu\big|_{\partial\Omega}$ in $C^\infty(\overline{\Omega})^n$ and let γ_τ be the map $\mathbf{u} \mapsto \mathbf{u} \cdot \tau\big|_{\partial\Omega}$ for $n = 2$ and $\mathbf{u} \mapsto \mathbf{u} \times \nu\big|_{\partial\Omega}$ for $n = 3$. Then these maps can be extended to linear continuous maps from $\mathbf{H}(\operatorname{div})$ into $\mathbf{H}^{-1/2}(\partial\Omega)$ and from $\mathbf{H}(\operatorname{curl})$ into $\mathbf{H}^{-1/2}(\partial\Omega)^r$, respectively, where $r = 1$ for $n = 2$ and $r = 3$ for $n = 3$. Moreover,*

$$
\mathbf{H}_0(\operatorname{div}) = \operatorname{Ker}(\gamma_\nu) = \left\{ \mathbf{u} \in \mathbf{H}(\operatorname{div}) : \mathbf{u} \cdot \nu\big|_{\partial\Omega} = 0 \right\}.
$$

Next, we define the spaces

$$
\mathbf{U} \equiv \mathbf{H}_0(\operatorname{div}) \cap \mathbf{H}(\operatorname{curl}) \quad \text{and} \quad \mathbf{V} \equiv \left\{ \mathbf{u} \in \mathbf{U} : \operatorname{curl} \mathbf{u} = 0 \right\}
$$

with scalar product

$$
(\mathbf{u}, \mathbf{v})_{\mathbf{U}} = (\mathbf{u}, \mathbf{v})_{L_2} + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_{L_2} + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v})_{L_2}, \qquad \mathbf{u}, \mathbf{v} \in \mathbf{U},
$$

and the corresponding norm $\|\mathbf{u}\|_{\mathbf{U}} = (\mathbf{u}, \mathbf{v})_{\mathbf{U}}^{1/2}$. We have the following result.

Lemma 2. *The space* $\mathbf{U}$ *is a Hilbert space and*

$$\|\mathbf{u}\|_{\mathbf{U}} \cong \|\operatorname{div}\mathbf{u}\|_0 + \|\operatorname{curl}\mathbf{u}\|_0, \qquad \mathbf{u}\in\mathbf{U}. \tag{5}$$

Proof. Both $\mathbf{H}_0(\mathrm{div})$ and $\mathbf{H}(\mathrm{curl})$ are Hilbert spaces [15], therefore $\mathbf{U}$ is a Hilbert space. The equivalence of the two norms (5) follows from the estimate $\|\mathbf{u}\|_0 \leqslant c(\Omega)(\|\operatorname{div}\mathbf{u}\|_0 + \|\operatorname{curl}\mathbf{u}\|_0)$, which is valid for $\mathbf{u}\in\mathbf{U}$ ([15], Lemma 3.6).

Here and below we denote by $c(\Omega)$ or c, c_0, c_1 constants that depend only on Ω.

Lemma 3. *The map* $\operatorname{div}\colon \mathbf{V}\to L_2/\mathbb{R}$ *is an isomorphism.*

Proof. Since $\mathbf{V}$ is a subspace of $\mathbf{U}$, it follows by (5) that $\operatorname{Ker}(\operatorname{div}) = \{0\}$. For an arbitrary $\varphi\in L_2/\mathbb{R}$ we now consider the Neumann problem

$$\begin{aligned} \Delta\psi &= \varphi \quad \text{in } \Omega,\\ \frac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

There exists a unique solution of this problem in $W_2^1(\Omega)/\mathbb{R}$, which satisfies the estimate $\|\psi\|_{W_2^1} \leqslant c\|\varphi\|_0$. We now set $\mathbf{u} = \nabla\psi$; then $\mathbf{u}\in\mathbf{V}$, $\operatorname{div}\mathbf{u} = \varphi$, and $\|\mathbf{u}\|_{\mathbf{U}} \leqslant c_1\|\varphi\|_0$, which proves the lemma.

We now denote by $\mathbf{U}^{-1}$ the space of linear bounded functionals on $\mathbf{U}$ with norm

$$\|\mathbf{f}\|_{-1} \equiv \sup_{0\neq\mathbf{v}\in\mathbf{U}} \frac{\langle\mathbf{f},\mathbf{v}\rangle}{\|\mathbf{v}\|_{\mathbf{U}}}, \qquad \mathbf{f}\in\mathbf{U}^{-1},$$

where $\langle\,\cdot\,,\cdot\,\rangle : \mathbf{U}^{-1}\times\mathbf{U}\to\mathbb{R}$ and $\langle\mathbf{f},\mathbf{v}\rangle = (\mathbf{f},\mathbf{v})_{L_2}$ for $\mathbf{f}\in L_2$ and $\mathbf{v}\in\mathbf{U}$.

For $p\in L_2/\mathbb{R}$ we can treat ∇p as an element of $\mathbf{U}^{-1}$. In fact, setting by definition $\langle\nabla p,\mathbf{u}\rangle = -(p,\operatorname{div}\mathbf{u})$ for arbitrary $p\in L_2/\mathbb{R}$ and $\mathbf{u}\in\mathbf{U}$, we obtain $\langle\nabla p,\mathbf{u}\rangle \leqslant \|p\|_0\|\operatorname{div}\mathbf{u}\|_0 \leqslant \|p\|_0\|\mathbf{u}\|_{\mathbf{U}}$, which gives $\|\nabla p\|_{-1}\leqslant\|p\|_0$.

Under our assumptions about Ω we can assert that $\mathbf{U}\subset W^1_{2,\mathrm{loc}}(\Omega)^n$ (see [15]), that is, each $\mathbf{u}\in\mathbf{U}$ belongs to the space W_2^1 in any subdomain lying strictly inside Ω. Suppose now that the boundary of Ω has a *greater* regularity, namely, condition (I) holds. Then $\mathbf{U}$ is continuously embedded in $W_2^1(\Omega)^n$ (see [15]) and

$$\|\mathbf{u}\|_{W_2^1} \cong \|\operatorname{div}\mathbf{u}\|_0 + \|\operatorname{curl}\mathbf{u}\|_0, \qquad \mathbf{u}\in\mathbf{U}.$$

Hence $\mathbf{U}$ is embedded in $\mathbf{H}_\nu$; the embedding $\mathbf{H}_\nu\subset\mathbf{U}$ is trivial, therefore the equalities

$$\mathbf{H}_\nu = \mathbf{U}, \qquad \mathbf{R}_\nu = \mathbf{V} \tag{6}$$

hold both algebraically and topologically.

§ 2. Problem with model boundary conditions

We return to the minimal assumptions on the domain: let Ω be a bounded simply connected domain with Lipschitz boundary. We say that a function $\mathbf{u} \in W_2^2(\Omega)^n \cap \mathbf{U}$ satisfies the *model* boundary conditions on $\partial\Omega$ and write $\mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0$ if

$$
\begin{gathered}
\operatorname{curl} \mathbf{u} = 0 \quad \text{on } \partial\Omega \quad \text{ for } n = 2, \\
\text{or} \\
\operatorname{curl} \mathbf{u} \cdot \tau_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, \quad \text{ for } n = 3.
\end{gathered}
\tag{7}
$$

Since for $n = 3$ the vectors τ_i, $i = 1, 2$, make up a basis in the tangent space to $\partial\Omega$ at each boundary point of the domain (except for a set of measure zero), the boundary conditions $\mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0$ mean for $n = 3$ the orthogonality of $\operatorname{curl} \mathbf{u}$ to $\partial\Omega$ at each point of $\partial\Omega$. They can also be written as

$$
\operatorname{curl} \mathbf{u} \times \nu = 0 \quad \text{on } \partial\Omega \quad \text{for } n = 3.
$$

Hence, in particular, the conditions $\mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0$, $n = 3$, are invariant with respect to the choice of a particular basis $\tau_i(P)$, $i = 1, 2$, of tangent vectors at points $P \in \partial\Omega$. It is easily seen that relations (7) give us one boundary condition on $\mathbf{u}$ for $n = 2$ and two boundary conditions on $\mathbf{u}$ for $n = 3$.

If $\mathbf{u}$ is an arbitrary function in $W_2^2(\Omega)^n \cap \mathbf{U}$ and $\mathcal{R}\mathbf{u} = 0$, then the equality

$$
-(\Delta \mathbf{u}, \mathbf{v}) = (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) \tag{8}
$$

holds for each $\mathbf{v} \in \mathbf{U}$. The converse is also true: if for some $\mathbf{u} \in W_2^2(\Omega)^n \cap \mathbf{U}$ relation (8) holds for each $\mathbf{v} \in \mathbf{U}$, then $\mathcal{R}\mathbf{u} = 0$. These two statements are easy to verify if we take into account the following equalities:

$$
\begin{aligned}
&(\nabla p, \mathbf{v}) + (p, \operatorname{div} \mathbf{v}) = 0 && \text{for all } \mathbf{v} \in \mathbf{U}, \quad p \in W_2^1(\Omega), \\
&(\operatorname{curl} \mathbf{u}, \mathbf{v}) + (\mathbf{u}, \operatorname{curl} \mathbf{v}) = \langle \mathbf{u}, \gamma_\tau \mathbf{v} \rangle && \text{for all } \mathbf{v} \in \mathbf{U}, \quad \begin{array}{l} \mathbf{u} \in W_2^1(\Omega), \quad n = 2, \\ \mathbf{u} \in W_2^1(\Omega)^3, \ n = 3, \end{array}
\end{aligned}
$$

where $\gamma_\tau \mathbf{v} = \mathbf{v} \cdot \tau \in \mathbf{H}^{-1/2}(\partial\Omega)$ for $n = 2$ and $\gamma_\tau \mathbf{v} = \mathbf{v} \times \nu \in \mathbf{H}^{-1/2}(\partial\Omega)^3$ for $n = 3$ (see Lemma 1).

For $\mathbf{f} \in \mathbf{U}^{-1}$ we consider the problem of finding a function $\mathbf{u}$ such that

$$
\begin{gathered}
-\Delta \mathbf{u} = \mathbf{f}, \\
\mathbf{u} \cdot \nu\big|_{\partial\Omega} = \mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0.
\end{gathered}
\tag{9}
$$

Thanks to (8), the problem (9) has the following generalization: find $\mathbf{u} \in \mathbf{U}$ such that

$$
(\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathbf{U}. \tag{10}
$$

We have the following result.

Lemma 4. *If* $\mathbf{f} \in \mathbf{U}^{-1}$, *then the problem* (9) *has a unique solution* $\mathbf{u}$ *in* $\mathbf{U}$, *which satisfies the estimate*

$$\|\mathbf{u}\|_{\mathbf{U}} \leqslant c(\Omega)\|\mathbf{f}\|_{-1}.$$

Proof. We consider the bilinear form

$$a(\mathbf{u},\mathbf{v}) = (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}), \qquad \mathbf{u},\mathbf{v} \in \mathbf{U},$$

from the statement of the generalized problem (10).

The estimate $a(\mathbf{u},\mathbf{v}) \leqslant \|\mathbf{u}\|_{\mathbf{U}}\|\mathbf{v}\|_{\mathbf{U}}$ and Lemma 2 show that the form $a(\mathbf{u},\mathbf{v})$ is continuous and coercive on $\mathbf{U}$. The existence and uniqueness of the generalized solution now follows from the Lax–Milgram theorem, and the a priori estimate follows from Lemma 2 and the formulae

$$a(\mathbf{u},\mathbf{u}) = \langle \mathbf{f},\mathbf{u}\rangle \leqslant \|\mathbf{f}\|_{-1}\|\mathbf{u}\|_{\mathbf{U}}.$$

This proves the lemma.

For an arbitrary $\mathbf{f} \in \mathbf{U}^{-1}$ we shall denote by $\Delta_\nu^{-1}\mathbf{f}$ the generalized solution of problem (9), the existence and uniqueness of which is guaranteed by Lemma 4.

We now consider the Stokes problem: given $\mathbf{f} \in \mathbf{U}^{-1}$ and $g \in L_2/\mathbb{R}$, find $\mathbf{u}$ and p such that

$$\begin{gathered} -\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{u} = g, \\ \mathbf{u}\cdot\nu\big|_{\partial\Omega} = \mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0. \end{gathered} \tag{11}$$

The generalized statement of the problem (11) consists in finding a pair $\{\mathbf{u}, p\}$ from $\mathbf{U} \times L_2/\mathbb{R}$ such that

$$\begin{gathered} (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle \quad \text{for all } \mathbf{v} \in \mathbf{U}, \\ (\operatorname{div}\mathbf{u}, q) = (g, q) \quad \text{for all } q \in L_2/\mathbb{R}\,. \end{gathered} \tag{12}$$

The solubility of problem (11) is established by the following lemma.

Lemma 5. *If* $\mathbf{f} \in \mathbf{U}^{-1}$ *and* $g \in L_2/\mathbb{R}$, *then the problem* (11) *has a unique generalized solution in the class* $\mathbf{U} \times L_2/\mathbb{R}$.

Proof. The coercivity of the bilinear form

$$(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})$$

on the space of solenoidal vector-valued functions from $\mathbf{U}$ follows from its coercivity on $\mathbf{U}$ established in the proof of Lemma 4. For the form $b(\,\cdot\,,\cdot\,)\colon \mathbf{U} \times L_2/\mathbb{R} \to \mathbb{R}$ defined by the formula $b(\mathbf{u}, p) = (\operatorname{div}\mathbf{u}, p)$ the inf-sup-condition is equivalent to the inequality

$$\|p\|_0 \leqslant c(\Omega)\|\nabla p\|_{-1}, \qquad p \in L_2/\mathbb{R}\,.$$

To prove this inequality we observe that for an arbitrary $p \in L_2/\mathbb{R}$ we have the following chain of inequalities:

$$\|p\|_0 \leqslant c_0 \sup_{\mathbf{v}\in\mathbf{H}_0} \frac{(p, \operatorname{div}\mathbf{v})}{\|\mathbf{v}\|_1} \leqslant \sqrt{2}\, c_0 \sup_{\mathbf{v}\in\mathbf{H}_0} \frac{(p, \operatorname{div}\mathbf{v})}{\|\operatorname{div}\mathbf{v}\|_0 + \|\operatorname{curl}\mathbf{v}\|_0}$$
$$\leqslant \sqrt{2}\, c_0 \sup_{\mathbf{v}\in\mathbf{U}} \frac{(p, \operatorname{div}\mathbf{v})}{\|\operatorname{div}\mathbf{v}\|_0 + \|\operatorname{curl}\mathbf{v}\|_0} \leqslant c_1 \sup_{\mathbf{v}\in\mathbf{U}} \frac{(p, \operatorname{div}\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{U}}} \equiv c(\Omega)\|\nabla p\|_{-1}.$$

The first inequality in this chain is well known (see, for instance, [16] or [17]). Next we use the obvious inequality $\|\mathbf{v}\|_1^2 = \|\operatorname{div}\mathbf{v}\|_0^2 + \|\operatorname{curl}\mathbf{v}\|_0^2$ for $\mathbf{v} \in \mathbf{H}_0$, the embedding $\mathbf{H}_0 \subset \mathbf{U}$, and Lemma 2. The assertion of the lemma now follows from Corollary 4.1 in [15].

A priori estimates for the solution of (11) will be obtained later. We now consider the Schur operator $\mathbf{A}_\nu \colon L_2/\mathbb{R} \to L_2/\mathbb{R}$,

$$\mathbf{A}_\nu p = \operatorname{div}\Delta_\nu^{-1}\nabla p$$

for the system (11). By Lemma 4, the operator $\mathbf{A}_\nu$ is defined for each $p \in L_2/\mathbb{R}$, and $\mathbf{A}_\nu p \in L_2/\mathbb{R}$. We have the following result.

Theorem 1. *If $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded simply connected domain with Lipschitz boundary, then $\mathbf{A}_\nu$ is the identity operator on $L_2/\mathbb{R}$, that is,*

$$\mathbf{A}_\nu p = p \text{ for all } p \in L_2/\mathbb{R}.$$

Proof. For an arbitrary $p \in L_2/\mathbb{R}$ we set $q = \mathbf{A}_\nu p \in L_2/\mathbb{R}$ and denote by $\mathbf{u}$ the vector-valued function $\Delta_\nu^{-1}\nabla p$ from $\mathbf{U}$. Then, by definition (10) of the generalized solution of (6) and the definition of the operator $\mathbf{A}_\nu$, we obtain

$$\begin{gathered}(p, \operatorname{div}\mathbf{v}) = (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) \quad \text{for each } \mathbf{v} \in \mathbf{U}, \\ q = \operatorname{div}\mathbf{u}.\end{gathered} \tag{13}$$

We take the scalar product of the second equality in (13) and $\operatorname{div}\mathbf{v}$ and consider functions $\mathbf{v}$ such that $\operatorname{curl}\mathbf{v} = 0$ (in other words, $\mathbf{v} \in \mathbf{V}$). Subtracting the second equality in (13) from the first we obtain

$$(p, \operatorname{div}\mathbf{v}) - (q, \operatorname{div}\mathbf{v}) = 0 \quad \text{for each } \mathbf{v} \in \mathbf{V}.$$

By Lemma 3 we obtain $(p, r) = (q, r)$ for all $r \in L_2/\mathbb{R}$. Since $L_2/\mathbb{R}$ is a Hilbert space, it follows from the last equality that $p = q$, which proves the theorem.

Corollary 1. *The generalized solution of problem* (11) *in $\mathbf{U} \times L_2/\mathbb{R}$ can be written in the following form*:

$$\mathbf{u} = \Delta_\nu^{-1}(\nabla p - \mathbf{f}), \qquad p = g + \operatorname{div}\Delta_\nu^{-1}\mathbf{f}. \tag{14}$$

This solution satisfies the following a priori estimates:

$$\|\mathbf{u}\|_{\mathbf{U}} \leqslant (\Omega)(\|g\|_0 + \|\mathbf{f}\|_{-1}), \qquad \|p\|_0 \leqslant (\Omega)(\|g\|_0 + \|\mathbf{f}\|_{-1}).$$

Proof. Equalities (14) follow from the statement of the theorem and the generalized formulations (10) and (12) of the corresponding problems. A priori estimates of the solution follow from equalities (14), the estimates in Lemma 4, and the inequality $\|\nabla p\|_{-1} \leqslant \|p\|_0$, which completes the proof.

Remark. We now present a simple example explaining why the above argument cannot be carried out in the framework of the space $W_2^1(\Omega)^n$ for arbitrary domains with Lipschitz boundaries.

We consider the domain $\Omega = \{x = \rho e^{i\theta},\ 0 < \rho < 1,\ 0 < \theta < \pi + \varepsilon\}$, $\varepsilon > 0$. It is clear that Ω is a bounded simply connected subdomain of $\mathbb{R}^2$ with Lipschitz boundary that does not satisfy condition (I). Consider a function q in $L_2/\mathbb{R}$ such that the generalized solution $\psi \in W_2^1(\Omega)/\mathbb{R}$ of the problem $\Delta\psi = q$, $\partial\psi/\partial\nu\big|_{\partial\Omega} = 0$ does not belong to $W_2^2(\Omega)$ (see, for instance, [18]). We consider the vector-valued function $\mathbf{u} = \nabla\psi$. It is clear that $\mathbf{u} \in \mathbf{U}$, but $\mathbf{u} \notin W_2^1(\Omega)$, while

$$(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) = (q, \operatorname{div}\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U},$$

that is, $\mathbf{u} = \Delta_\nu^{-1}\nabla q$ for some function $q \in L_2/\mathbb{R}$.

§ 3. Regularity of solutions

At the present time, there exists a well-developed $W_p^m(\Omega)$-theory of the Stokes problem in a bounded domain with Dirichlet boundary conditions, which establishes, in particular, that if $\mathbf{f} \in W_2^{m-2}(\Omega)^n$, $g \in W_2^{m-1}(\Omega) \cap L_2/\mathbb{R}$, $m \geqslant 1$, and $\partial\Omega \in C^m$, then the generalized solution $\{\mathbf{u}, p\}$ satisfies the following relations: $\mathbf{u} \in W_2^m(\Omega)^n$, $p \in W_2^{m-1}(\Omega) \cap L_2/\mathbb{R}$, and

$$\|\mathbf{u}\|_m + \|p\|_{m-1} \leqslant c(m, \Omega)(\|\mathbf{f}\|_{m-2} + \|g\|_{m-1})$$

(see, for instance, [1] and [19]). In this section we obtain a priori estimates for the smoothness class and the norm of the generalized solution of the problem (11) with model boundary conditions.

We suppose at first that a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, satisfies condition (I) in § 1. Then by (6) the results of § 1 and § 2 remain valid if the spaces $\mathbf{U}$ and $\mathbf{V}$ are replaced by $\mathbf{H}_\nu$ and $\mathbf{R}_\nu$, respectively, and the norm $\|\cdot\|_{\mathbf{U}}$ is replaced by $\|\cdot\|_1$. Moreover, the generalized problems in question can be set in terms of the space $\mathbf{H}_\nu$. Thus, we have the following result.

Lemma 6. *Let Ω be a bounded simply connected subdomain of $\mathbb{R}^n$, $n = 2, 3$, satisfying condition* (I). *If $\mathbf{f} \in \mathbf{H}_\nu^{-1}$, then the problem* (9) *has a unique solution $\mathbf{u}$ in the space $\mathbf{H}_\nu$ and*

$$\|\mathbf{u}\|_1 \leqslant c(\Omega)\|\mathbf{f}\|_{-1}.$$

If $\mathbf{f} \in \mathbf{H}_\nu^{-1}$ and $g \in L_2/\mathbb{R}$, then the problem (11) *has a unique solution $\{\mathbf{u}, p\}$ in the space $\mathbf{H}_\nu \times L_2/\mathbb{R}$, which satisfies the estimates*

$$\|\mathbf{u}\|_1 \leqslant (\Omega)(\|g\|_0 + 2\|\mathbf{f}\|_{-1}), \qquad \|p\|_0 \leqslant (\Omega)(\|g\|_0 + \|\mathbf{f}\|_{-1}).$$

The operator $\mathbf{A}_\nu$ is the identity operator in $L_2/\mathbb{R}$.

By Theorem 3.1.1.1 in [18] and Theorem 3.9 in [15], the inequality

$$(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{u}) \geqslant \alpha\|\mathbf{u}\|_1^2, \qquad \mathbf{u} \in \mathbf{H}_\nu,$$

holds in all convex domains Ω with $\alpha = 1$, therefore the constant $c(\Omega)$ in Lemma 6 can be set equal to 1 for such domains.

We now consider the problem of the regularity of solutions in the case when we impose additional conditions on the right-hand side and the boundary of the domain. We require the following lemma from [20].

Lemma 7. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, and let $\partial\Omega \in C^{m+1}$, $m \geqslant 1$. Then*

$$W_2^m(\Omega)^n = \{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div}\mathbf{u} \in W_2^{m-1}(\Omega), \operatorname{curl}\mathbf{u} \in W_2^{m-1}(\Omega)^r, \mathbf{u}\cdot\nu \in W_2^{m-1/2}(\partial\Omega)\}$$

and

$$\|\mathbf{u}\|_m \leqslant c(m,\Omega)\big(\|\mathbf{u}\|_0 + \|\operatorname{div}\mathbf{u}\|_{m-1} + \|\operatorname{curl}\mathbf{u}\|_{m-1} + \|\mathbf{u}\cdot\nu\|_{m-1/2,\partial\Omega}\big),$$

where $r = 1$ for $n = 2$ and $r = 3$ for $n = 3$.

We see from Corollary 1 to Theorem 1 that the entire information on the smoothness of the generalized solution of (12) can be obtained from the knowledge of the smoothness of $\mathbf{f}$, g, and the solutions of (9). Therefore the following lemma is important.

Lemma 8. *Suppose that $m \geqslant 1$ and let Ω be a bounded simply connected subdomain of $\mathbb{R}^n$, let $\partial\Omega \in C^r$, where $r = \max(m, 2)$, and let $\mathbf{f} \in W_2^{m-2}(\Omega)^n$. Then the problem (9) has a unique solution $\mathbf{u}$ in $W_2^m(\Omega)^n \cap \mathbf{H}_\nu$ and*

$$\|\mathbf{u}\|_m \leqslant c(m,\Omega)\|\mathbf{f}\|_{m-2}. \tag{15}$$

Proof. We first consider the case $n = 2$. We prove the lemma by induction on m. For $m = 1$ the assertion of the lemma follows from Lemma 6. Assume now that the lemma is proved for $m = 1, \dots, k-1$; we shall prove it for $m = k \geqslant 2$.

Since $\partial\Omega \in C^k$, it follows that the vector field ν belongs to the class $C^{k-1}(\partial\Omega)^2$ and can be continued to a vector field in $C^{k-1}(\overline{\Omega})^2$. We denote this continuation by $\overline{\nu} = (\overline{\nu}_1, \overline{\nu}_2)$. Note also the embeddings $C^{k-1}(\overline{\Omega})^2 \subset W_\infty^{k-1}(\Omega)^2 \subset W_2^k(\Omega)^2$ (see [21]).

Suppose that $\mathbf{u}$ is the solution of (9) in $W_2^{k-1}(\Omega)^2 \cap \mathbf{H}_\nu$, which exists by the induction hypothesis and satisfies estimate (15) with $m = k-1$. We denote by Ω_σ, $\sigma > 0$, the set of points in Ω whose distance from $\partial\Omega$ is less than σ. It is sufficient to prove that the lemma holds for the restriction of $\mathbf{u}$ to Ω_σ for some $\sigma > 0$. In fact, if this is the case, then we can consider the domain Ω' such that $\overline{\Omega}' \subset \Omega$, $\overline{\Omega \setminus \Omega_\sigma} \subset \Omega'$ and $\partial\Omega' \in C^k$. The function $\mathbf{u}$ is the solution in Ω' of the vector Laplace equation with Dirichlet boundary conditions on $\partial\Omega'$; in addition $\mathbf{u}|_{\partial\Omega'} \in W_2^{k-1/2}(\partial\Omega')^2$ and

$$\|\mathbf{u}|_{\partial\Omega'}\|_{k-1/2,\partial\Omega'} \leqslant c(\Omega,\Omega',\sigma)\|\mathbf{u}\|_{k,\Omega_\sigma} \leqslant c\|\mathbf{f}\|_{k-2}.$$

By the classical results on the regularity of the solutions of elliptic equations we obtain $\mathbf{u} \in W_2^k(\Omega')^2$ and $\|\mathbf{u}\|_{k,\Omega'} \leqslant c\|\mathbf{f}\|_{k-2,\Omega'}$, from which the assertion of the lemma follows if we take into account the fact that $\mathbf{u}$ belongs to $W_2^k(\Omega_\sigma)^2$ and $\|\mathbf{u}\|_{k,\Omega_\sigma} \leqslant c\|\mathbf{f}\|_{k-2,\Omega_\sigma}$.

Hence it suffices to prove that $\mathbf{u} \in W_2^k(\Omega_{\sigma_0})^2$ and $\|\mathbf{u}\|_{k,\Omega_{\sigma_0}} \leqslant c\|\mathbf{f}\|_{k-2}$ for some $\sigma_0 > 0$. Consider the function $q = \mathbf{u} \cdot \overline{\nu} = u_1\overline{\nu}_1 + u_2\overline{\nu}_2$ in $W_2^{k-1}(\Omega) \cap W_{2,0}^1(\Omega)$. It follows from the definition of q that

$$\begin{gathered} \Delta q = \mathbf{f} \cdot \overline{\nu} + \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial \overline{\nu}_i}{\partial x_j}, \\ q\big|_{\partial\Omega} = 0. \end{gathered} \tag{16}$$

Since $\mathbf{f} \in W_2^{k-2}(\Omega)^2$, $\overline{\nu} \in W_2^k(\Omega)^2$, and $\mathbf{u} \in W_2^{k-1}(\Omega)^2$ by the induction hypothesis, it follows that the right-hand side of system (16) belongs to $W_2^{k-2}(\Omega)$ and therefore $q \in W_2^k(\Omega)$ and

$$\|q\|_k \leqslant c\left\|\mathbf{f} \cdot \overline{\nu} + \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial \overline{\nu}_i}{\partial x_j}\right\|_{k-2} \leqslant c_1(\|\mathbf{f}\|_{k-2} + \|\mathbf{u}\|_{k-1}) \leqslant c_2\|\mathbf{f}\|_{k-2}. \tag{17}$$

For the function

$$\varphi = \left(\frac{\partial u_1}{\partial x_1}\overline{\nu}_1 + \frac{\partial u_2}{\partial x_1}\overline{\nu}_2,\ \frac{\partial u_1}{\partial x_2}\overline{\nu}_1 + \frac{\partial u_2}{\partial x_2}\overline{\nu}_2\right) = \nabla q - \mathbf{u}^t \cdot (\nabla \overline{\nu})$$

we now see that $\varphi \in W_2^{k-1}(\Omega)$ and

$$\|\varphi\|_{k-1} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2} \tag{18}$$

by (17).

We also consider the function

$$\psi = \left(-\frac{\partial u_1}{\partial x_1}\overline{\nu}_2 + \frac{\partial u_2}{\partial x_1}\overline{\nu}_1 - \frac{\partial u_2}{\partial x_2}\overline{\nu}_2 - \frac{\partial u_1}{\partial x_2}\overline{\nu}_1,\ \frac{\partial u_1}{\partial x_1}\overline{\nu}_1 + \frac{\partial u_2}{\partial x_1}\overline{\nu}_2 + \frac{\partial u_2}{\partial x_2}\overline{\nu}_1 - \frac{\partial u_1}{\partial x_2}\overline{\nu}_2\right).$$

Equations (9) and the induction hypothesis immediately show that $\operatorname{div}\psi$ and $\operatorname{curl}\psi$ belong to $W_2^{k-2}(\Omega)$, $\psi \cdot \nu\big|_{\partial\Omega} = 0$ and

$$\|\psi\|_0 + \|\operatorname{div}\psi\|_{k-2} + \|\operatorname{curl}\psi\|_{k-2} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2}.$$

Hence $\psi \in W_2^{k-1}(\Omega)^2$ by Lemma 7 and

$$\|\psi\|_{k-1} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2}. \tag{19}$$

We note that

$$\frac{\partial u_1}{\partial x_1}(\overline{\nu}_1^2 + \overline{\nu}_2^2) = \varphi_1\overline{\nu}_1 - \varphi_2\overline{\nu}_2 - \psi_1\overline{\nu}_2 \in W_2^{k-1}(\Omega).$$

In a similar way we obtain for all other partial derivatives

$$\frac{\partial u_i}{\partial x_j}(\overline{\nu}_1^2+\overline{\nu}_2^2) \in W_2^{k-1}(\Omega), \qquad i,j=1,2. \tag{20}$$

Moreover,

$$\left\| \frac{\partial u_i}{\partial x_j}(\overline{\nu}_1^2+\overline{\nu}_2^2) \right\|_{k-1} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2} \tag{21}$$

by (18) and (19).

We now consider the function $\overline{\nu}_1^2+\overline{\nu}_2^2$ in $C^{k-1}(\overline{\Omega})$. Note that $\overline{\nu}_1^2+\overline{\nu}_2^2\big|_{\partial\Omega}=1$. We choose $\sigma_0>0$ so that $\overline{\nu}_1^2+\overline{\nu}_2^2 \geqslant \frac{1}{2}$ in Ω_{σ_0}. By (20) and (21),

$$\frac{\partial u_i}{\partial x_j} \in W_2^{k-1}(\Omega_{\sigma_0})^2,$$
$$\left\| \frac{\partial u_i}{\partial x_j} \right\|_{k-1,\Omega_{\sigma_0}} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2}, \qquad i,j=1,2.$$

Hence $\mathbf{u}\in W_2^k(\Omega_{\sigma_0})^2$ and $\|\mathbf{u}\|_{k,\Omega_{\sigma_0}} \leqslant c(\Omega)\|\mathbf{f}\|_{k-2}$ for some $\sigma_0>0$, which proves the lemma for $n=2$.

The proof in the case $n=3$ is similar. Here it suffices to consider the function $q=\mathbf{u}\cdot\overline{\nu}$ and the vector-valued functions

$$\psi_1=(-n_2,n_1,0)\cdot(\nabla\mathbf{u})-\bigl(-[\overline{\nu}\cdot(\nabla\mathbf{u})]_2,[\overline{\nu}\cdot(\nabla\mathbf{u})]_1,0\bigr),$$
$$\psi_2=(-n_3,0,n_1)\cdot(\nabla\mathbf{u})-\bigl(-[\overline{\nu}\cdot(\nabla\mathbf{u})]_3,0,[\overline{\nu}\cdot(\nabla\mathbf{u})]_1\bigr),$$

where $[\,\cdot\,]_i$ is the ith component of the corresponding vector. The lemma is proved.

Theorem 2 below immediately follows from Lemma 8 and Corollary 1 to Theorem 1.

Theorem 2. *Suppose that $m\geqslant 1$ and let Ω be a bounded simply connected subdomain of $\mathbb{R}^n$ with boundary $\partial\Omega\in C^r$, where $r=\max(m,2)$. Let $\mathbf{f}\in W_2^{m-2}(\Omega)^n$ and let $g\in W_2^{m-1}(\Omega)\cap L_2/\mathbb{R}$. Then the problem (11) has a unique solution $\{\mathbf{u},\,p\}$ in the class $(W_2^m(\Omega)^n\cap\mathbf{H}_\nu)\times(W_2^{m-1}(\Omega)\cap L_2/\mathbb{R})$ and*

$$\|\mathbf{u}\|_m \leqslant c(m,\Omega)(\|g\|_{m-1}+\|\mathbf{f}\|_{m-2}),$$
$$\|p\|_{m-1} \leqslant c(m,\Omega)(\|g\|_{m-1}+\|\mathbf{f}\|_{m-2}).$$

§ 4. Several observations and consequences

The problem (11) with inhomogeneous conditions on curl $\mathbf{u}$ was considered in [6] for two-dimensional bounded simply connected domains with smooth boundaries and used there for the construction of an Uzawa-type algorithm of the numerical solution of (1), (2). The following theorem is a consequence of Theorem 1 and is similar to the corresponding result in [6].

Theorem 3. *Let Ω be a bounded simply connected domain satisfying condition* (I), *let $\mathbf{f} \in L_2(\Omega)^n$ and let $g \in W_2^1(\Omega) \cap L_2/\mathbb{R}$. Then the solution p of* (11) *belongs to $W_2^1(\Omega) \cap L_2/\mathbb{R}$ and satisfies (in the generalized sense) the system of equations*

$$\begin{gathered} \Delta p = \Delta g + \operatorname{div} \mathbf{f}, \\ \left.\frac{\partial p}{\partial \nu}\right|_{\partial\Omega} = \left.\left(\frac{\partial g}{\partial \nu} + \mathbf{f} \cdot \nu\right)\right|_{\partial\Omega}. \end{gathered} \tag{22}$$

Proof. From Theorem 2 we know that under our assumptions on Ω, $\mathbf{f}$ and g the solution p of (11) belongs to the class $W_2^1(\Omega) \cap L_2/\mathbb{R}$. By Corollary 1 we obtain $p = g + \operatorname{div} \mathbf{w}$, where $\mathbf{w} = \Delta_\nu^{-1}\mathbf{f}$. The function $\mathbf{w}$ is in the class $W_2^2(\Omega)^n \cap \mathbf{H}_\nu$ and satisfies the equality

$$(\nabla \operatorname{div} \mathbf{w}, \mathbf{v}) + (\operatorname{curl} \mathbf{w}, \operatorname{curl} \mathbf{v}) + \langle \operatorname{curl} \mathbf{w}, \gamma_\tau \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in W_2^1(\Omega)^n.$$

Since $\mathcal{R}\mathbf{w}\big|_{\partial\Omega} = 0$, it follows that $\langle \operatorname{curl} \mathbf{w}, \gamma_\tau \mathbf{v} \rangle = 0$. Further,

$$(\nabla p, \mathbf{v}) = (\nabla g, \mathbf{v}) + (\operatorname{curl} \mathbf{w}, \operatorname{curl} \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \qquad \forall\, \mathbf{v} \in W_2^1(\Omega)^n.$$

For an arbitrary $\psi \in W_2^2(\Omega)$ we set $\mathbf{v} = \nabla\psi$ in the last equality and find that

$$(\nabla p, \nabla\psi) = (\nabla g, \nabla\psi) + (\mathbf{f}, \nabla\psi).$$

Passing here to ψ from the closure of $W_2^2(\Omega)$ with respect to the norm of $W_2^1(\Omega)$ we obtain that (22) holds in the generalized sense, which proves the theorem.

We now present two other consequences of Theorem 1. We denote by $\mathbf{W}$ the following subspace of $\mathbf{U}$:

$$\mathbf{W} \equiv \left\{\mathbf{u} \in \mathbf{U} : \mathbf{u} = \Delta_\nu^{-1}\nabla p \quad \text{for some } p \in L_2/\mathbb{R}\right\}.$$

Theorem 4. *Under the hypothesis of Theorem* 1,

$$\mathbf{W} = \mathbf{V}$$

as subspaces of $\mathbf{U}$. Moreover, the operator $\Delta_\nu^{-1}\nabla$ is an isomorphism between $L_2/\mathbb{R}$ and $\mathbf{V}$ and

$$\|\operatorname{div} \mathbf{u}\|_0 = \|p\|_0$$

for $\mathbf{u} = \Delta_\nu^{-1}\nabla p$.

Proof. We recall that the equality $\mathbf{u} = \Delta_\nu^{-1}\nabla p$ means that

$$(p, \operatorname{div} \mathbf{v}) = (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) + (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{U}. \tag{23}$$

The embedding $\mathbf{V} \subset \mathbf{W}$ obviously follows from (23) if we set $p = \operatorname{div} \mathbf{u}$. To see that $\mathbf{W} \subset \mathbf{V}$ we observe that for each $\mathbf{u} \in \mathbf{W}$ it follows from (23) and Theorem 1 that

$$(\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) = 0 \quad \text{for all } \mathbf{v} \in \mathbf{U}.$$

We now set $\mathbf{v} = \mathbf{u}$. Then $\|\operatorname{curl}\mathbf{u}\|_0^2 = 0$ and therefore $\mathbf{u} \in \mathbf{V}$. Hence $\mathbf{W} = \mathbf{V}$. The second part of the assertion of the theorem follows from the relations

$$\|p\|_0 = \|\mathbf{A}_\nu p\|_0 = \|\operatorname{div}\mathbf{u}\|_0 \leqslant \|\mathbf{u}\|_{\mathbf{U}},$$

which are valid for each $p \in L_2/\mathbb{R}$ and $\mathbf{u} = \Delta_\nu^{-1}\nabla p$. The theorem is proved.

We now consider the inequality

$$\|p\|_0 \leqslant c_0(\Omega) \sup_{0 \neq \mathbf{u} \in \mathbf{H}_0} \frac{(p, \operatorname{div}\mathbf{u})}{\|\mathbf{u}\|_1} \quad \text{for each } p \in L_2/\mathbb{R}. \tag{24}$$

As mentioned in the proof of Lemma 5, inequality (24) holds with a certain constant $1 < c_0(\Omega) < \infty$ depending only on the domain. We also point out the equality $c_0(\Omega) = \lambda_{\min}^{-1/2}$ (see, for instance, [6]), where $\lambda_{\min}$ is the smallest eigenvalue of $\mathbf{A}_0$ (see the introduction). The discrete analogue of (24) is widely known as the LBB (or inf-sup-) condition (see, for instance, [15] and [22]). We denote by $\mathcal{H}_\nu$ the following subspace of $\mathbf{U}$:

$$\mathcal{H}_\nu \equiv \bigl\{\mathbf{u} \in \mathbf{U} : \Delta\mathbf{u} = 0\bigr\}.$$

We now define the operator $\mathbf{D}\colon \mathbf{V} \to \mathcal{H}_\nu$ as follows: $\mathbf{v} = \mathbf{D}\mathbf{u}$ if $\gamma_\nu\mathbf{v}\big|_{\partial\Omega} = 0$, $\gamma_\tau\mathbf{v}\big|_{\partial\Omega} = \gamma_\tau\mathbf{u}\big|_{\partial\Omega}$, and $\mathbf{v} \in \mathcal{H}_\nu$. For $\mathbf{u} \in \mathbf{V}$ the function $\mathbf{v} = \mathbf{D}\mathbf{u}$ is uniquely defined as the solution of the problem

$$\begin{gathered} \mathbf{v} = \mathbf{r} + \mathbf{u}, \qquad \mathbf{r} \in \mathbf{H}_0, \\ -(\nabla\mathbf{r}, \nabla\mathbf{w}) = (\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{w}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{H}_0. \end{gathered} \tag{25}$$

We use the following notation for functions $\mathbf{u}$ in $\mathbf{U}$:

$$|\mathbf{u}|_{\mathbf{U}} \equiv \bigl(\|\operatorname{div}\mathbf{u}\|_0^2 + \|\operatorname{curl}\mathbf{u}\|_0^2\bigr)^{1/2}.$$

For $\mathbf{v} = \mathbf{D}\mathbf{u}$ it immediately follows from (25) that

$$|\mathbf{v}|_{\mathbf{U}}^2 = (\operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{u}) + (\operatorname{curl}\mathbf{v}, \operatorname{curl}\mathbf{u}), \tag{26}$$

$$|\mathbf{v}|_{\mathbf{U}} \leqslant c_1(\Omega)|\mathbf{u}|_{\mathbf{U}}, \tag{27}$$

where $c_1(\Omega) \leqslant 1$. We have the following result.

Theorem 5. *Under the hypothesis of Theorem* 1 *inequality* (24) *is equivalent to the relation*

$$|\mathbf{D}\mathbf{u}|_{\mathbf{U}}^2 \leqslant \bigl(1 - c_0^{-2}(\Omega)\bigr)\|\operatorname{div}\mathbf{u}\|_0^2 \quad \text{for all } \mathbf{u} \in \mathbf{V}. \tag{28}$$

The equalities in (24) *and* (28) *can hold simultaneously only for* $p \in L_2/\mathbb{R}$ *and* $\mathbf{u}$ *equal to* $\Delta_\nu^{-1}\nabla p \in \mathbf{V}$, *respectively. The constant* $c_0(\Omega)$ *in* (24) *and* (28) *is the same.*

Proof. From Theorem 4 it follows that for each $p \in L_2/\mathbb{R}$ and $\mathbf{u} = \Delta_\nu^{-1}\nabla p \in \mathbf{V}$ we have

$$\|p\|_0 = \|\operatorname{div}\mathbf{u}\|_0. \tag{29}$$

On the other hand,

$$\sup_{0\neq\mathbf{v}\in\mathbf{H}_0}\frac{(p,\operatorname{div}\mathbf{v})^2}{\|\mathbf{v}\|_1^2}=\sup_{0\neq\mathbf{v}\in\mathbf{H}_0}\frac{(\Delta_0^{-1}\nabla p,\mathbf{v})_1^2}{\|\mathbf{v}\|_1^2}=\|\mathbf{w}\|_1^2=|\mathbf{w}|_{\mathbf{U}}^2,\tag{30}$$

where $\mathbf{w}=\Delta_0^{-1}\nabla p$ is the solution of the Dirichlet problem $\Delta\mathbf{w}=\nabla p$, $\mathbf{w}\big|_{\partial\Omega}=0$. We now consider $\mathbf{v}=\mathbf{u}-\mathbf{w}$. We have the equalities $\gamma_\nu\mathbf{v}\big|_{\partial\Omega}=\gamma_\nu\mathbf{u}|_{\partial\Omega}=0$, $\gamma_\tau\mathbf{v}\big|_{\partial\Omega}=\gamma_\tau\mathbf{u}\big|_{\partial\Omega}$, and $\Delta\mathbf{v}=0$, therefore $\mathbf{v}=\mathbf{D}\mathbf{u}$. Now,

$$|\mathbf{w}|_{\mathbf{U}}^2=|\mathbf{u}-\mathbf{v}|_{\mathbf{U}}^2=|\mathbf{u}|_{\mathbf{U}}^2+|\mathbf{v}|_{\mathbf{V}}^2-2\big((\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v})+(\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v})\big).$$

Using (26) we obtain

$$|\mathbf{w}|_{\mathbf{U}}^2=|\mathbf{u}|_{\mathbf{U}}^2-|\mathbf{v}|_{\mathbf{U}}^2=\|\operatorname{div}\mathbf{u}\|_0^2-|\mathbf{D}\mathbf{u}|_{\mathbf{U}}^2.\tag{31}$$

Equalities (31) together with (29) and (30) show that if (28) holds, then so does (24). Since the operator $\Delta_\nu^{-1}\nabla$ is an isomorphism between $L_2/\mathbb{R}$ and $\mathbf{V}$ by Theorem 4, it follows from (29)–(31) that the converse assertion also holds, which proves the theorem.

The following result is an immediate consequence of (24), (27) and the equality $\|\operatorname{div}\mathbf{u}\|_0^2=|\mathbf{u}|_{\mathbf{U}}^2$, $\mathbf{u}\in\mathbf{V}$.

Corollary 2. *The constants $c_0(\Omega)$ in (24) and $c_1(\Omega)$ in (27) are related by the equality $c_0^2(\Omega)=(1-c_1^2(\Omega))^{-1}$.*

We note that in the notation introduced above the model boundary conditions (7) can be written as follows:

$$\begin{aligned}&\tau^t\cdot(\nabla\mathbf{u})\cdot\nu-\nu^t\cdot(\nabla\mathbf{u})\cdot\tau=0\quad\text{on }\partial\Omega\quad\text{for }n=2;\\&\tau_i^t\cdot(\nabla\mathbf{u})\cdot\nu-\nu^t\cdot(\nabla\mathbf{u})\cdot\tau_i=0\quad\text{on }\partial\Omega,\ i=1,2,\quad\text{for }n=3.\end{aligned}\tag{32}$$

In the case $n=2$ the equivalence of (7) and (32) can be verified directly; if $n=3$, then we must set the vectors τ_i in (7) equal to $\tau_i'\times\nu$, where the τ_i', $i=1,2$, are the tangent vectors in (32).

In the case of a polygon the term $\tau^t\,(\nabla\mathbf{u})\,\nu$ in (32) corresponds to the conventional notation $\partial(\mathbf{u}\cdot\tau)/\partial\nu$ used for setting the periodic boundary conditions in a rectangle (see (4)), while $\nu^t(\nabla\mathbf{u})\tau=\partial(\mathbf{u}\cdot\nu)/\partial\tau=0$. Writing the model boundary conditions in the form (32) makes more explicit their generalization to the case of $n>3$.

§ 5. Stokes problem with a parameter

We now consider the Stokes problem with model boundary conditions and with a parameter in a bounded simply connected domain with Lipschitz boundary $\Omega\subset\mathbb{R}^n$, $n=2,3$:

$$\begin{gathered}-\Delta\mathbf{u}+\alpha\mathbf{u}+\nabla p=\mathbf{f},\\\operatorname{div}\mathbf{u}=g,\\\mathbf{u}\cdot\nu\big|_{\partial\Omega}=\mathcal{R}\mathbf{u}\big|_{\partial\Omega}=0,\end{gathered}\tag{33}$$

where $\alpha\in[0,\infty)$, $\mathbf{f}\in\mathbf{U}^{-1}$, and $g\in L_2/\mathbb{R}$.

The problem with a parameter arises, for example, in computational hydrodynamics, in the analysis of semi-implicit schemes for the Stokes and Navier–Stokes equations. In this case $\alpha \sim (\upsilon\,\delta t)^{-1}$, where υ is the kinematic viscosity and δt is the time step.

The generalized formulation of (33) consists in finding $\{\mathbf{u}, p\}$ in $\mathbf{U} \times L_2/\mathbb{R}$ such that

$$(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle\,,$$
$$(\operatorname{div}\mathbf{u}, q) = (g, q) \quad \text{for all } \mathbf{v} \in \mathbf{U} \quad \text{and } q \in L_2/\mathbb{R}.$$

The results on the solubility of the problem (33) and the regularity of its solutions are literally the same as for problem (11). We now consider the Schur complement

$$\mathbf{A}_\nu(\alpha) \equiv \operatorname{div}(\Delta - \alpha\mathbf{I})_\nu^{-1}\nabla,$$

where $\mathbf{I}$ is the identity operator. The following result is a generalization of Theorem 1.

Theorem 6. *Under the hypothesis of Theorem* 1, *for any $\alpha \in [0, \infty)$, $p \in L_2/\mathbb{R}$ and for $q = \mathbf{A}_\nu(\alpha)p$, $q \in L_2/\mathbb{R}$, we have*

$$p = q - \alpha\Delta_N^{-1}q, \tag{34}$$

where $r = \Delta_N^{-1}q$ is the solution of the Neumann problem

$$\Delta r = q, \qquad \left.\frac{\partial r}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Proof. If $\alpha = 0$, then the assertion of the theorem coincides with Theorem 1. We now suppose that $\alpha > 0$. Then an arbitrary function p in $L_2/\mathbb{R}$ and $q = \mathbf{A}_\nu(\alpha)p$ are related by the formulae

$$\begin{gathered} -\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p = 0, \\ \operatorname{div}\mathbf{u} = q, \\ \mathbf{u}\cdot\nu\big|_{\partial\Omega} = \mathcal{R}\mathbf{u}\big|_{\partial\Omega} = 0. \end{gathered} \tag{35}$$

We consider the functions $p_1 = -\alpha\Delta_N^{-1}q$ and $\mathbf{u}_1 = \alpha^{-1}\nabla p_1$ and we define $\mathbf{u}_2 \in \mathbf{U}$ from the system

$$-\Delta\mathbf{u}_2 = \nabla q, \qquad \mathbf{u}_2\cdot\nu\big|_{\partial\Omega} = \mathcal{R}\mathbf{u}_2\big|_{\partial\Omega} = 0.$$

Then $\Delta\mathbf{u}_1 = \nabla\operatorname{div}\mathbf{u}_1 = -\nabla q = \Delta\mathbf{u}_2$, and since $\mathbf{u}_1 \in \mathbf{U}$ and $\mathcal{R}\mathbf{u}_1\big|_{\partial\Omega} = 0$, it follows that $\mathbf{u}_1 = \mathbf{u}_2$.

We now set $\widetilde{p} = q + p_1$. Then the functions $\widetilde{\mathbf{u}} = \mathbf{u}_1 = \mathbf{u}_2$ and $\widetilde{p}$ satisfy the system

$$\begin{gathered} -\Delta\widetilde{\mathbf{u}} + \alpha\widetilde{\mathbf{u}} + \nabla\widetilde{p} = 0, \\ \operatorname{div}\widetilde{\mathbf{u}} = q, \\ \widetilde{\mathbf{u}}\cdot\nu\big|_{\partial\Omega} = \mathcal{R}\widetilde{\mathbf{u}}\big|_{\partial\Omega} = 0. \end{gathered} \tag{36}$$

From the uniqueness of the solution of (33) and formulae (35) and (36) we see that $p = \widetilde{p} = q - \alpha\Delta_N^{-1} q$, which proves the theorem.

We observe that equality (34) makes it possible to generalize the construction and analysis of the efficient preconditioning of the Uzawa algorithm for the numerical solution of the Stokes problem with a parameter and with Dirichlet boundary conditions (see [9], [14], and [23]) to a wide class of domains.

The following result is a generalization of Theorem 3.

Theorem 7. *Let* Ω *be a bounded simply connected domain satisfying condition* (I). *Let* $\mathbf{f} \in L_2(\Omega)^n$ *and let* $g \in W_2^1(\Omega) \cap L_2/\mathbb{R}$. *Then the solution* $p \in W_2^1(\Omega) \cap L_2/\mathbb{R}$ *of the problem* (33) *solves the equation*

$$\Delta p = \Delta g - \alpha g + \operatorname{div} \mathbf{f},$$
$$\frac{\partial p}{\partial \nu} = \frac{\partial g}{\partial \nu} + \mathbf{f} \cdot \nu$$

in the generalized sense.

The proof of Theorem 7 is the same as that of Theorem 3.

Conclusion

From the variational standpoint, we can treat the pressure in the Stokes system as a Lagrange multiplier corresponding to the constraint $\operatorname{div} \mathbf{u} = 0$ on the velocity field. Hence the conventional point of view that, under the incompressibility condition, the velocity and the pressure are inseparable in principle is quite justified. Our example of the problem with model boundary conditions shows that there are exceptions to this rule. Corollary 1 and Theorem 2 demonstrate that in the Stokes problem with model boundary conditions we can find the pressure by solving one vector Poisson equation with homogeneous boundary conditions of the third kind or one scalar Poisson equation with inhomogeneous Neumann conditions. The velocity field is found separately, which additionally requires the solution of a vector Poisson equation.

The relaxation of the conditions imposed on the domain to the weakest possible ones (from our point of view) has required additional mathematical 'effort', in particular, the consideration of function spaces less 'familiar' than the Sobolev spaces. This has brought us to a greater completeness of results.

Finally, the natural generalization of the results to the Stokes problem with a parameter seems to be important in the further analysis of this problem with Dirichlet boundary conditions from the point of view of numerical analysis.

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