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Error analysis of a projection method for the Navier–Stokes equations with Coriolis force

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Abstract. In this paper a projection method for the Navier–Stokes equations with Coriolis force is considered. This time-stepping algorithm takes into account the Coriolis terms both on prediction and correction steps. We study the accuracy of its semi-discretized form and show that the velocity is weakly first-order approximation and the pressure is weakly order $\frac{1}{2}$ approximation.

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1. Introduction

In many physical and industrial applications there is the necessity of numerical simulations for CFD models with moving geometries. In the literature one can find several techniques for handling such type of problems. Among them are fictitious domain [4], resp., fictitious boundary [18, 19] and arbitrary lagrangian eulerian [3] methods. Although being quite popular these methods require often a large amount of CPU time to simulate even 2D benchmark models if high accuracy is desired. Moreover, their handling of geometry and meshes serves as a source of additional errors in velocity and pressure fields. For example, the fictitious boundary approach often uses a fixed mesh and therefore may capture boundaries of a moving object not sufficiently accurate unless the mesh is very fine. At the same time, there is a large class of "rotating" models, when the application of

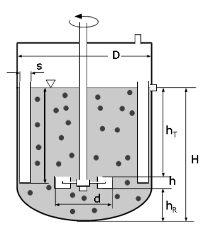

Fig. 1. STR geometry.

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the above methods can be avoided by some modifications of the underlying PDEs and/or by special transformations of the model that allow considering a static computational domain. As an example, let us consider the numerical simulation of a Stirred Tank Reactor benchmark problem (Fig. 1).

The motion of an incompressible Newtonian fluid in the tank is modeled by the system of Navier–Stokes equations

$$\begin{aligned} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla q &= \boldsymbol{f} \\ \operatorname{div}\boldsymbol{v} &= 0 \end{aligned} \quad \text{in } \ \Omega\times(0,T], \tag{1}$$

where Ω is an open bounded domain with sufficiently smooth boundary Γ, $\boldsymbol{f}$ is a given force and $\nu > 0$ is a kinematic viscosity. Changing the inertial frame of reference to the noninertial frame rotating with the blades leads to the following system:

$$\begin{aligned} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + 2\boldsymbol{\omega}\times\boldsymbol{u} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) + \nabla q &= \boldsymbol{f}, \\ \operatorname{div}\boldsymbol{u} &= 0, \end{aligned} \tag{2}$$

where $\boldsymbol{\omega}$ is the angular velocity vector, $\boldsymbol{r}$ is the radius vector from the center of coordinates, $2\boldsymbol{\omega}\times\boldsymbol{u}$ and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ are the so-called Coriolis and centrifugal forces, respectively, and $\boldsymbol{u} = \boldsymbol{v} + (\boldsymbol{\omega}\times\boldsymbol{r})$. For a more detailed derivation of (2) see, e.g., [1]. Using the equality

$$\boldsymbol{\omega}\times\ (\boldsymbol{\omega}\times\boldsymbol{r}) = -\nabla\frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{r})^2$$

and setting $p = q - \frac{1}{2}(\boldsymbol{\omega}\times\boldsymbol{r})^2$ in (2), we get the following system of equations

$$\begin{aligned} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + 2\boldsymbol{\omega}\times\boldsymbol{u} + \nabla p &= \boldsymbol{f} \\ \operatorname{div}\boldsymbol{u} &= 0 \end{aligned} \quad \text{in } \ \Omega\times(0,T]\ , \tag{3}$$

which will be treated in this paper. Exclusively for the purpose of analysis we assume homogeneous Dirichlet boundary conditions $\boldsymbol{u}|_\Gamma = 0$.

To handle effectively the possibly dominating Coriolis force we modify the classical projection scheme [2, 15] in the following way: Given $\boldsymbol{u}^n \approx \boldsymbol{u}(t_n)$

Step 1: Find intermediate velocity $\tilde{\boldsymbol{u}}^{n+1}$ from

$$\left\{ \begin{aligned} \frac{1}{k}(\tilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n) - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} + \boldsymbol{\omega}\times\tilde{\boldsymbol{u}}^{n+1} &= \boldsymbol{f}(t_{n+1}), \\ \tilde{\boldsymbol{u}}^{n+1}|_\Gamma &= 0. \end{aligned} \right. \tag{4}$$

Step 2: Find new velocity and pressure $\{\boldsymbol{u}^{n+1}, p^{n+1}\}$ as the result of the projection into the divergence-free subspace

$$\left\{ \begin{aligned} \frac{1}{k}(\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}) + \boldsymbol{\omega}\times(\boldsymbol{u}^{n+1} - \tilde{\boldsymbol{u}}^{n+1}) + \nabla p^{n+1} &= 0, \\ \operatorname{div}\boldsymbol{u}^{n+1} &= 0, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}|_\Gamma &= 0, \end{aligned} \right. \tag{5}$$

where k is the time step, $t_{n+1} = (n+1)k$, and $\boldsymbol{n}$ is the normal vector to Γ. One notes that the essential modification of the well-known Chorin–Temam method is introduced on the correction step 2, which is not an orthogonal projection any more. The rationale and motivation of this modified scheme can be found in [13], where the scheme is treated as an incomplete LU factorization of the transition operator for fully implicit time discretization. Numerical experiments from [13, 14] show that including $\boldsymbol{\omega}$-terms in (5) enhances stability and accuracy of the scheme for the case of dominating Coriolis forces. The present paper deals with convergence analysis for the method (4)–(5).

A well established framework for numerical analysis of projection schemes is the following, see [9, 10]: one deduces an equivalent pseudo-compressibility or penalty method and further treats a projection scheme as the discretization of perturbed Navier–Stokes equations. However, applying this approach to (4)–(5) leads to a number of additional terms depending on $\boldsymbol{\omega}$, which are not easy to handle. Therefore we analyse the problem using the techniques developed by J. Shen in [11, 12] for the case of $\boldsymbol{\omega} = 0$. Although the arguments in [11, 12] essentially use the fact that the projection on step 2 is orthogonal, we show that the similar convergence results can be proved for the modified method (4)–(5). Finally, although we discuss only the first order scheme in this paper, the second order modification of (4)–(5) can be build in a standard way, cf. [13].

2. Preliminaries

Below we use the following notation:

$$|\cdot|^2 = \int_\Omega |\cdot|^2 dx, \quad \|\cdot\|^2 = \int_\Omega |\nabla\cdot|^2 dx, \quad \|\cdot\|_s\text{-norm in } H^s(\Omega).$$

By $(\cdot,\cdot)$ we will denote the inner product in $L^2(\Omega)$ and by $\langle\cdot,\cdot\rangle$ – the duality between H^{-s} and H_0^{-s} for all $s > 0$. We also define

$$\begin{aligned} H &= \left\{\boldsymbol{u} \in \left(L^2(\Omega)\right)^d : \operatorname{div} \boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\boldsymbol{n}|_\Gamma = 0\right\}, \\ V &= \left\{\boldsymbol{v} \in \left(H_0^1(\Omega)\right)^d : \operatorname{div} \boldsymbol{v} = 0\right\}. \end{aligned}$$

In the following, we assume

$$\begin{cases} \boldsymbol{u}_0 \in (H^2(\Omega))^d \cap V, \\ \boldsymbol{f} \in L^\infty\big(0,T;(L^2(\Omega))^d\big) \cap L^2\big(0,T;(H^1(\Omega))^d\big), \\ \boldsymbol{f}_t \in L^2(0,T;H^{-1}), \\ \sup_{t\in[0,T]} \|\boldsymbol{u}(t)\| \le m_1. \end{cases} \tag{6}$$

We will use c or C as a generic positive constant which depends only on Ω, ν, T, and constants from various Sobolev inequalities. We will denote m or M as a generic positive constant which may additionally depend on $\boldsymbol{u}_0$, $\boldsymbol{f}$, $\boldsymbol{\omega}$ and the solution $\boldsymbol{u}$ through the constant m_1 in (6).

Under the assumption (6) one can prove the following inequalities

$$\sup_{t\in[0,T]}\{\|\boldsymbol{u}(t)\|_2+|\boldsymbol{u}_t(t)|+|\nabla p(t)|\}\le M, \tag{7}$$

$$\int_0^T \|\boldsymbol{u}_t(t)\|^2+t|\boldsymbol{u}_{tt}|^2 dt\le M, \tag{8}$$

which will be used in the sequel. Indeed, in [6] the estimates (7)–(8) were proved for the Navier–Stokes equations (1) without Coriolis term. However adding *linear skew-symmetric* term $\boldsymbol{\omega}\times\boldsymbol{u}$ to the momentum equation does not change arguments from [6], but leads to (7)–(8) with constant M depending, in general, on $\boldsymbol{\omega}$. Further we often use the following well-known [16] estimates for the bilinear form $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}$:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\le\begin{cases} c\|\boldsymbol{u}\|\|\boldsymbol{v}\|^{\frac12}|\boldsymbol{v}|^{\frac12}\|\boldsymbol{w}\|,\\ c\|\boldsymbol{u}\|_2|\boldsymbol{v}|\|\boldsymbol{w}\|,\\ c\|\boldsymbol{u}\|\|\boldsymbol{v}\|_2|\boldsymbol{w}|.\end{cases} \tag{9}$$

and $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$ for $\boldsymbol{u}\in H$.

Let P_H be the orthogonal projector in $\left(L^2(\Omega)\right)^d$ onto H and define the Stokes operator $A\boldsymbol{u}=-P_H\Delta\boldsymbol{u},\ \ \forall\boldsymbol{u}\in D(A)=V\cap\left(H^2(\Omega)\right)^d$. We will use the following properties: A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$, and its inverse A^{-1} is compact in H and satisfies the following relations [11, 12][1]:

$$\exists c,C>0,\ \text{such that}\ \forall\boldsymbol{u}\in H:\begin{cases}\|A^{-1}\boldsymbol{u}\|_2\le c|\boldsymbol{u}|\ \text{ and }\ \|A^{-1}\boldsymbol{u}\|\le c\|\boldsymbol{u}\|_{V'},\\ \quad c\|\boldsymbol{u}\|^2_{V'}\le(A^{-1}\boldsymbol{u},\boldsymbol{u})\le C\|\boldsymbol{u}\|^2_{V'}.\end{cases} \tag{10}$$

Further in this section we will prove several auxiliary lemmas. The first lemma shows that the projection (5) is uniformly (with respect to k) stable in H^1. Another two preliminary lemmas extend the results of Lemma 2 from [11] and Lemma A1 from [12] for the case of $\boldsymbol{\omega}\neq0$ and non-orthogonal projection in (5). We also note that in [12] the similar result was proved only for the Stokes case (no nonlinear terms has been treated). We include the nonlinear terms in the analysis and encounter additional assumption on the size of the time step.

Lemma 2.1. *The estimate*

$$\|\boldsymbol{u}^{n+1}\|_1\le\tilde m\|\tilde{\boldsymbol{u}}^{n+1}\|_1$$

holds with some $\tilde m$ *independent of* $k\in(0,1]$.

[1] In [11] the estimates (10) are given with H^{-1} norm instead of V'. In [12] it was noted that the low bound for $(A^{-1}\boldsymbol{u},\boldsymbol{u})$ should actually involve the V' norm. Other estimates in (10) can be equally used with both norms.

Proof. First note that the pressure p^{n+1} from (5) satisfies the following elliptic equation

$$\operatorname{div} \mathcal{M}^{-1} \nabla p^{n+1} = \frac{1}{k} \operatorname{div} \tilde{\boldsymbol{u}}^{n+1}, \tag{11}$$

$$\left[\mathcal{M}^{-1} \nabla p^{n+1}\right] \cdot \boldsymbol{n}|_{\Gamma} = 0 \tag{12}$$

with $\mathcal{M} = [I + k\boldsymbol{\omega}\times]$. One can verify [8] that for $d = 3$ it holds

$$\mathcal{M}^{-1} = (1 + |\tilde{\boldsymbol{\omega}}|^2)^{-1}[I + \tilde{\boldsymbol{\omega}} \otimes \tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\times], \quad \tilde{\boldsymbol{\omega}} = k\,\boldsymbol{\omega}, \tag{13}$$

where $(\tilde{\boldsymbol{\omega}} \otimes \tilde{\boldsymbol{\omega}})_{ij} = \tilde{\boldsymbol{\omega}}_i\, \tilde{\boldsymbol{\omega}}_j$. (For the 2D case the identity (13) holds without $\tilde{\boldsymbol{\omega}} \otimes \tilde{\boldsymbol{\omega}}$ term.) Since $\tilde{\boldsymbol{\omega}}$ is a constant vector one has $\tilde{\boldsymbol{\omega}} \times \nabla q = \nabla \times (q\tilde{\boldsymbol{\omega}})$ for a scalar function q. Therefore $\operatorname{div}(\tilde{\boldsymbol{\omega}} \times \nabla q) = 0$ and the equation (11) can be written as

$$\operatorname{div} \mathcal{B}\, \nabla p^{n+1} = \frac{1}{k} \operatorname{div} \tilde{\boldsymbol{u}}^{n+1} \tag{14}$$

with the *symmetric* diffusion tensor $\mathcal{B} = \dfrac{1}{1 + |\tilde{\boldsymbol{\omega}}|^2}[I + \tilde{\boldsymbol{\omega}} \otimes \tilde{\boldsymbol{\omega}}]$. One can easily see that the inequalities

$$m_1 |\xi|^2 \le (\mathcal{B}\xi, \xi) \le m_2 |\xi|^2 \tag{15}$$

hold with m_1 and m_2 independent on k, e.g. $m_1 = \dfrac{1}{1 + |\tilde{\boldsymbol{\omega}}|^2}$, $m_2 = 1$. (For the 2D case $\mathcal{B}$ is the scaled identity matrix.) Furthermore, the boundary condition (12) can be rewritten as

$$\left.\frac{\partial p^{n+1}}{\partial \boldsymbol{l}}\right|_{\Gamma} = 0 \quad \text{with} \quad \boldsymbol{l} = \mathcal{M}^{-1}\boldsymbol{n}.$$

The angle $\phi(\boldsymbol{x})$ between the vector $\boldsymbol{l}(\boldsymbol{x})$ and tangential plane to Γ at $\boldsymbol{x} \in \Gamma$ is uniformly bounded from below. Indeed, it holds:

$$|\sin \phi| = \frac{|\boldsymbol{l}^T \cdot \boldsymbol{n}|}{\boldsymbol{l}^T \cdot \boldsymbol{l}} = \frac{|\boldsymbol{n}^T \mathcal{M}^{-1} \boldsymbol{n}|}{\boldsymbol{n}^T \mathcal{M}^{-T} \mathcal{M}^{-1} \boldsymbol{n}} \ge \frac{|\boldsymbol{n}^T \mathcal{B} \boldsymbol{n}|}{\|\mathcal{M}^{-1}\|^2} \ge \frac{m_1}{4}. \tag{16}$$

Here we used the identity $\mathcal{M}^{-T} + \mathcal{M}^{-1} = 2\mathcal{B}$, inequalities (15) and $\|\mathcal{M}^{-1}\| \le 2$. Thus the smoothness assumption on Ω, (15) and (16) imply the following H^2 estimate for the solution of (11)–(12) [7]:

$$\|p^{n+1}\|_2 \le m\, k^{-1} |\operatorname{div} \tilde{\boldsymbol{u}}^{n+1}| \le m\, k^{-1} \|\tilde{\boldsymbol{u}}^{n+1}\|_1$$

with some constant c independent of k. Finally, using this result we get from (5) and the triangle inequality

$$\begin{aligned} \|\boldsymbol{u}^{n+1}\|_1 &\le \|\tilde{\boldsymbol{u}}^{n+1}\|_1 + k \|\mathcal{M}^{-1} \nabla p^{n+1}\| \\ &\le \|\tilde{\boldsymbol{u}}^{n+1}\|_1 + k \|\mathcal{M}^{-1}\| \|p^{n+1}\|_2 \le m\, \|\tilde{\boldsymbol{u}}^{n+1}\|_1. \end{aligned}$$

□

It is straightforward to check that the solution to (11)–(12) satisfies the estimate

$$|\mathcal{M}^{-1}\nabla p^{n+1}| \le m\, k^{-1}|\tilde{\boldsymbol{u}}^{n+1}|.$$

Thus the projection (5) is also uniformly stable in L^2:

$$|\boldsymbol{u}^{n+1}| \le |\tilde{\boldsymbol{u}}^{n+1}| + k|\mathcal{M}^{-1}\nabla p^{n+1}| \le m\,|\tilde{\boldsymbol{u}}^{n+1}|. \tag{17}$$

Lemma 2.2. *Denote*

$$e^{n+1} = \boldsymbol{u}(t_{n+1}) - \boldsymbol{u}^{n+1} \quad \text{and} \quad \tilde{e}^{n+1} = \boldsymbol{u}(t_{n+1}) - \tilde{\boldsymbol{u}}^{n+1}.$$

Assume (6) *and* $2k|\boldsymbol{\omega}|^2 \le 1$. *It holds:*

$$\begin{aligned}|e^{N+1}|^2 + |\tilde{e}^{N+1}|^2 + k\nu\sum_{n=0}^{N}\big\{\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2\big\}\\ + \sum_{n=0}^{N}\big\{|e^{n+1}-\tilde{e}^{n+1}|^2 + |\tilde{e}^{n+1}-e^n|^2\big\} \le m\,k,\ \ \forall\ 0 \le N \le T/k-1.\end{aligned} \tag{18}$$

Proof. Let R^n be the truncation error defined by

$$\begin{aligned}\frac{1}{k}(\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)) - \nu\Delta\boldsymbol{u}(t_{n+1}) + \boldsymbol{\omega}\times\boldsymbol{u}(t_{n+1})\\ + (\boldsymbol{u}(t_{n+1})\cdot\nabla)\boldsymbol{u}(t_{n+1}) + \nabla p(t_{n+1}) = \boldsymbol{f}(t_{n+1}) + R^n,\end{aligned} \tag{19}$$

where R^n is the integral residual of the Taylor series, i.e.,

$$R^n = \frac{1}{k}\int_{t_n}^{t_{n+1}}(t-t_n)\boldsymbol{u}_{tt}(t)dt.$$

By subtracting (4) from (19), we obtain

$$\begin{aligned}\frac{1}{k}(\tilde{e}^{n+1} - e^n) - \nu\Delta\tilde{e}^{n+1} + \boldsymbol{\omega}\times\tilde{e}^{n+1}\\ = (\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1} - (\boldsymbol{u}(t_{n+1})\cdot\nabla)\boldsymbol{u}(t_{n+1}) + R^n - \nabla p(t_{n+1}).\end{aligned} \tag{20}$$

Taking the inner product of (20) with $2k\tilde{e}^{n+1}$ and using the identity

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2,$$

we derive

$$\begin{aligned}|\tilde{e}^{n+1}|^2 - |e^n|^2 + |\tilde{e}^{n+1} - e^n|^2 + 2k\nu\|\tilde{e}^{n+1}\|^2 + \big(\boldsymbol{\omega}\times\tilde{e}^{n+1}, 2k\tilde{e}^{n+1}\big)\\ = 2k(R^n, \tilde{e}^{n+1}) - 2k\big(\nabla p(t_{n+1}), \tilde{e}^{n+1}\big) - 2kb\big(e^n, \tilde{\boldsymbol{u}}^{n+1}, \tilde{e}^{n+1}\big)\\ + 2kb\big(\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n+1}), \tilde{\boldsymbol{u}}^{n+1}, \tilde{e}^{n+1}\big) - 2kb\big(\boldsymbol{u}(t_{n+1}), \tilde{e}^{n+1}, \tilde{e}^{n+1}\big).\end{aligned} \tag{21}$$

Since the Coriolis term vanishes: $(\boldsymbol{\omega}\times\tilde{e}^{n+1}, 2k\tilde{e}^{n+1}) = 0$, using the same arguments as in [11], see pages 64–65, and applying inequality (3) from [12] one deduces from (21) the estimate

$$\begin{aligned}&|\tilde{e}^{n+1}|^2 - |e^n|^2 + |\tilde{e}^{n+1} - e^n|^2 + 2k\nu\|\tilde{e}^{n+1}\|^2 \\ &\leq \bar{m}\,k\left(\int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}\|_{-1}^2 \mathrm{d}t + k\int_{t_n}^{t_{n+1}} |\boldsymbol{u}_t|^2 \mathrm{d}t\right) + 2k^2|\nabla p(t_{n+1})|^2 + \bar{m}\,k|e^n|^2. \quad (22)\end{aligned}$$

It follows from (5)

$$\frac{1}{k}(e^{n+1} - \tilde{e}^{n+1}) - \nabla p^{n+1} + \boldsymbol{\omega}\times(e^{n+1} - \tilde{e}^{n+1}) = 0. \quad (23)$$

Taking the inner product of (23) with $2ke^{n+1}$, we get

$$|e^{n+1}|^2 - |\tilde{e}^{n+1}|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2 - 2k(\boldsymbol{\omega}\times\tilde{e}^{n+1}, e^{n+1} - \tilde{e}^{n+1}) = 0.$$

Then

$$|e^{n+1}|^2 - |\tilde{e}^{n+1}|^2 + |e^{n+1} - \tilde{e}^{n+1}|^2 - 2k^2|\boldsymbol{\omega}|^2|\tilde{e}^{n+1}|^2 - \frac{1}{2}|e^{n+1} - \tilde{e}^{n+1}|^2 \leq 0,$$

$$|e^{n+1}|^2 - (1 + k\tilde{m})|\tilde{e}^{n+1}|^2 + \frac{1}{2}|e^{n+1} - \tilde{e}^{n+1}|^2 \leq 0, \quad (24)$$

with $\tilde{m} = 2k|\boldsymbol{\omega}|^2$. Inequality (24) yields

$$(1 + k\tilde{m})^{-1}|e^{n+1}|^2 - |\tilde{e}^{n+1}|^2 + \frac{1}{2(1 + k\tilde{m})}|e^{n+1} - \tilde{e}^{n+1}|^2 \leq 0. \quad (25)$$

Since $k\tilde{m} = 2k^2|\boldsymbol{\omega}|^2 \leq 1$ under the natural assumption $k \leq 1$ and using $(1 - b) \leq (1 + b)^{-1}$, for $b \in [0, 1]$, from (25) we derive

$$|e^{n+1}|^2 - |\tilde{e}^{n+1}|^2 + \frac{1}{2(1 + k\tilde{m})}|e^{n+1} - \tilde{e}^{n+1}|^2 \leq k\tilde{m}|e^{n+1}|^2. \quad (26)$$

Taking the sum of (22) and (26) for $n = 0, \ldots, N$ $(0 \leq N \leq T/k - 1)$, we obtain

$$\begin{aligned}&|e^{N+1}|^2 + \sum_{n=0}^{N}\left\{\frac{1}{2(1 + k\tilde{m})}|e^{n+1} - \tilde{e}^{n+1}|^2 + \frac{1}{2}|\tilde{e}^{n+1} - e^n|^2 + k\nu\|\tilde{e}^{n+1}\|^2\right\} \\ &\leq \bar{m}\,k\sum_{n=0}^{N}|e^n|^2 + \bar{m}\,k\left(\int_0^T t\|\boldsymbol{u}_{tt}\|_{-1}^2 dt + k\int_0^T |\boldsymbol{u}_t|^2 dt + \sup_{t\in[0,T]}|\nabla p(t)|^2\right) \\ &\quad + \sum_{n=0}^{N} k\tilde{m}|e^{n+1}|^2.\end{aligned}$$

Denoting $m=\max\{\bar m,\tilde m\}$, we can rewrite the previous inequality as

$$\begin{aligned}&|e^{N+1}|^2+\sum_{n=0}^{N}\left\{\frac{1}{2(1+k\tilde m)}|e^{n+1}-\tilde e^{n+1}|^2+\frac12|\tilde e^{n+1}-e^n|^2+k\nu\|\tilde e^{n+1}\|^2\right\}\\&\le m\,k\sum_{n=0}^{N}|e^n|^2+m\,k\left(\int_0^T t\|\boldsymbol{u}_{tt}\|_{-1}^2dt+k\int_0^T|\boldsymbol{u}_t|^2dt+\sup_{t\in[0,T]}|\nabla p(t)|^2\right)\\&\quad+k\tilde m|e^{N+1}|^2.\end{aligned}$$

Thanks to the condition $2k|\boldsymbol{\omega}|^2\le 1$ and (7)–(8) one can write

$$\begin{aligned}&|e^{N+1}|^2+\sum_{n=0}^{N}\left\{|e^{n+1}-\tilde e^{n+1}|^2+\frac12|\tilde e^{n+1}-e^n|^2+k\nu\|\tilde e^{n+1}\|^2\right\}\\&\le m\,k\sum_{n=0}^{N}|e^n|^2+m\,k\left(\int_0^T t\|\boldsymbol{u}_{tt}\|_{-1}^2dt+k\int_0^T|\boldsymbol{u}_t|^2dt+\sup_{t\in[0,T]}|\nabla p(t)|^2\right)\\&\le m\,k\sum_{n=0}^{N}|e^n|^2+m\,k.\end{aligned}$$

Applying the discrete Gronwall lemma to the last inequality, we arrive at

$$|e^{N+1}|^2+\sum_{n=0}^{N}\left\{|e^{n+1}-\tilde e^{n+1}|^2+|\tilde e^{n+1}-e^n|^2+k\nu\|\tilde e^{n+1}\|^2\right\}\le m\,k. \tag{27}$$

Further, Lemma 2.1 provides the estimate

$$\|e^{n+1}\|_1\le\tilde m\|\tilde e^{n+1}\|_1. \tag{28}$$

Applying (28) and the triangle inequality $|\tilde e^{n+1}|\le|e^{n+1}|+|e^{n+1}-\tilde e^{n+1}|$ and (27), we also obtain

$$|\tilde e^{N+1}|^2+k\nu\sum_{n=0}^{N}\|e^{n+1}\|^2\le m\,k.$$

This proves the lemma. □

Lemma 2.3. *Assume* (6) *and*

$$\int_0^T|\nabla p_t|^2\le m\,. \tag{29}$$

Moreover, assume that k is sufficiently small, then it holds

$$\sum_{n=0}^{N}|\tilde e^{n+1}-\tilde e^n|^2+k\|\tilde e^{N+1}\|^2\le m\,k^2\qquad\forall\ 0\le N\le T/k-1.$$

Proof. We shift the index $n+1 \to n$ in (23) and take the sum with (20). This brings us to

$$
\begin{aligned}
&\frac{1}{k}(\tilde e^{n+1}-\tilde e^n)-\nu\Delta\tilde e^{n+1}+\boldsymbol{\omega}\times(\tilde e^{n+1}-\tilde e^n)\\
&=(\boldsymbol{u}^n\cdot\nabla)\tilde{\boldsymbol{u}}^{n+1}-(\boldsymbol{u}(t_{n+1})\cdot\nabla)\boldsymbol{u}(t_{n+1})+R^n-\nabla(p(t_{n+1})-p^n)-\boldsymbol{\omega}\times e^n. \quad (30)
\end{aligned}
$$

We take the inner product of (30) with $k(\tilde e^{n+1}-\tilde e^n)$ and obtain

$$
\begin{aligned}
&|\tilde e^{n+1}-\tilde e^n|^2+\frac{k\nu}{2}\left(\|\tilde e^{n+1}\|^2-\|\tilde e^n\|^2+\|\tilde e^{n+1}-\tilde e^n\|^2\right)\\
&=-k\big(\boldsymbol{\omega}\times e^n,\tilde e^{n+1}-\tilde e^n\big)+k\big(R^n,\tilde e^{n+1}-\tilde e^n\big)+k\big(p(t_{n+1})-p^n,\operatorname{div}(\tilde e^{n+1}-\tilde e^n)\big)\\
&\qquad+kb\big(\boldsymbol{u}^n,\tilde{\boldsymbol{u}}^{n+1},\tilde e^{n+1}-\tilde e^n\big)-kb\big(\boldsymbol{u}(t_{n+1}),\boldsymbol{u}(t_{n+1}),\tilde e^{n+1}-\tilde e^n\big). \quad (31)
\end{aligned}
$$

Now we estimate terms on the right-hand side of (31). Below δ is a positive constant to be determined later. Using (18) we get

$$
-k(\boldsymbol{\omega}\times e^n,\tilde e^{n+1}-\tilde e^n)\le\delta|\tilde e^{n+1}-\tilde e^n|^2+m\,k^2|e^n|^2\le\delta|\tilde e^{n+1}-\tilde e^n|^2+m\,k^3. \quad (32)
$$

Thanks to the estimate $|R^n|^2\le c\int_{t_n}^{t_{n+1}}t|\boldsymbol{u}_{tt}|^2\mathrm{d}t$ from [12] we have

$$
k\langle R^n,\tilde e^{n+1}-\tilde e^n\rangle\le\delta|\tilde e^{n+1}-\tilde e^n|^2+c\,k^2\int_{t_n}^{t_{n+1}}t|\boldsymbol{u}_{tt}|^2\mathrm{d}t. \quad (33)
$$

Let us estimate the pressure-dependent term. Denote $q^n=p(t_{n+1})-p^n$, since $\operatorname{div}e^{n+1}=0$, resp. div $e^n=0$, we obtain

$$
k(p(t_{n+1})-p^n,\operatorname{div}(\tilde e^{n+1}-\tilde e^n))=k(\nabla q^n,\tilde e^n-\tilde e^{n+1}-e^n+e^{n+1}). \quad (34)
$$

Then we deduce from (23) and (34):

$$
\begin{aligned}
&k\big(p(t_{n+1})-p^n,\operatorname{div}(\tilde e^{n+1}-\tilde e^n)\big)\\
&=k^2\big(\nabla q^n,\nabla(p^{n+1}-p^n)\big)+k^2\big(\nabla q^n,\boldsymbol{\omega}\times(\tilde e^{n+1}-e^{n+1}-\tilde e^n+e^n)\big)\\
&\le-k^2\big(\nabla q^n,\nabla(q^{n+1}-q^n)\big)+k^2\big(\nabla q^n,\nabla(p(t_{n+2})-p(t_{n+1}))\big)\\
&\quad+k^2\big(\nabla q^n,\boldsymbol{\omega}\times(\tilde e^{n+1}-e^{n+1})-k^2(\nabla q^n,\boldsymbol{\omega}\times(\tilde e^n-e^n)\big). \quad (35)
\end{aligned}
$$

We estimate the terms on the right-hand side of (35) separately:

$$
-k^2\big(\nabla q^n,\nabla(q^{n+1}-q^n)\big)=\frac{k^2}{2}\big(\|q^n\|^2-\|q^{n+1}\|^2+\|q^{n+1}-q^n\|^2\big). \quad (36)
$$

We obtain from (23) the following relation:

$$
k\mathcal{M}^{-1}\nabla(q^{n+1}-q^n)=(\tilde e^{n+1}-e^{n+1})-(\tilde e^n-e^n)+k\mathcal{M}^{-1}\nabla\big(p(t_{n+2})-p(t_{n+1})\big).
$$

Multiplying by $\nabla(q^{n+1}-q^n)$ and using (15) and condition $k|\boldsymbol{\omega}|\le\frac{1}{2}$ we get

$$
\begin{aligned}
&k^2\|q^{n+1}-q^n\|^2 \le \frac{5}{4}k^2\big(\mathcal{M}^{-1}\nabla(q^{n+1}-q^n),\nabla(q^{n+1}-q^n)\big)\\
&\le \frac{5}{4}k\big(\tilde e^{n+1}-\tilde e^n,\nabla(q^{n+1}-q^n)\big)\\
&\quad+\frac{5}{4}k^2\big(\mathcal{M}^{-1}\nabla\big(p(t_{n+2})-p(t_{n+1})\big),\nabla(q^{n+1}-q^n)\big)\\
&\le \frac{1}{2}k^2\|q^{n+1}-q^n\|^2+\frac{5}{4}\left(\frac{5}{8}+\sigma\right)|\tilde e^{n+1}-\tilde e^n|^2+m\,k^2\int_{t_{n+1}}^{t_{n+2}}|\nabla p_t|^2\mathrm{d}t,\quad\forall\ \sigma>0.
\end{aligned}
$$

Thus, choosing sufficiently small σ we obtain:

$$
\frac{k^2}{2}\|q^{n+1}-q^n\|^2\le\frac{5}{6}|\tilde e^{n+1}-\tilde e^n|^2+m\,k^2\int_{t_{n+1}}^{t_{n+2}}|\nabla p_t|^2\mathrm{d}t. \tag{37}
$$

The second term on the right-hand side of (35) we estimate as follows:

$$
k^2\big(\nabla q^n,\nabla(p(t_{n+2})-p(t_{n+1}))\big)\le k^3\|q^n\|^2+m\,k^2\int_{t_{n+1}}^{t_{n+2}}|\nabla p_t|^2\mathrm{d}t. \tag{38}
$$

For the third and the fourth terms on the right-hand side of (35) we have:

$$
\begin{aligned}
k^2\big(\nabla q^n,\boldsymbol{\omega}\times(\tilde e^{n+1}-e^{n+1})\big)-k^2\big(\nabla q^n,\boldsymbol{\omega}\times(\tilde e^n-e^n)\big)&\\
\le k^3\|q^n\|^2+m\,k\sum_{i=0,1}|\tilde e^{n+i}-e^{n+i}|^2.&
\end{aligned} \tag{39}
$$

Now estimates (35)–(39) give

$$
\begin{aligned}
&k\big(p(t_{n+1})-p^n,\operatorname{div}(\tilde e^{n+1}-\tilde e^n)\big)\\
&\le\frac{5}{6}|\tilde e^{n+1}-\tilde e^n|^2+m\,k^3\|q^n\|^2+k^2\big(\|q^n\|^2-\|q^{n+1}\|^2\big)\\
&\quad+m\,k^2\int_{t_{n+1}}^{t_{n+2}}|\nabla p_t|^2\mathrm{d}t+m\,k\sum_{i=0,1}|\tilde e^{n+i}-e^{n+i}|^2.
\end{aligned} \tag{40}
$$

Further, consider the following splitting:

$$
\begin{aligned}
&\boldsymbol{u}(t_{n+1})\cdot\nabla\boldsymbol{u}(t_{n+1})-\boldsymbol{u}^n\cdot\nabla\tilde{\boldsymbol{u}}^{n+1}=\boldsymbol{u}(t_{n+1})\cdot\nabla\tilde e^{n+1}\\
&\qquad+(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n))\cdot\nabla\tilde{\boldsymbol{u}}^{n+1}+e^n\cdot\nabla\boldsymbol{u}(t_{n+1})-e^n\cdot\nabla\tilde e^{n+1}.
\end{aligned} \tag{41}
$$

Based on this splitting we estimate the last two terms on the right-hand side of (31). The first three resulting terms can be estimated in the straightforward manner with the help of (9) and a priori estimates (7), (8):

$$
\begin{aligned}
kb(\boldsymbol{u}(t_{n+1}),\tilde e^{n+1},\tilde e^{n+1}-\tilde e^n)&\le\delta|\tilde e^{n+1}-\tilde e^n|^2+k^2m\,\|\boldsymbol{u}(t_{n+1})\|_2^2\|\tilde e^{n+1}\|^2\\
&\le\delta|\tilde e^{n+1}-\tilde e^n|^2+k^2m\,\|\tilde e^{n+1}\|^2,
\end{aligned} \tag{42}
$$

$$kb\big(\boldsymbol{u}(t_{n+1})-\boldsymbol{u}(t_n),\tilde{\boldsymbol{u}}^{n+1},\tilde{e}^{n+1}-\tilde{e}^n\big) \le \delta|\tilde{e}^{n+1}-\tilde{e}^n|^2 + m\,k^3\|\tilde{\boldsymbol{u}}^{n+1}\|\int_{t_n}^{t_{n+1}}\|u_t\|_2^2$$
$$\le \delta|\tilde{e}^{n+1}-\tilde{e}^n|^2 + k^3 m\,, \tag{43}$$

$$kb(e^n,\boldsymbol{u}(t_{n+1}),\tilde{e}^{n+1}-\tilde{e}^n) \le \delta|\tilde{e}^{n+1}-\tilde{e}^n|^2 + k^2 m\,\|\boldsymbol{u}(t_{n+1})\|_2^2\|e^n\|^2$$
$$\le \delta|\tilde{e}^{n+1}-\tilde{e}^n|^2 + k^2 m\,\|e^n\|^2. \tag{44}$$

Due to (9) the last term on the right hand of (41) is treated as follows:

$$kb(e^n,\tilde{e}^{n+1},\tilde{e}^{n+1}-\tilde{e}^n) \le m\,k\|e^n\|\|\tilde{e}^{n+1}\|\|\tilde{e}^{n+1}-\tilde{e}^n\|^{\frac12}|\tilde{e}^{n+1}-\tilde{e}^n|^{\frac12}$$
$$\le m\,k^{\frac32}\|e^n\|^2\|\tilde{e}^{n+1}\|^2 + \sqrt{k\nu\delta}\|\tilde{e}^{n+1}-\tilde{e}^n\||\tilde{e}^{n+1}-\tilde{e}^n|$$
$$\le m\,k^{\frac32}\|e^n\|^2\|\tilde{e}^{n+1}\|^2 + \frac{k\nu}{2}\|\tilde{e}^{n+1}-\tilde{e}^n\|^2 + \delta|\tilde{e}^{n+1}-\tilde{e}^n|^2. \tag{45}$$

Finally (31) with (32)–(33) and (40)–(45) yield for sufficiently small $\delta > 0$:

$$|\tilde{e}^{n+1}-\tilde{e}^n|^2 + \frac{k\nu}{2}\big(\|\tilde{e}^{n+1}\|^2 - \|\tilde{e}^n\|^2\big) + k^2\big(\|q^{n+1}\|^2 - \|q^n\|^2\big)$$
$$\le M\bigg(k^3 + k^2\int_{t_{n+1}}^{t_{n+2}}|\nabla p_t|^2\mathrm{d}t + k^2\big(\|\tilde{e}^{n+1}\|^2 + \|e^{n+1}\|^2\big) + k^{\frac32}\|e^n\|^2\|\tilde{e}^{n+1}\|^2$$
$$+ k^3\|q^n\|^2 + k\sum_{i=0,1}|\tilde{e}^{n+i} - e^{n+i}|^2\bigg). \tag{46}$$

We sum up the last inequalities for $n = 0,\ldots,N$ and use the assumption (29) and the estimate (18). This gives

$$\sum_{n=0}^{N}|\tilde{e}^{n+1}-\tilde{e}^n|^2 + k^2\|q^{N+1}\|^2 + \frac{k\nu}{2}\|\tilde{e}^{N+1}\|^2$$
$$\le M\bigg(k^2 + \sum_{n=0}^{N}k^3\|q^n\|^2 + \sum_{n=0}^{N}k^{\frac32}\|e^n\|^2\|\tilde{e}^{n+1}\|^2\bigg).$$

Now we assume that k is sufficiently small such that $2M\sqrt{k}\|e^N\|^2\nu^{-1} < 1$ holds (note that $\|e^N\|$ is uniformly bounded due to Lemma 2.2), then the application of the discrete Gronwall inequality and (18) yields

$$\sum_{n=0}^{N}|\tilde{e}^{n+1}-\tilde{e}^n|^2 + \frac{k\nu}{2}\|\tilde{e}^{N+1}\|^2 \le m\,k^2\exp\bigg(\sqrt{k}\sum_{n=0}^{N}\|e^n\|^2\bigg)$$
$$\le m\,k^2\,\exp\big(\sqrt{k}M\big). \qquad \square$$

Thanks to the embedding $L^2 \hookrightarrow H^{-1}$ and the L^2 stability of projection, see (17), we conclude:

$$\|e^{n+1} - e^n\|_{-1} \le m\,|e^{n+1} - e^n| \le m\,|\tilde{e}^{n+1} - \tilde{e}^n|.$$

Therefore Lemma 2.3 yields

$$\sum_{n=0}^{N} \|e^{n+1} - e^n\|_{-1}^2 \le m\,k^2 \qquad \forall\ 0 \le N \le T/k - 1. \tag{47}$$

3. Error estimate

In this section we show that the scheme (4)–(5) for the Navier–Stokes equations with the Coriolis force (2) has the same order of accuracy as the classical projection scheme [2, 15] for the Navier–Stokes equations (1). The following theorem is the main result of the paper.

Theorem 3.1. *Assume* (6) *and* $2k|\boldsymbol{\omega}|^2 \le 1$*, then both* $\tilde{\boldsymbol{u}}^{n+1}$ *and* $\boldsymbol{u}^{n+1}$ *are weakly first-order approximations to* $\boldsymbol{u}(t_{n+1})$ *in* $L^2(\Omega)^d$*:*

$$k\nu \sum_{n=0}^{T/k-1} \left\{ |e^{n+1}|^2 + |\tilde{e}^{n+1}|^2 \right\} \le m\,k^2. \tag{48}$$

Additionally assume that k *is sufficiently small and* $\int_0^T |\nabla p_t|^2 \le m$*, then* p^{n+1} *as well as* $(I - k\,\nu\Delta)p^{n+1}$ *are weakly order* $\frac{1}{2}$ *approximations to* $p(t_{n+1})$ *in* $L^2(\Omega)/R$*:*

$$k \sum_{n=0}^{T/k-1} \left\{ |p^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} + |(I - k\,\nu\Delta)p^{n+1} - p(t_{n+1})|^2_{L^2(\Omega)/R} \right\} \le m\,k. \tag{49}$$

Proof. (i) *Error estimate for the velocity.* Taking the sum of (4) and (5), we obtain

$$\frac{1}{k}(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) - \nu\Delta\tilde{\boldsymbol{u}}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1} + \boldsymbol{\omega} \times \boldsymbol{u}^{n+1} + \nabla p^{n+1} = \boldsymbol{f}(t_{n+1}). \tag{50}$$

Let us denote $\tilde{q}^{n+1} = p(t_{n+1}) - p^{n+1}$. Subtracting (50) from (19), we obtain

$$\begin{aligned} &\frac{1}{k}(e^{n+1} - e^n) - \nu\Delta\tilde{e}^{n+1} + \boldsymbol{\omega} \times e^{n+1} + \nabla\tilde{q}^{n+1} \\ &\qquad = (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1} - (\boldsymbol{u}(t_{n+1}) \cdot \nabla)\boldsymbol{u}(t_{n+1}) + R^n. \end{aligned} \tag{51}$$

Taking the inner product of (51) with $2kA^{-1}e^{n+1}$, splitting the nonlinear term into three parts, using (10) and noticing that

$$(A^{-1}\boldsymbol{u}, \nabla p) = 0, \quad \forall \boldsymbol{u} \in H,$$

we derive (for details see [11] p. 67)

$$\begin{aligned}
&\|e^{n+1}\|_{V'}^2 - \|e^n\|_{V'}^2 + \|e^{n+1}-e^n\|_{V'}^2 + \frac{15k\nu}{8}|e^{n+1}|^2 \\
&\quad \le -2k(\boldsymbol{\omega}\times e^{n+1}, A^{-1}e^{n+1}) + 2k(R^n, A^{-1}e^{n+1}) - 2kb(e^n, \tilde{\boldsymbol{u}}^{n+1}, A^{-1}e^{n+1}) \\
&\quad\quad - 2kb(\boldsymbol{u}(t_{n+1}), \tilde{e}^{n+1}, A^{-1}e^{n+1}) + 2kb(\boldsymbol{u}(t_n)-\boldsymbol{u}(t_{n+1}), \tilde{\boldsymbol{u}}^{n+1}, A^{-1}e^{n+1}) \\
&\quad\quad + m\,k|e^{n+1}-\tilde{e}^{n+1}|^2. \qquad (52)
\end{aligned}$$

The Coriolis term is estimated as follows

$$\begin{aligned}
|2k(\boldsymbol{\omega}\times e^{n+1}, A^{-1}e^{n+1})| &\le m\,k\|A^{-1}e^{n+1}\||e^{n+1}| \\
&\le m\,k\|e^{n+1}\|_{V'}|e^{n+1}| \le \frac{\nu k}{8}|e^{n+1}|^2 + m\,k\|e^{n+1}\|_{V'}^2. \qquad (53)
\end{aligned}$$

Applying the same arguments as in [11, 12] we deduce from (52) and (53) the estimate

$$\begin{aligned}
&\|e^{n+1}\|_{V'}^2 - \|e^n\|_{V'}^2 + \nu k|e^{n+1}|^2 + \|e^{n+1}-e^n\|_{V'}^2 \\
&\le m\,k\|e^{n+1}\|_{V'}^2 + m\,(k^2+k^3)\|\tilde{e}^{n+1}\|^2 + m\,k|\tilde{e}^{n+1}-e^n| \\
&\quad + m\,k|e^{n+1}-\tilde{e}^{n+1}|^2 + m\,k\left(\int_{t_n}^{t_{n+1}} t\|\boldsymbol{u}_{tt}\|_{-1}^2 \mathrm{d}t + k\int_{t_n}^{t_{n+1}} |\boldsymbol{u}_t|^2 \mathrm{d}t\right). \qquad (54)
\end{aligned}$$

The only modification of the arguments from [11, 12] here is that instead of identity

$$|\tilde{e}^{n+1}|^2 = |e^{n+1}|^2 + |e^{n+1}-\tilde{e}^{n+1}|^2,$$

which is no longer true we use the triangle inequality

$$|\tilde{e}^{n+1}|^2 \le |e^{n+1}|^2 + |e^{n+1}-\tilde{e}^{n+1}|^2, \qquad (55)$$

Taking the sum of (54) for $n = 0,\dots,N$, $N \in [0, T/k-1]$, we derive from Lemma 2.2 that

$$\|e^{N+1}\|_{V'}^2 + \sum_{n=0}^{N}\left\{\|e^{n+1}-e^n\|_{V'}^2 + k\,\nu|e^{n+1}|^2\right\} \le m\,k^2 + m\,k\sum_{n=0}^{N+1}\|e^n\|_{V'}^2.$$

By applying the discrete Gronwall lemma to the last inequality, we obtain

$$\|e^{N+1}\|_{V'}^2 + \sum_{n=0}^{N}\left\{\|e^{n+1}-e^n\|_{V'}^2 + k\,\nu|e^{n+1}|^2\right\} \le m\,k^2 \quad \forall 0 \le N \le T/k-1.$$

Then, from (55) and Lemma 2.2 we arrive at

$$k\sum_{n=0}^{N}|\tilde{e}^{n+1}|^2 \le k\sum_{n=0}^{N}\left\{|e^{n+1}|^2 + |\tilde{e}^{n+1}-e^{n+1}|^2\right\} \le m\,k^2 \quad \forall 0 \le N \le T/k-1. \quad (56)$$

(ii) *Error estimate for the pressure.* The skeleton of our derivations for the pressure estimate remains the same as in [11]. Remarks from [12] are applied through Lemma 2.3.

We start from rearranging (51) to

$$\nabla q_*^{n+1} = \frac{1}{k}(e^{n+1} - e^n) - \nu \Delta e_*^{n+1} + \boldsymbol{\omega} \times e^{n+1} \\ + (\boldsymbol{u}(t_{n+1}) \cdot \nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}(t_n) \cdot \nabla)\tilde{\boldsymbol{u}}(t_{n+1}) - R^n, \quad (57)$$

where $\{e_*^{n+1}, q_*^{n+1}\} = \{\tilde{e}^{n+1}, \tilde{q}^{n+1}\}$.

If we denote $q^{n+1} = p(t_{n+1}) - (I - k\nu\nabla)p^{n+1}$, we derive that (57) is also true for $\{e_*^{n+1}, q_*^{n+1}\} = \{e^{n+1}, q^{n+1}\}$. Hence we can consider simultaneously both pressure approximations.

We split the nonlinear term on the right-hand side of (57) as

$$(\boldsymbol{u}(t_{n+1}) \cdot \nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1} \\ = ((\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)) \cdot \nabla)\boldsymbol{u}(t_{n+1}) + (e^n \cdot \nabla)\boldsymbol{u}(t_{n+1}) + (\boldsymbol{u}^n \cdot \nabla)\tilde{e}^{n+1}.$$

From Lemma 2.2 we derive that

$$\|\boldsymbol{u}^n\| \le \|e^n\| + \|\boldsymbol{u}(t_n)\| \le m \quad \forall n.$$

By using (9) we obtain that, for all $\boldsymbol{v} \in H_0^1(\Omega)^d$,

$$\big((\boldsymbol{u}(t_{n+1}) \cdot \nabla)\boldsymbol{u}(t_{n+1}) - (\boldsymbol{u}^n \cdot \nabla)\tilde{\boldsymbol{u}}^{n+1}, \boldsymbol{v}\big) \\ \le c\,|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)| \|\boldsymbol{u}(t_{n+1})\|_2 \|\boldsymbol{v}\| + c\|e^n\| \|\boldsymbol{u}(t_{n+1})\| \|v\| + c\|\boldsymbol{u}^n\| \|\tilde{e}^{n+1}\| \|\boldsymbol{v}\| \\ \le \bar{c}\big\{\|\tilde{e}^{n+1}\| + \|e^n\| + |\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)|\big\} \|\boldsymbol{v}\|. \quad (58)$$

Using the Schwarz inequality we have also, for all $\boldsymbol{v} \in H_0^1(\Omega)^d$,

$$\left(\frac{1}{k}(e^{n+1} - e^n) - \nu \Delta e_*^{n+1} + \boldsymbol{\omega} \times e^{n+1} - R^n, \boldsymbol{v}\right) \\ \le \left(\frac{1}{k}\|e^{n+1} - e^n\|_{-1} + \nu\|e_*^{n+1}\| + \tilde{m}\,\|e^{n+1}\| + \|R^n\|_{-1}\right)\|\boldsymbol{v}\|. \quad (59)$$

From the inequalities (57), (58), (59) and

$$|p|_{L_2(\Omega)/R} \le \hat{c} \sup_{\boldsymbol{v} \in H_0^1(\Omega)^d} \frac{(\nabla p, \boldsymbol{v})}{\|\boldsymbol{v}\|},$$

we obtain that

$$|q_*^{n+1}|_{L^2(\Omega)/R} \le \hat{c} \sup_{v \in H_0^1(\Omega)^d} \frac{(\nabla q_*^{n+1}, v)}{\|v\|} \le \frac{m}{k}\|e^{n+1} - e^n\|_{-1} \\ + m\left(\|R^n\|_{-1} + \|\tilde{e}^{n+1}\| + \|e^n\| + (1 + \tilde{m})\|e^{n+1}\| + |\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)|\right).$$

Therefore, applying Lemmas 2.2 and 2.3, and the inequality (56), we derive

$$
\begin{aligned}
& k \sum_{n=0}^{T/k-1} |q_*^{n+1}|^2_{L^2(\Omega)/R} \\
& \leq m\, k \sum_{n=0}^{T/k-1} \left\{ \|\tilde{e}^{n+1}\|^2 + (1+\tilde{m})\|e^{n+1}\|^2 + \|R^n\|^2_{-1} + |\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)|^2 \right\} \\
& \quad + \frac{1}{k} \sum_{n=0}^{T/k-1} \|e^{n+1} - e^n\|^2_{-1} \leq m\, k.
\end{aligned}
$$

The proof of Theorem 3.1 is complete. □

Remark 1. It was discussed in [6] that the assumption $\int_0^T |\nabla p_t|^2 \leq m$, which we need to prove pressure error estimate does not hold for general flows, but requires a compatibility condition on given data, cf. [6]. The sufficient assumption for this condition to be valid is $\boldsymbol{f}(\boldsymbol{x}, t)|_{t=0} = \boldsymbol{0}$.

4. Numerical results

In this chapter we examine the accuracy in time of the pressure and velocity for the modified projection scheme and compare results with those of the nonmodified scheme. We take a test model of a unit square domain $[-1,1] \times [-1,1]$ and solve the system of the incompressible Stokes equations with the Coriolis force term

$$
\begin{aligned}
\boldsymbol{u}_t - \nu \Delta \boldsymbol{u} + 2\boldsymbol{\omega} \times \boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}
\tag{60}
$$

with homogeneous Dirichlet boundary conditions on the velocity. The exact solution $(\boldsymbol{u}, p)$ of (60) is chosen as follows [5]

$$
u_1(x,y) = \pi \sin(t) \sin(2\pi y) \sin^2(\pi x), \tag{61}
$$

$$
u_2(x,y) = -\pi \sin(t) \sin(2\pi x) \sin^2(\pi y), \tag{62}
$$

$$
p(x,y) = \sin(t) \cos(\pi x) \sin(\pi y). \tag{63}
$$

A uniform cartesian mesh with the mesh-size $1/32$ is used.

We denote

$$
v_{\mathrm{mean}} = \frac{1}{NDF} \sum_{k=1}^{NDF} |v_{\mathrm{analyt}}(k) - v_{\mathrm{numer}}(k)|,
$$

where v_{analyt} is some value (the velocity magnitude $|\boldsymbol{u}|$ or the pressure p from (61)–(63)) and v_{numer} is a corresponding numerical value, NDF is a number of degrees of freedom. To examine the error at some time $T_{\rm mes}$ we apply the formula:

$$v_{\rm err}(\Delta t) = \left(\frac{1}{N} \sum_{k=1}^{N} v_{\rm mean}^2 (k\,\Delta t) \right)^{1/2}, \quad N = \frac{T_{\rm mes}}{\Delta t}.$$

The following setting is chosen: $\nu = 1$, $T_{\rm mes} = 1.8$, $\Delta t \in \{0.025,\ 0.05,\ 0.1,\ 0.15\}$ and $|\boldsymbol{\omega}| = 10$. In Fig. 2 we show graphics for $\boldsymbol{u}_{\rm err}(\Delta t)$ and $p_{\rm err}(\Delta t)$. An exemplary graphic for $\boldsymbol{u}_{\rm mean}(t)$ and $p_{\rm mean}(t)$ for the time-step $\Delta t = 0.1$ is shown in Fig. 3.

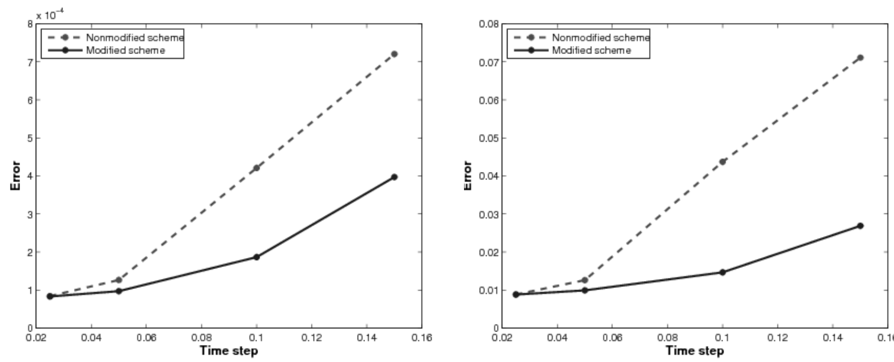

Fig. 2. Accuracy in time as a function on Δt, (LEFT) $\boldsymbol{u}_{\rm err}(\Delta t)$, (RIGHT) $p_{\rm err}(\Delta t)$.

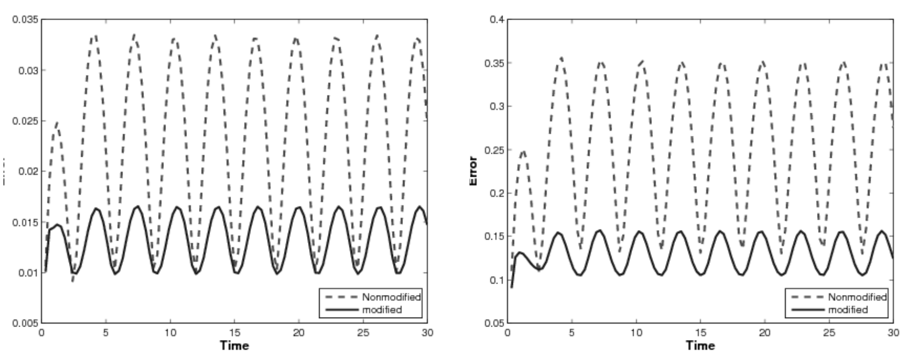

Fig. 3. Error distribution, $\Delta t = 0.1$, (LEFT) $\boldsymbol{u}_{\rm mean}$, (RIGHT) $p_{\rm mean}$.

From the graphics above one can observe that the modified projection scheme for the system of incompressible Stokes equations with the Coriolis force term is more accurate than the standard one. Moreover, difference in accuracy between modified and nonmodified schemes increases if larger Δt, resp. $\boldsymbol{\omega}$, is used. For further numerical studies of the discrete projection method for a model problem and for 3D flow simulations in stirred tank reactors the reader is referred to [13, 14].

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