A robust iterative solver in simulation of unsteady incompressible Navier-Stokes flow

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Abstract. Incompressible unsteady Navier-Stokes equations in primitive variables are considered together with a numerical method for them. The linearization of nonlinear terms at each time step is combined with the selecting of Bernoulli pressure. For the iterative solution of the resulting nonsymmetric saddle-point system an efficient preconditioning strategy is proposed. The method benefits both by strongly viscous and slightly viscous unsteady flows.

1 INTRODUCTION

The objective of the research is to contribute in constructing an efficient and robust numerical solver for unsteady laminar incompressible flow problem. Although the great effort was made by many scientists to establish suitable methods (based on multigrid, preconditioning, domain decomposition, etc. ideas) (see, e.g., [6],[8]), there is still the crucial problem of building a robust, flexible, optimal (in some sense), and efficient algorithm. These imply that an ideal method should be robust with respect to viscosity, time step, and spatial mesh size parameters, it should be readily implemented for 3D, complex geometries, and different discretization techniques, it should benefit from extreme situations (e.g. strongly viscous, convective or reactive problems), finally it should provide sufficient convergence and could be used as a 'black box' solver in appropriate applications.

Bearing in mind this ultimate goal, we concentrate further on the numerical solution of linearized incompressible Navier-Stokes equations (Oseen problem). The Schur complement technique is used and a nonsymmetric preconditioner for solving initially nonsymmetric problem is constructed. This new preconditioner is deemed to predict cases of strongly viscous, convective, and reactive nature as well as to be robust with respect to mesh parameters.

The key points of the method are following:

- distinguish Bernoulli pressure variable as the basic iterated unknown;
- relax tangent boundary condition on velocity and use these equations for the deducing of a preconditioner;
- construct final preconditioner for the Schur complement in an additive manner, considering separately purely viscous and reactive-convective problems.

Where it is not specified below we feel the user to be free with the particular choice of his favorite discretization method.

2 BASIC SCHEME

2.1 Governing equations

Let us consider the unsteady incompressible Navier-Stokes equations in a bounded 2D or 3D domain Ω:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \qquad (1)$$

$$\operatorname{div}\mathbf{u} = 0, \qquad (2)$$

with given force field $\mathbf{f}$, kinematic viscosity $\nu > 0$. The vector function $\mathbf{u}(t,\mathbf{x})$ (velocity) and scalar function $p(t,\mathbf{x})$ (kinematic pressure) are to be found, and subjected to some conditions. The classical case is the Dirichlet boundary conditions for velocity:

$$\mathbf{u} = \varphi \quad \text{on } \partial\Omega \times [0,T], \qquad (3)$$

(for simplicity we take further $\varphi \equiv 0$), initial condition for velocity at $t = 0$

$$\mathbf{u} = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \bar{\Omega}, \qquad (4)$$

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and integral condition $\int_\Omega p(\mathbf{x},t)\,d\mathbf{x} = 0,\ \forall t \in (0,T]$ to ensure the unique choice of pressure. We also assume for given data to satisfy div $\mathbf{u}_0 = 0$.

A common way of the numerical treating of the problem is the following. First apply some finite difference discretization in time or a method from ODE theory. Further, apply in space some finite difference, finite element or another method and solve the resulting finite system by an appropriate iterative algorithm. Undertaking here the same way, we note that explicit schemes for (1)-(2) cause severe stability problems for fine spatial grids and/or low viscosity. Hence, the effective time-stepping techniques lead (as a rule) to some nonlinear or linearized problem on each time step. Furthermore, most algorithms for solving nonlinear problem require the solution of linearized one on each iteration.

2.2 Time stepping techniques

Our choice of time stepping technique is the following fully implicit scheme. Given $\mathbf{u}^n$ and $\mathbf{f}^{n+1}$ find $\mathbf{u}^{n+1}$ and p^{n+1} from

$$\begin{aligned} \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\triangle t} - \nu\Delta\mathbf{u}^{n+1} & \\ +(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f}^{n+1}, \quad (5)\\ \operatorname{div}\mathbf{u}^{n+1} &= 0, \quad (6)\\ \mathbf{u}^{n+1}|_{\partial\Omega} &= 0, \quad (7) \end{aligned}$$

where $\mathbf{u}^n = \mathbf{u}(n\triangle t),\ p^n = p(n\triangle t)$, and $\triangle t$ is a time step. The scheme is known to be very robust and unconditionally stable.

Consider two possibilities of further calculations:

1. Linearization
The first one is related to widely used linearization of convective terms. We act as follows. Observe the equality

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\Big(\frac{\mathbf{u}^2}{2}\Big), \qquad (8)$$

where $\mathbf{u}^2 = u_1^2 + \ldots + u_n^2$ is a scalar function and in 2D case $\nabla\times\mathbf{u}$ is also a scalar function equals $-\partial u_1/\partial x_2 + \partial u_2/\partial x_1$. Denote by P the new pressure $P = p + \frac{\mathbf{u}^2}{2}$, sometimes refereed as Bernoulli (or dynamic) pressure. Now, replace $(\nabla\times\mathbf{u})\times\mathbf{u}$ by $(\nabla\times U)\times\mathbf{u}$, where U is given and corresponds to some extrapolation of velocity from previous times (e.g. linear $U = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ for equidistant time steps).
The resulting problem on every time step finally reads

$$\begin{aligned} \frac{1}{\triangle t}\mathbf{u}^{n+1} - \nu\Delta\mathbf{u}^{n+1} + (\nabla\times U)\times\mathbf{u}^{n+1} & \\ +\nabla P^{n+1} &= F(\mathbf{f},\mathbf{u}^n), \quad (9)\\ \operatorname{div}\mathbf{u}^{n+1} &= 0, \quad (10)\\ \mathbf{u}^{n+1}|_{\partial\Omega} &= 0, \quad (11) \end{aligned}$$

Note that $((\nabla\times U)\times\mathbf{u},\mathbf{v}) = -((\nabla\times U)\times\mathbf{v},\mathbf{u})$ thanks to the properties of the vector product, hence the corresponding bilinear form is skewsymmetric. Therefore, the resulting scheme (9)- (11) is unconditionally stable in the sense that it solution satisfies the discrete analogue of energy estimate for (1)-(4).

2. Nonlinear iterations
Relations (5)-(7) define a nonlinear problem with unknown $\mathbf{u}^{n+1}$ and p^{n+1}. Hence, the second possibility is to apply several nonlinear iterations for the direct solution of (5)-(7): given $\{\mathbf{u}_0,p_0\}$ (e.g. $\mathbf{u}_0 = \mathbf{u}^n, p_0 = p^n$) iterate $k = 1,2,\ldots$

$$\begin{pmatrix}\mathbf{u}_k\\ p_k\end{pmatrix} = \begin{pmatrix}\mathbf{u}_{k-1}\\ p_{k-1}\end{pmatrix} - \kappa_{k-1}F(\mathbf{u}_{k-1})^{-1}\begin{pmatrix} res(\mathbf{u}_{k-1})\\ \operatorname{div}\mathbf{u}_{k-1}\end{pmatrix}, \qquad (12)$$

where $F(\mathbf{u}_{k-1})$ is the Freshet-derivative in $\mathbf{u}_{k-1}$ and $res(\mathbf{u}_{k-1})$ is the nonlinear residual for $\mathbf{u}_{k-1}$; κ_{k-1} is a relaxation parameter that can be chosen, for example, as in the adaptive fixed point defect correction method. When the desired accuracy is obtained set $\mathbf{u}^{n+1} = \mathbf{u}_k,\ p^{n+1} = p_k$.
The crucial point in (12) is the applying of $F(\mathbf{v})^{-1}$ for some given vector function $\mathbf{v}$. The common way is to replace $F(\mathbf{v})$ with the approximate $\tilde{F}(\mathbf{v})$, but easily inverted. To this end, consider the contribution of nonlinear terms in $F(\mathbf{v})\mathbf{u}$ that is

$$(\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v} \qquad (13)$$

and observe the relation

$$(\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v} = (\nabla\times\mathbf{v})\times\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{v} + \nabla(\mathbf{v},\mathbf{u}), \qquad (14)$$

where $(\mathbf{v},\mathbf{u})$ is a scalar function defined as $(\mathbf{v},\mathbf{u}) = v_1u_1 + v_2u_2 + v_3u_3$. Now, the first term in the right-hand side of (14) is scewsymmetric and is retained in $\tilde{F}(\mathbf{v})$, the second one is dropped and the third one is also retained and added to the new pressure variable.

3 ALGORITHM FOR LINEARIZED PROBLEM

Finally the problem to be solved on each step of scheme (12) is similar to (9)-(11) and reads: given $\mathbf{f}$, g, and $\mathbf{w}$ from apropriate function spaces, find $\mathbf{u}$ and p from

$$\begin{aligned} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p &= \mathbf{f}, && (15)\\ \operatorname{div}\mathbf{u} &= g, && (16)\\ \mathbf{u}|_{\partial\Omega} &= 0 && (17) \end{aligned}$$

with $\alpha \approx \frac{1}{(\triangle t)} > 0, \nu > 0$; note that $\mathbf{w}$ is a vector function for 3D problem and a scalar one for 2D. Problem (15) - (17) is linear, nonsymmetric, and of saddle point type. The weak formulation of (15) - (17) is straightforward.

We solve problem (15) - (17) iteratively. Although a variety of techniques to solve saddle point problems are known (see, e.g., [1],[2],[5],[9]), most of them were established for symmetric problems. Hence, to treat nonsymmetric one, we use classical Uzawa approach, which includes variants of exact and inexact Uzawa algorithms. To this end, consider the operator

$$\mathcal{A} = -\operatorname{div}(\alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u})_0^{-1}\nabla, \qquad (18)$$

where $(\alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u})_0^{-1}$ is the solution operator for the convection-diffusion problem with the Dirichlet homogenous boundary conditions. $\mathcal{A}$ is a linear positive operator on the space of pressure functions and pressure p satisfies equation

$$\mathcal{A}p = F \qquad (19)$$

with given $F \equiv g - \operatorname{div}(\alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u})_0^{-1}\mathbf{f}$. Equation (19) can be effectively solved by iterations if some 'good' preconditioner for $\mathcal{A}$ is available.

To avoid the solution of the convection-diffusion equation on every iterative step one can use so called *inexact* Uzawa algorithm that iterates both pressure and velocity (see details in [4],[5]).

4 PRECONDITIONING

4.1 Diffusive pressure problem

A robust and optimal preconditioner for $\mathcal{A}$ should take care of diffusive, reactive, and convective effects in (15) (and resp. in (19)).

It is well-known that strongly viscous flow is nearly symmetric and the effect of convective terms is negligible. For *symmetric* problem ($\mathbf{w} = 0$) in [3] it was proved that operator $\mathcal{B}^{-1} = \nu\mathrm{I} - \alpha\Delta_N^{-1}$ (here Δ_N^{-1} is a solution operator for the scalar Neumann problem) provides the estimate $cond(\mathcal{B}^{-1}\mathcal{A}) \le c$, with c independent of ν, α and mesh size. However, for strongly convective problem such symmetric preconditioner is far from being optimal (see numerical results in Sec.5).

To construct a preconditioner which works well for all type of flows, we recall [7] that $\mathcal{B}$ can be realized as a Schur complement of the Stokes type problem after relaxing tangential boundary conditions on velocity and posing some natural boundary conditions on vorticity. Below we utilize the similar ideas in another extreme case of $\nu \ll 1$.

Rewriting equation (19) and dropping the diffusive terms, we arrive at

$$\begin{aligned} \alpha\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p &= \mathbf{0}, && (20)\\ \operatorname{div}\mathbf{u} &= F, && (21)\\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} &= 0. && (22) \end{aligned}$$

Due to the lack of high order derivatives for the velocity in (20) only the boundary conditions on normal velocity component are preserved. The appropriate weak saddle point formulation of (20) - (22) is posed in $\mathbf{H}_0(\operatorname{div})$ (for $\mathbf{u}$) and L_2^0 (for p).

$$\begin{aligned} \mathbf{H}_0(\operatorname{div}) &= \{\mathbf{u} \in \mathbf{L}_2(\Omega) : \operatorname{div}\mathbf{u} \in L_2(\Omega), \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0\},\\ L_2^0 &= \{p \in L_2(\Omega) : \int_\Omega p\, d\mathbf{x} = 0\}. \end{aligned}$$

Now we note that (20) - (22) can be considered as a mixed formulation of some elliptic problem for pressure. Indeed, we can formally eliminate velocity from (20). Using (21) and (22), we get the following diffusive problem with the Neumann conditions

$$\begin{aligned} -\frac{1}{\alpha}\operatorname{div}(A(\mathbf{x})\nabla p) &= F \quad \text{in } \Omega, && (23)\\ \frac{\partial p}{\partial N} &= 0 \quad \text{on } \partial\Omega, && (24) \end{aligned}$$

where $A(\mathbf{x}) = \{a_{ij}(\mathbf{x})\}$, $i,j = 1,2,3$ is the 'diffusive' matrix detailed below and $\frac{\partial p}{\partial N} = a_{ij}\frac{\partial p}{\partial x_i}n_j$.

Matrix $A(\mathbf{x})$ is expressed in the terms of α and $\mathbf{w}$ as follows.

- 2D case

$$A(\mathbf{x}) = \frac{\alpha^2}{\alpha^2 + \mathbf{w}^2}\mathrm{I} - \frac{\alpha}{\alpha^2 + \mathbf{w}^2}(\mathbf{w}\times) \qquad (25)$$

- 3D case

$$A(\mathbf{x}) = \frac{\alpha^2}{\alpha^2 + \mathbf{w}^2}(\mathrm{I} + \alpha^{-2}(\mathbf{w}\otimes\mathbf{w})) - \frac{\alpha}{\alpha^2 + \mathbf{w}^2}(\mathbf{w}\times) \qquad (26)$$

Here I stands for the identity matrix, $(\mathbf{w} \otimes \mathbf{w})$ for the one with ij-element equals $w_i(\mathbf{x})w_j(\mathbf{x})$; $(\mathbf{w}\times)$ stands for the matrix corresponding to the vector product with $\mathbf{w}$:

- 2D case
$$(\mathbf{w}\times) = \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix}$$
- 3D case
$$(\mathbf{w}\times) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

The first term in (25) or (26) is the symmetric part of $A(\mathbf{x})$ and the second is skewsymmetric.

The following weak formulation of (23) - (24) makes sense. For given $F \in L_2^0$ find $p \in H^1(\Omega)$ such that

$$\int_\Omega A(\mathbf{x})\nabla p \nabla q \, d\mathbf{x} = \int_\Omega F \, q \, d\mathbf{x}, \forall q \in H^1(\Omega). \quad (27)$$

For sufficiently smooth $\mathbf{w}$ problem (27) is well-posed. Moreover, the following statement is valid.

For sufficiently smooth $\mathbf{w}$ the problems (20) - (22) and (23) - (24) have a unique weak solutions. Moreover, the pressure component p of the solution of (20) - (22) belongs to $H^1(\Omega) \cap L_2^0(\Omega)$ and solves problem (23) - (24) .

4.2 Preconditioner

Denote by $\mathcal{D}(\mathbf{w})^{-1}$ the solution operator for problem (23) - (24) and consider

$$\mathcal{B}(\mathbf{w})^{-1} = \nu \mathrm{I} + \mathcal{D}(\mathbf{w})^{-1} \quad (28)$$

as a preconditioner for $\mathcal{A}$. On one hand, for highly viscous flows (when we can ignore the nonlinear effects) new preconditioner coincides with $\nu\mathrm{I} - \alpha\Delta_N^{-1} = \mathcal{B}(0)^{-1}$, which is known to be optimal in this case. On the other hand, for convective dominant flows the omitting of the diffusion terms in (20) does not alter much the global properties of the problem and $\mathcal{D}(\mathbf{w})$ is close to $\mathcal{A}$. Hopefully, the choice of $\mathcal{B}(\mathbf{w})$ covers all intermediate cases as well.

5 NUMERICAL RESULTS

Below we report the convergence rates of MINRES or conjugate gradient (in symmetric case) preconditioned iterative algorithms applied to solve pressure problem (19). Finite difference 'MAC' scheme was used to discretize equations (15)-(17). A special attention was paid to preserve the discrete equivalence of the equations (20)-(22) and (23)-(24). In all tests we consider 2D problem in a square domain $\Omega = (0,1) \times (0,1)$. As an exact solution of (19) we choose nonsmooth pressure function $p(x)$ which was obtained via random distribution over [-1,1] in every grid point.

First, some results for purely symmetric case ($\mathbf{w} = 0$) can be found in table 1. Viscosity was set $\nu = 1$, since it can be always done for symmetric problem by scaling pressure. The convergence rates were obtained for various values of $\alpha = (\nu\triangle t)^{-1}$ and mesh size. Note that for $\alpha \gg 1$ the limit ($h \to 0$) of convergence rate is achieved for very fine mesh size.

Table 1. Average convergence factor for preconditioned CG.

	Mesh size				
α	1/32	1/64	1/128	1/256	1/512
0	0.20	0.22	0.24	0.24	0.24
32	0.17	0.19	0.22	0.20	0.21
128	0.11	0.17	0.18	0.18	0.20
512	0.08	0.12	0.15	0.17	0.18
2048	0.04	0.08	0.11	0.14	0.17

Further we present results for nonsymmetric problem. Two possibilities for function $\mathbf{w}$ were tested:

1. set simply $\mathbf{w} = \kappa(\equiv const)$
2. set $\mathbf{w} = \nabla \times \mathbf{v}$, where $\mathbf{v} = (v_1, v_2)$,

$$\begin{aligned} v_1 &= \kappa(2y-1)x(1-x), \\ v_2 &= \kappa(2x-1)y(1-y). \end{aligned}$$

In the second case $\mathbf{v}$ represents a rotating vortex in Ω and serves as an example of multidirectional flow. In both cases $\kappa \gg 1$ means the convection dominant problem. Generally the problem has four parameters: h,ν,α, and κ. Fixing $h = 1/64$, we try below different values for the other parameters.

In tables 2-5 the results for the first choice of $\mathbf{w}$ are presented. Tables 3 and 5 show the convergence rates of MINRES method for *nonsymmetric* operator $\mathcal{A}$ with *symmetric* preconditioner $\mathcal{B}(0)$, while tables 2 and 4 show the convergence rates for the method with new *nonsymmetric* preconditioner $\mathcal{B}(\mathbf{w})$. Since for $\kappa \to \infty$ the problem (19) as well as auxiliary problems becomes strongly nonsymmetric, we use "inexact" variant of the iterative method to calculate the values from the tables 4 and 5, i.e. the less was residual $\mathcal{B}^{-1}(\mathcal{A}p^k - F)$, more accurate the evaluation of $\mathcal{B}^{-1}$ and $\mathcal{A}$ was done.

For the second choice of $\mathbf{w}$ the results are presented in tables 6-7. Table 6 shows the convergence rates for the method with symmetric preconditioner, while table 7

Table 2. Average convergence factor for MINRES with nonsymmetric preconditioning, $\mathbf{w} = const$ ($\kappa = 10$).

	Viscosity				
α	1.0	0.1	0.01	10^{-3}	10^{-4}
10	0.41	0.35	0.24	0.11	0.05
20	0.37	0.28	0.15	0.05	0.01
100	0.30	0.18	0.05	8e-3	8e-4

Table 3. Average convergence factor for MINRES with symmetric preconditioning, $\mathbf{w} = const$ ($\kappa = 10$).

	Viscosity				
α	1.0	0.1	0.01	10^{-3}	10^{-4}
10	0.40	0.51	0.60	0.64	0.67
20	0.37	0.32	0.34	0.37	0.43
100	0.31	0.19	0.08	0.08	0.08

Table 4. Average convergence factor for 'inexact' MINRES with nonsymmetric preconditioning, $\mathbf{w} = const$ ($\nu = 0.01$).

	Convection magnitude			
α	1	10	50	100
10	0.19	0.30	0.53	0.72[1]
20	0.15	0.27	0.26	0.43[1]
100	0.06	0.09	0.16	0.25

[1] the desired accuracy in evaluating $\mathcal{B}^{-1}$ was not obtained

Table 5. Average convergence factor for 'inexact' MINRES with symmetric preconditioning, $\mathbf{w} = const$ ($\nu = 0.01$).

	Convection magnitude			
α	1	10	50	100
10	0.20	0.63	*	*
20	0.15	0.26	0.90	*
100	0.06	0.09	0.36	0.65

* the method fails to converge

shows the convergence rates for the method with new nonsymmetric one.

Table 6. Average convergence factor for MINRES with nonsymmetric preconditioning, $\mathbf{w} = \nabla \times \mathbf{v}$ ($\kappa = 10$).

	Viscosity				
α	1.	0.1	0.01	10^{-3}	10^{-4}
10	0.39	0.34	0.50	0.41	0.10
20	0.37	0.27	0.41	0.23	0.04
100	0.30	0.18	0.20	0.04	4e-3

Table 7. Average convergence factor for MINRES with symmetric preconditioning, $\mathbf{w} = \nabla \times \mathbf{v}$ ($\kappa = 10$).

	Viscosity				
α	1.	0.1	0.01	10^{-3}	10^{-4}
10	0.39	0.35	0.61	0.75	0.84
20	0.37	0.30	0.59	0.65	0.69
100	0.30	0.19	0.53	0.56	0.59

6 CONCLUSIONS

The iterative solver has been considered as an inner engine in numerical simulation of unsteady incompressible viscous flow. As follows from numerical experiments, for small enough time steps the satisfactory convergence is achieved even if symmetric preconditioner is used to solve nonsymmetric problem on each time step. However, the stability of implicit schemes for unsteady problem (1)-(4) implies that large enough time step can be used for integrating. In this case the symmetric preconditioner does not provide a satisfactory convergence of the method for low viscosity and/or high convection. In contrast, new nonsymmetric preconditioner seems to provide good convergence in the latter case. Moreover, for $\nu \to 0$ (as well as for $\triangle t \to 0$) the convergence results with new preconditioner improve. Some degradation is still observed in the case of $||\mathbf{w}|| \to \infty$ with other parameters fixed. However, since $\mathbf{w}$ is some interpolation of vorticity, the dependence of $||\mathbf{w}||$ on ν is rather smart, say $||\mathbf{w}||_{L_2} = O(\nu^{-\frac{1}{4}})$ in a boundary layer. Furthermore high initial and boundary values of velocity implicitly imply the decrease of a time step $\triangle t$.

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