A Connection Between Filter Stabilization and Eddy Viscosity Models

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Recently, a new approach for the stabilization of the incompressible Navier–Stokes equations for high Reynolds numbers was introduced based on the nonlinear differential filtering of solutions on every time step of a discrete scheme. In this article, the stabilization is shown to be equivalent to a certain eddy-viscosity model in Large Eddy Simulation. This allows a refined analysis and further understanding of desired filter properties. We also consider the application of the filtering in a projection (pressure correction) method, the standard splitting algorithm for time integration of the incompressible fluid equations. The article proves an estimate on the convergence of the filtered numerical solution to the corresponding Navier-Stokes solution.
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I. INTRODUCTION

A stabilization of a numerical time-integration algorithm for the incompressible Navier–Stokes equations

$$\begin{aligned} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{aligned} \quad \text{in } \Omega\times(0,T], \tag{1}$$

for large Reynolds numbers with the help of an additional filtering step was recently introduced in Ref. [1]. Denote by w^n or u^n approximations to the Navier–Stokes system velocity solution at time t_n, and similarly p^n approximates pressure $p(t_n)$. Let $\triangle t = t_{n+1} - t_n$. The algorithm, referred to further as (A1), reads: For $n = 0, 1, \ldots$ and $u^0 = u(t^0)$

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1. compute intermediate velocity w^{n+1} from

$$\begin{cases} \dfrac{1}{\triangle t}(w^{n+1} - u^n) + (w^{n+1} \cdot \nabla) w^{n+1} + \nabla p^{n+1} - \nu \Delta w^{n+1} & = f^{n+1}, \\ \operatorname{div} w^{n+1} & = 0, \end{cases}$$

subject to appropriate boundary conditions;
2. filter the intermediate velocity, $\overline{w^{n+1}} := F\, w^{n+1}$;
3. relax $u^{n+1} := (1 - \chi) w^{n+1} + \chi\, \overline{w^{n+1}}$, with a relaxation parameter $\chi \in [0, 1]$.

Here, F is a generic nonlinear filter acting from $L^2(\Omega)^3$ to $H^1(\Omega)^3$. We shall consider further in the article several examples of differential filters. The convergence of the finite element solutions of (A1) to the smooth Navier–Stokes solution has been analyzed in Ref. [1]. One advantage of the approach is the convenience of implementation within an existing CFD code for laminar flows and flexibility in the choice of a filter. Numerical results from Refs. [1–5] with composite nonlinear differential filters, as defined in Section III, consistently show more precise localization of model viscosity and its more precise correlation with the action of nonlinearity on the smallest resolved scales than plain Smagorinsky-type LES or Variational Multiscale methods. Thus, we deem the approach deserves further study, should be put into perspective and related to developing LES models.

In this article, we show that introducing the filter stabilization is closely related (and even equivalent in a sense which is made precise further in the article) to adapting a certain eddy-viscosity model for LES. The connection to a LES model helps us to quantify the model dissipation introduced by the filter stabilization (Theorem 1), formulate stability criteria [see (6) and (8)], and gives insight into the choice of the filter and the relaxation parameter. In particular, it provides an explanation why the stabilization by the filtering avoids adding excessive model viscosity in regions of larger velocity gradients, unlike most other eddy viscosity models.

The entire approach is specifically designed for treating high Reynolds number flows. Therefore, it is natural to extend it to the Chorin–Temam–Yanenko-type splitting algorithms, which are the prevailing method for the time-integration of the incompressible Navier–Stokes equations for fast unsteady flows. Such (rather natural) extension is presented in the article together with the relevant error analysis. We note right away that the analysis demonstrates the convergence of numerical solutions to the Navier–Stokes smooth solution, while it would be also interesting to analyze the error of the numerical solutions to a (presumably smoother) solution of the corresponding LES model. However, the specific difficulty we faced in the latter case is the lacking of the monotonicity property by most of eddy viscosity indicator functionals, which were numerically proved to be useful in defining the filter F, see Section III. Though practically attractive, introducing such functionals makes the mathematical well-posedness of the LES model and accordingly the error analysis hard to accomplish and we are unaware of relevant results in this direction.

II. FILTER STABILIZATION AND LES MODEL

It is well known, see, for example, Refs. [6] or [7], that explicit filtering is related to adding eddy or artificial viscosity. The connection of the filter stabilization as defined above to LES modeling is easily recovered by noting that shifting the index $n + 1 \to n$ on Steps 2 and 3 and using Step 1 gives the implicit discretization of the Navier–Stokes equations, with explicitly treated nonlinear dissipation term:

$$\begin{cases} \frac{1}{\triangle t}(w^{n+1} - w^n) + (w^{n+1} \cdot \nabla)w^{n+1} + \nabla p^{n+1} - \nu \Delta w^{n+1} + \frac{\chi}{\triangle t} G w^n &= f^{n+1}, \\ \operatorname{div} w^{n+1} &= 0, \end{cases} \tag{2}$$

with

$$G := I - F, \quad I \text{ is the identity operator.}$$

Assume $\chi = \chi_0 \triangle t$, where χ_0 is a time- and mesh-independent constant, then (2) can be treated as the time-stepping scheme for

$$\begin{cases} w_t + (w \cdot \nabla)w + \nabla p - \nu \Delta w + \chi_0 G\, w &= f, \\ \operatorname{div} w &= 0. \end{cases} \tag{3}$$

These arguments show that the numerical integrator (A1) with filter stabilization is the splitting scheme for solving (3). Furthermore, (3) can be observed as a LES model, with $\chi_0 G\, w$ corresponding to the Reynolds stress tensor closure:

$$\nabla \cdot (\overline{w \otimes w} - \overline{w} \otimes \overline{w}) \approx \chi_0 G\, w.$$

This simple observation leads to a refined analysis and better interpretation of the numerical results and the method properties.

We note that $\chi = O(\triangle t)$ is exactly the scaling of relaxation parameter which allows us to prove optimal convergence result for a time-stepping splitting method (Theorem 3). Furthermore, numerical experiments in Refs. [3, 8] suggested that $\chi = O(\triangle t)$ is indeed the right scaling of the relaxation parameter with respect to numerical solution accuracy.

We start by showing several numerical properties of the approach. Throughout the article we use $(\cdot, \cdot)$ and $\|\cdot\|$ to denote the L^2 scalar product and the norm, respectively. For the sake of analysis, assume the homogeneous Dirichlet boundary conditions for velocity. Taking the L^2 scalar product of (2) with $2\triangle t w^{n+1}$ and integrating by parts gives

$$\|w^{n+1}\|^2 - \|w^n\|^2 + \frac{1}{2}\|w^{n+1} - w^n\|^2 + \nu \triangle t \|\nabla w^{n+1}\|^2 + \chi(G w^n, w^{n+1}) = \triangle t(f^{n+1}, w^{n+1}). \tag{4}$$

For a self-adjoint filtering operator, that is, $(Gu, v) = (Gv, u)$ for any $u, v \in H_0^1(\Omega)^3$, the equality (4) can be alternatively written as

$$\|w^{n+1}\|^2 - \|w^n\|^2 + \nu \triangle t \|\nabla w^{n+1}\|^2 + \frac{\chi}{2}\left((G w^{n+1}, w^{n+1}) + (G w^n, w^n)\right)$$
$$= \triangle t(f, w^{n+1}) + \frac{1}{2}\left(\chi(G(w^{n+1} - w^n), w^{n+1} - w^n) - \|w^{n+1} - w^n\|^2\right). \tag{5}$$

Considering the last two terms on the right-hand side, we immediately get the sufficient condition of the energy stability of (2) for the case of self-adjoint filters:

$$\chi(Gu, u) \le \|u\|^2 \quad \forall\, u \in H_0^1(\Omega)^3. \tag{6}$$

If G is not necessarily self-adjoint, one may rewrite (4) as

$$\|w^{n+1}\|^2 - \|w^n\|^2 + \frac{1}{2}\|w^{n+1} - w^n\|^2 + \nu\Delta t\|\nabla w^{n+1}\|^2 + \chi(Gw^n, w^n)$$
$$= \Delta t(f, w^{n+1}) + \chi(Gw^n, w^n - w^{n+1}).$$

Thanks to the Cauchy inequality one gets for any $\theta \in \mathbb{R}$:

$$\|w^{n+1}\|^2 - \|w^n\|^2 + \nu\Delta t\|\nabla w^{n+1}\|^2 + (1-\theta)\chi(Gw^n, w^n)$$
$$\leq \Delta t(f, w^{n+1}) - \chi\left(\theta(Gw^n, w^n) - \frac{\chi}{2}(Gw^n, Gw^n)\right). \quad (7)$$

In this more general case, one may consider the following sufficient condition for the energy stability. Fixing, for example, $\theta = \frac{1}{2}$, assures the sum of the last two terms in (7) is positive if

$$\chi(Gu, Gu) \leq (Gu, u) \quad \forall\, u \in H_0^1(\Omega)^3. \quad (8)$$

Assume G is self-adjoint and w^n approximates a smooth in time Navier–Stokes solution, then (5) leads to the following energy balance relation of the numerical method:

$$\|w^N\|^2 + \nu\sum_{n=1}^{N}\Delta t\|\nabla w^n\|^2 + \chi_0\sum_{n=1}^{N}\Delta t(Gw^n, w^n) = \|w^0\|^2 + \sum_{n=1}^{N}\Delta t(f^n, w^n) + O(\Delta t).$$

In particular, we may conclude that the filter stabilization introduces the model dissipation of

$$\chi_0\sum_{n=1}^{N}\Delta t(Gw^n, w^n). \quad (9)$$

Finally, we notice that the filtering and relaxation steps in (A1) can be rearranged as

$$\frac{u^{n+1} - w^{n+1}}{\Delta t} = -\chi_0 G\, w^{n+1},$$

which is the explicit Euler method for integrating

$$u_t = -\chi_0 G\, u \quad \text{on } [t_n, t_{n+1}], \text{ with } u(t_n) = w(t_{n+1}). \quad (10)$$

The coupling of a numerical method with the evolution Eq. (10) is known as another way of introducing explicit filtering in modeling of dynamical systems, for example, Ref. [6]. This suggests that an improvement leading to higher-order methods for integrating (10) might be possible.

In the next section, we shall study properties of the operator G for a class of nonlinear differential filters.

III. NONLINEAR DIFFERENTIAL FILTERS

Linear differential filters have a long history in LES, see Ref. [9]. We also point to Ref. [10] and references therein for applications of linear differential filters in the Lagrange-averaging turbulence models. In this section, we consider a family of nonlinear differential filters for the filtering procedure. Some conclusions will be drawn concerning the stability conditions (6), (8), and equivalence to other approaches in the LES modelling. We use the following notation:

$$V := \left\{v \in H_0^1(\Omega)^3 \ : \ \operatorname{div} v = 0\right\}, \quad H = \left\{v \in \ L^2(\Omega)^3 : \operatorname{div} v = 0, v \cdot n|_{\partial\Omega} = 0\right\}.$$

By $\mathbb{P}$ we denote the L^2 orthogonal projector from $L^2(\Omega)^3$ onto H.

For a given sufficiently smooth vector function u and $w \in L^2(\Omega)^3$, we define $F\,w$ as the solution to

$$(\delta^2 a(u)\nabla(F\,w), \nabla v) + (F\,w, v) = (w, v) \quad \forall v \in X, \tag{11}$$

with an indicator functional $0 \le a(u) \le 1$ and filtering radius δ^2, which generally may depend on x and t, $\delta_{\max} = \max_{x,t} |\delta|$. Here, $X = H_0^1(\Omega)^3$ or $X = V$, if the filter is div-free preserving. We note that it is not immediately clear if the problem (11) is well-posed. In practice, this is not an issue, as in a finite dimension setting, for example, for a finite element method, the bilinear form from the left-hand side of (11) is elliptic and thus (11) is well-posed. Otherwise, we may assume $0 < \varepsilon \le a(u) \le 1$ for some sufficiently small positive ε. If we assume this, none of our results further in the article depend on the parameter ε. It is standard to base the indicator functional on the input function w itself, that is $u = w$ and we will denote $\overline{w} := F\,w$ in this case. However, in the course of analysis, we need to consider (auxiliary) filtering with $u \neq w$. If we need to show explicitly the function used for the indicator, we shall write $F(u)w$ instead of $F\,w$ or $F(w)w$ instead of $\overline{w}$.

The action of $G = I - F$, $w_g := G\,w$, is defined formally as the solution to

$$(\delta^2 a(u)\nabla w_g, \nabla v) + (w_g, v) = (\delta^2 a(u)\nabla w, \nabla v) \quad \forall v \in X. \tag{12}$$

The operator G is self-adjoint on X and in the operator notation it can be written as

$$G = -\left[I - \Delta_a\right]^{-1} \Delta_a, \tag{13}$$

with

$$\Delta_a := \begin{cases} \operatorname{div}(\delta^2 a(u)\nabla) & \text{if } X = H_0^1(\Omega)^3, \\ \mathbb{P}\operatorname{div}(\delta^2 a(u)\nabla) & \text{if } X = V. \end{cases}$$

Because operator Δ_a is self-adjoint and positive definite, one see from (13) that $G \le I$ and thus the sufficient stability condition (6) holds for any $\chi \in [0, 1]$. This can be easily verified in a formal way by substituting $v = F\,w$ in (11) to get $(w, F\,w) \ge 0$ and thus $(w, Gw) = (w, w - F\,w) \le \|w\|^2$ for any $w \in H_0^1(\Omega)^3$. Moreover, varying θ in (7) and using (8), one shows the energy stability estimate for any $\chi \in [0, 2]$. However, such refinement is not important for our further analysis.

With the help of (9) and (13), we now quantify the model dissipation introduced by the differential filters. To make notation shorter and without loss of generality, let $\chi = \chi_0 \Delta t$.

First, representation (13) immediately implies $G \le -\Delta_a$. Thus, the additional dissipation introduced by the differential filtering does not exceed those introduced by the LES closure model:

$$\operatorname{div}(\overline{w \otimes w} - \overline{w} \otimes \overline{w}) \approx -\chi_0 \Delta_a w. \tag{14}$$

It is easy to show that for a discrete case and if the condition

$$\delta \lesssim \text{ spatial mesh width}$$

holds and $0 \le a(u) \le 1$, then the dissipation introduced by the differential filtering (11) is equivalent to the dissipation of the closure model (14).

We make the above statement more precise for a finite element discretization. To this end, assume a consistent triangulation $\mathcal{T}$ of Ω, satisfying the minimal angle condition

$$\inf_{K\in\mathcal{T}} \rho(K)/r(K) =: \alpha_0 > 0$$

where $\rho(K)$ and $r(K)$ are the diameters of inscribed and superscribed circles (spheres in 3D) for a triangle (tetrahedron) K. We have the following result.

Theorem 1. *Assume X is the finite element space of continuous functions which are polynomials of degree $p \ge 1$ on every element K and* $\max_{x\in K} |\delta(x)| \le C_\delta\, r(K)$ *for any $K \in \mathcal{T}$, with a constant C_δ independent of K. Then for any $w \in X$ the equivalence*

$$\widetilde{c}\,(\delta^2 a(u)\nabla w, \nabla w) \le (G\, w, w) \le (\delta^2 a(u)\nabla w, \nabla w) \tag{15}$$

holds with a constant $\widetilde{c} > 0$ independent of w, the indicator $a(\cdot)$, and the filtering radius δ. The constant $\widetilde{c} > 0$ may depend on p, C_δ, and α_0.

Proof. Consider the finite element inverse inequality

$$\|\nabla w\|_{L^2(K)} \le c_0 \rho(K)^{-1} \|w\|_{L^2(K)} \quad \forall\, w \in X, \tag{16}$$

where the constant c_0 depends only on the polynomial degree p and α_0. The inequality (16), the assumption on δ, and the minimal angle condition imply

$$\|\delta \nabla w\|_{L^2(K)} \le \widetilde{C} \|w\|_{L^2(K)}, \tag{17}$$

where the constant $\widetilde{C}$ depends only on p, C_δ, and α_0. Squaring (17), summing over all $K \in \mathcal{T}$, and recalling that $a(\cdot) \le 1$, implies

$$(\delta^2 a(u)\nabla w, \nabla w) \le \widetilde{C}^2 \|w\|^2. \tag{18}$$

Denote $w_g = G\, w$ for some $w \in X$. We set $v = w_g$ and $v = -w$ in (12) and sum up the equalities to get

$$\begin{aligned} 0 &= (\delta^2 a(u)\nabla w_g, \nabla w_g) + (w_g, w_g) - 2(\delta^2 a(u)\nabla w, \nabla w_g) - (w_g, w) + (\delta^2 a(u)\nabla w, \nabla w) \\ &= \|w_g\|^2 - (w_g, w) + (\delta^2 a(u)\nabla(w - w_g), \nabla(w - w_g)). \end{aligned}$$

Thus, it holds $\|w_g\|^2 \le (w_g, w)$, that is, the condition (8). Now we set $v = w$ in (12) and use (8) and (18) to estimate

$$\begin{aligned}(\delta^2 a(u)\nabla w, \nabla w) &= (\delta^2 a(u)\nabla w_g, \nabla w) + (w_g, w)\\ &\le \frac{1}{2}(\delta^2 a(u)\nabla w_g, \nabla w_g) + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w) + (w_g, w)\\ &\le \frac{1}{2}\widetilde{C}^2\|w_g\|^2 + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w) + (w_g, w)\\ &\le \left(\frac{1}{2}\widetilde{C}^2 + 1\right)(w_g, w) + \frac{1}{2}(\delta^2 a(u)\nabla w, \nabla w).\end{aligned}$$

We proved the lower bound in (15).

To show the upper bound, we set $v = w_g$ and $v = w$ in (12) and sum up the equalities to get

$$0 = (\delta^2 a(u)\nabla w_g, \nabla w_g) + (w_g, w_g) + (w_g, w) - (\delta^2 a(u)\nabla w, \nabla w).$$

This yields the upper bound in (15): $(w_g, w) \le (\delta^2 a(u)\nabla w, \nabla w)$. ■

Few conclusions can be drawn from the equivalence result (15) concerning the relation of the filter stabilization to some other eddy-viscosity models.

The use of the linear differential filter ($a \equiv 1$), as considered in Ref. [3], is equivalent to the method of artificial viscosity. This means that the model dissipation is equivalent to the isotropic diffusion scaled with $\chi_0\delta^2$. Given what is known about the method of artificial viscosity, it is not surprising that the method is not very accurate in this case. Thus, more elaborated indicator functionals should be used. Generally, we may think of $a(u)$ as a real valued functional, depending on $u, \nabla u$, and selected with the intent that

$$\begin{aligned} a(u(x)) &\approx 0 \quad \text{for laminar regions or persistent flow structures,}\\ a(u(x)) &\approx 1 \quad \text{for flow structures which decay rapidly.}\end{aligned}$$

The choice of the Smagorinsky-type indicator function, $a(u) = |\nabla u|$, does not necessarily satisfy the condition $a(u) \le 1$. In this case, we do not have the equivalence result of the filter stabilization to the Smagorinsky LES model. Only the upper bound in (15) is guaranteed to hold. Thus, the dissipation introduced by the filtering with $a(u) = |\nabla u|$ is likely less than that of the Smagorinsky model. This can be a desirable property, as the Smagorinsky LES model is known to be severely over-diffusive for certain flows, see, for example, Ref. [11], and several ad hoc corrections were introduced such as the van Driest damping, dynamic models, and others, see Refs. [12–14].

Several reasonable indicator functions $a(u)$ are known to satisfy the boundedness condition: $0 \le a(u) \le 1$. These are the renormalized Smagorinsky-type indicator [15], the indicator based on the Q-criteria [16], and the Vreman indicators [17]; also an indicator based on the normalized helical density distribution was considered in Ref. [2]. Given several indicators $a_i(\cdot), i = 1, \dots, N$, the combined indicator can be defined as the geometric mean: $a(\cdot) := (\prod_{i=1}^N a_i(\cdot))^{\frac{1}{N}}$.

We remark, that the convergence results proved further in this article do not rely on any smoothness properties or particular form of $a(\cdot)$.

The last remark in this section is that Theorem 1 does not give much insight if enforcing the divergence constraint in the filter is important or not. However, if we assume $X = V$ in (11), that is, the filtered velocity satisfies the divergence free condition, then this slightly simplifies the error analysis in Section VI.

IV. PROJECTION SCHEME WITH FILTER STABILIZATION

One idea behind introducing the filter stabilization or explicit filtering was to provide CFD software users and developers with a simple way to enhance existing codes for laminar incompressible flows to compute high Reynolds number flows. This goal is accomplished by making the filtering procedure algorithmically independent of a time integration method. Driven by this intention, we consider the Chorin [18] splitting (projection) scheme with an additional separate filtering step. Projection methods are the common numerical approach to the incompressible Navier–Stokes equations and form a family of splitting algorithms, see Refs. [19, 20]. We perform the numerical analysis for the simplest first-order method given below. From the algorithmic standpoint, the generalization to higher-order projection methods is straightforward, although analysis may become considerably more involved.

Projection methods split the time evolution of the velocity vector field according to the momentum equation and the projection of the velocity to satisfy the divergence-free condition. The filtering step can be introduced before or after the projection step. In the former case, it is not necessary to augment the filter with the div-free constraint, as the projection step takes care of the keeping the approximates in the subspace of div-free functions. If the filter is div-free preserving, then it is reasonable to put it after the projection. In this article, we consider the constrained filter. We shall study the following algorithm:

Step 1: Solve the convection-diffusion-type problem: Given u^n, w^*, find $\widetilde{w^{n+1}}$:

$$\begin{cases} \frac{1}{\triangle t}(\widetilde{w^{n+1}} - u^n) + (w^* \cdot \nabla)\widetilde{w^{n+1}} - \nu\Delta\widetilde{w^{n+1}} = f^{n+1}, \\ \widetilde{w^{n+1}}|_{\partial\Omega} = 0. \end{cases} \tag{19}$$

The velocity w^* is typically an interpolation from previous times, for example, $w^* := w^n$ or a higher-order interpolation. For the sake of analysis, we consider $w^* = w^n$.

Step 2: Project $\widetilde{w^{n+1}}$ on the div-free subspace: Find p^{n+1} and w^{n+1} solving the Neumann pressure Poisson problem:

$$\begin{cases} \frac{1}{\triangle t}(w^{n+1} - \widetilde{w^{n+1}}) + \nabla p^{n+1} = 0, \\ \operatorname{div} w^{n+1} = 0, \\ n \cdot w^{n+1}|_{\partial\Omega} = 0. \end{cases} \tag{20}$$

Step 3: Filter: $\overline{w^{n+1}} := F\, w^{n+1}$;

Step 4: Relax:

$$u^{n+1} := (1 - \chi)w^{n+1} + \chi\,\overline{w^{n+1}}, \tag{21}$$

with some $\chi \in [0, 1]$.

Similar to what was shown in section II, shifting the index $n+1 \to n$ on steps 2–4 and substituting into (19) gives for $\chi = \chi_0 \triangle t$

$$\begin{cases} \frac{1}{\triangle t}(\widetilde{w^{n+1}} - \widetilde{w^n}) + (w^* \cdot \nabla)\widetilde{w^{n+1}} + \nabla p^{n+1} - \nu \Delta \widetilde{w^{n+1}} + \chi_0 G \widetilde{w^n} - \triangle t \chi_0 G \nabla p^{n+1} = f^{n+1}, \\ \qquad \operatorname{div} \widetilde{w^{n+1}} - \triangle t \Delta p^{n+1} = 0. \end{cases} \tag{22}$$

From (22), we see that the splitting scheme (19)–(21) is formally the first-order accurate time-discretization of the LES model (3).

Further, we show that the splitting scheme (19)–(21) is stable. There are two well-known approaches to accomplish the error analysis of projection methods. The one of Rannacher and Prohl [20, 21] uses the relation between projection and quasicompressibility methods as it is seen from (22). However, this analysis needs considerable effort to get extended to equations different from the plain Navier–Stokes equations. Another framework is mainly due to Shen (see Refs. [22, 23]), where convergence results were shown based on energy-type estimates. In our error analysis, we follow (to a certain extent) arguments from these two papers.

V. STABILITY

To show the stability of the splitting scheme, we need the following simple auxiliary result:

Lemma 1. *For w^{n+1} and u^{n+1} from the algorithm* (19)–(21) *and the filter F defined in* (11)*, it holds*

$$\|w^{n+1}\| \geq \|u^{n+1}\|.$$

Proof. From the definition (11), we obtain:

$$\begin{aligned}(\delta^2 a(w^{n+1})\nabla\overline{w^{n+1}}, \nabla\overline{w^{n+1}}) + \|\overline{w^{n+1}}\|^2 &= (w^{n+1}, \overline{w^{n+1}}) \\ &= \frac{1}{2}(\|w^{n+1}\|^2 + \|\overline{w^{n+1}}\|^2 - \|w^{n+1} - \overline{w^{n+1}}\|^2).\end{aligned}$$

This yields

$$\|w^{n+1}\|^2 = 2(\delta^2 a(w^{n+1})\nabla\overline{w^{n+1}}, \nabla\overline{w^{n+1}}) + \|\overline{w^{n+1}}\|^2 + \|\overline{w^{n+1}} - w^{n+1}\|^2.$$

Hence, $\|w^{n+1}\| \geq \|\overline{w^{n+1}}\|$. From (21), we get

$$\|u^{n+1}\| \leq (1-\chi)\|w^{n+1}\| + \chi\|\overline{w^{n+1}}\| \leq \|w^{n+1}\| \quad \text{for } \chi \in [0,1].$$

■

Denote by $\|\cdot\|_{-1}$ the L^2-dual norm for $H_0^1(\Omega)^3$. Now, we are ready to prove the following stability result.

Theorem 2. *The algorithm* (19)–(21) *is stable in the sense of the following a priori estimate:*

$$\|w^l\|^2 + \sum_{n=0}^{l-1} \|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \sum_{n=0}^{l-1} \|\widetilde{w^{n+1}} - u^n\|^2 + \sum_{n=0}^{l-1} \nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2$$
$$\leq \|w^0\|^2 + \sum_{n=0}^{l-1} \nu^{-1}\Delta t \|f(t_{n+1})\|_{-1}^2 \tag{23}$$

for any $l = 1, 2, \ldots$.

Proof. Take the L^2 scalar product of (19) with $2\Delta t \widetilde{w^{n+1}}$:

$$2(\widetilde{w^{n+1}} - u^n, \widetilde{w^{n+1}}) + 2\nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2 = 2\Delta t (f^{n+1}, \widetilde{w^{n+1}}) \leq \nu^{-1}\Delta t \|f^{n+1}\|_{-1}^2 + \nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2.$$

Rewriting and simplifying this leads to:

$$\|\widetilde{w^{n+1}}\|^2 - \|u^n\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2 \leq \nu^{-1}\Delta t \|f^{n+1}\|_{-1}^2. \tag{24}$$

The L^2 scalar of (20) with $2\Delta t\, w^{n+1}$ and div $w^{n+1} = 0$ gives

$$2(w^{n+1} - \widetilde{w^{n+1}}, w^{n+1}) = 0 \quad \Longrightarrow \quad \|w^{n+1}\|^2 - \|\widetilde{w^{n+1}}\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 = 0.$$

Substituting $\|\widetilde{w^{n+1}}\|^2$ with $\|w^{n+1}\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2$ in (24) yields

$$\|w^{n+1}\|^2 - \|u^n\|^2 + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2 \leq \nu^{-1}\Delta t \|f^{n+1}\|_{-1}^2.$$

The application of Lemma 1 gives

$$(\|w^{n+1}\|^2 - \|w^n\|^2) + \|w^{n+1} - \widetilde{w^{n+1}}\|^2 + \|\widetilde{w^{n+1}} - u^n\|^2 + \nu\Delta t \|\nabla \widetilde{w^{n+1}}\|^2 \leq \nu^{-1}\Delta t \|f^{n+1}\|_{-1}^2.$$

Summing up the inequality from $n = 0, \ldots, l-1$, we arrive at (23). ■

VI. ERROR ESTIMATES

We shall use $\langle\cdot,\cdot\rangle$ to denote the duality product between H^{-s} and $H_0^s(\Omega)$ for all $s \geq 0$. In the following, we assume that the given data and solution to the Eq. (1) subject to the homogeneous Dirichlet velocity boundary conditions satisfy

$$\begin{cases} u_0 \in (H^2(\Omega))^d \cap V, \\ f \in L^\infty(0,T;(L^2(\Omega))^d) \cap L^2(0,T;(H^1(\Omega))^d), \\ f_t \in L^2(0,T;H^{-1}), \\ \sup_{t\in[0,T]} \|\nabla u(t)\| \leq \tilde{C}. \end{cases} \tag{25}$$

We shall use c and C as a generic positive constant which may depend on Ω, ν, T, constants from various Sobolev inequalities, u_0, f, and the solution u through the constant $\tilde{C}$ in (25).

Under the assumption (25), one can prove the following inequalities, see Ref. [24]:

$$\sup_{t\in[0,T]} \{\|u(t)\|_2 + \|u_t(t)\| + \|\nabla p(t)\|\} \le C, \tag{26}$$

$$\int_0^T \|\nabla u_t(t)\|^2 + t\|u_{tt}\|^2 dt \le C, \tag{27}$$

which will be used in the sequel. Further, we often use the following well-known [25] estimates for the bilinear form $b(u, v, w) = \int_\Omega (u \cdot \nabla)v \cdot w \,\mathrm{d}x$:

$$b(u, v, w) \le \begin{cases} c\|\nabla u\|\|\nabla v\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|\nabla w\|, \\ c\|u\|_2\|v\|\|\nabla w\|, \\ c\|\nabla u\|\|v\|_2\|w\|. \end{cases}$$

and $b(u, v, w) = -b(u, w, v)$ for $u \in H$.

Define the Stokes operator $Au = -\mathbb{P}\Delta u,\ \forall u \in D(A) = V \cap H^2(\Omega)^3$. We will use the following properties: A is an unbounded positive self-adjoint closed operator in H with domain $D(A)$, and its inverse A^{-1} is compact in H and satisfies the following relations [22, 23]:

$$\exists\, c, C > 0,\ \text{ such that } \forall u \in H : \begin{cases} \|A^{-1}u\|_2 \le c\|u\| \text{ and } \|A^{-1}u\| \le c\|u\|_{V'}, \\ c\|u\|^2_{V'} \le (A^{-1}u, u) \le C\|u\|^2_{V'}. \end{cases}$$

Before we proceed with the error analysis, we prove several auxiliary results given below in Lemma 2. The lemma gives estimates on the difference between a velocity w and the filtered velocity $F(u)w$.

Lemma 2. *Consider the differential filter F defined in* (11) *with some sufficiently smooth vector function u. For any $w \in V$ and $Fw \in V$ it holds*

$$\|w - Fw\| \le \delta_{\max}\|\nabla w\|, \tag{28}$$

$$\|w - Fw\|_{V'} \le \delta^2_{\max}\|\nabla w\|. \tag{29}$$

Proof. Denote $e = w - Fw$. The Eq. (11) gives

$$(\delta^2 a(u)\nabla e, \nabla v) + (e, v) = (\delta^2 a(u)\nabla w, \nabla v) \quad \forall\, v \in V.$$

Letting $v = e$ yields

$$\|\delta\sqrt{a(u)}\nabla e\|^2 + \|e\|^2 = (\delta^2 a(u)\nabla w, \nabla e) \le \|\delta\sqrt{a(u)}\nabla w\|\|\delta\sqrt{a(u)}\nabla e\|$$
$$\le \|\delta\sqrt{a(u)}\nabla e\|^2 + \frac{1}{4}\|\delta\sqrt{a(u)}\nabla w\|^2 \le \|\delta\sqrt{a(u)}\nabla e\|^2 + \frac{1}{4}\delta^2_{\max}\|\nabla w\|^2.$$

This proves (28). To show (29), we note that setting $v = F\,w - w$ in (11) gives

$$(\delta^2 a(u)\nabla F\,w, \nabla(Fw - w)) = -\|Fw - w\|^2 \le 0.$$

Hence, we obtain:

$$\|\delta\sqrt{a(u)}\nabla Fw\|^2 \le \|\delta\sqrt{a(u)}\nabla w\|^2. \tag{30}$$

Allowing $v = A^{-1}(w - Fw)$ in (11) leads to the following relations:

$$\begin{aligned}\|w - Fw\|_{V'}^2 &= (w - Fw, A^{-1}(w - Fw)) = (\delta^2 a(u)\nabla F\, w, \nabla A^{-1}(w - Fw)) \\ &\le \|\delta^2 a(u)\nabla F w\|\|\nabla A^{-1}(w - Fw)\| \le \frac{1}{2}(\|\delta^2 a(u)\nabla F w\|^2 + \|w - Fw\|_{V'}^2) \\ &\le \frac{1}{2}\delta_{\max}^2\|\delta\sqrt{a(u)}\nabla F w\|^2 + \frac{1}{2}\|w - Fw\|_{V'}^2.\end{aligned}$$

The last estimate and (30) implies (29). ■

Further in this section, we show that $\overline{w^{n+1}}$, w^{n+1}, and u^{n+1} are all strongly $O((\triangle t)^{\frac{1}{2}} + \delta)$ approximations to $u(t_{n+1})$ in $L^2(\Omega)^3$ provided $\chi = \chi_0\triangle t$. Then, we use this result to improve the error estimates to weakly $O(\triangle t + \delta^2)$ approximations. This analysis largely follows the framework from Refs. [22] and [23] for the pure (non-filtered) Navier–Stokes equations, so we shall refer to these papers and Ref. [26] for some arguments which do not depend on the filtering procedure.

Lemma 3. *Let u be the solution to the Navier–Stokes system, satisfying* (25). *Denote*

$$\widetilde{\epsilon^{n+1}} = u(t_{n+1}) - \widetilde{w^{n+1}}, \quad \epsilon^{n+1} = u(t_{n+1}) - w^{n+1}, \quad \text{and} \quad e^{n+1} = u(t_{n+1}) - u^{n+1}.$$

The following estimate holds

$$\|\widetilde{\epsilon^l}\|^2 + \sum_{n=0}^{l-1}(\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \|\widetilde{\epsilon^{n+1}} - e^n\|^2) + \sum_{n=0}^{l-1} 2\nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \le C(\triangle t + \delta_{\max}^2). \tag{31}$$

Proof. Let R^n denote the truncation error defined by

$$\frac{1}{\triangle t}(u(t_{n+1}) - u(t_n)) - \nu\triangle u(t_{n+1}) + (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + \nabla p(t_{n+1}) = f^{n+1} + R^n, \tag{32}$$

where R^n is the integral residual of the Taylor series, that is,

$$R^n = \frac{1}{\triangle t}\int_{t_n}^{t_{n+1}}(t - t_n)u_{tt}(t)dt.$$

By subtracting (19) from (32), we obtain

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}} - e^n) - \nu\triangle\widetilde{\epsilon^{n+1}} = (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) - \nabla p(t_{n+1}) + R^n. \tag{33}$$

Taking the L^2 scalar product of (33) with $2\triangle t\widetilde{\epsilon^{n+1}}$, we get

$$\begin{aligned}\|\widetilde{\epsilon^{n+1}}\|^2 - \|e^n\|^2 + \|\widetilde{\epsilon^{n+1}} - e^n\|^2 + 2\nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 &= 2\triangle t(R^n, \widetilde{\epsilon^{n+1}}) - 2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}}) \\ &\quad + 2\triangle t b^*(w^n, \widetilde{w^{n+1}}, \widetilde{\epsilon^{n+1}}) - 2\triangle t b^*(u(t_{n+1}), u(t_{n+1}), \widetilde{\epsilon^{n+1}}).\end{aligned} \tag{34}$$

The terms on the right-hand side are bounded exactly the same way as in Ref. [22] p.64 and Ref. [23] p.512, leading to the estimates:

$$\triangle t|b^*(w^n, \widetilde{w^{n+1}}, \widetilde{\epsilon^{n+1}}) - b^*(u(t_{n+1}), u(t_{n+1}), \widetilde{\epsilon^{n+1}})|$$
$$\leq \frac{\nu\triangle t}{2}\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + C\triangle t\|\epsilon^n\|^2 + C(\triangle t)^2\int_{t_n}^{t_{n+1}}\|u_t\|^2 dt, \tag{35}$$

$$2\triangle t(R^n, \widetilde{\epsilon^{n+1}}) \leq \frac{\nu\triangle t}{4}\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + C(\triangle t)^2\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt, \tag{36}$$

$$2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}}) = 2\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - e^n) \leq \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2. \tag{37}$$

Combining the inequalities (34), (35), (36), (37), and rearranging terms, we obtain

$$\|\widetilde{\epsilon^{n+1}}\|^2 - \|e^n\|^2 + \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2$$
$$\leq 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2 + C\triangle t\|\epsilon^n\|^2 + C(\triangle t)^2\left(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt + \int_{t_n}^{t_{n+1}}\|u_t\|^2 dt\right). \tag{38}$$

The Step 4 of the algorithm (19)–(21) yields

$$e^n = (1-\chi)\epsilon^n + \chi F(w^{n+1})\epsilon^n + \chi(u(t_n) - F(w^{n+1})u(t_n)). \tag{39}$$

The definition of the filter and recalling that ϵ^n is the L^2 projection of $\widetilde{\epsilon^n}$ give $\|F(w^{n+1})\epsilon^n\| \leq \|\epsilon^n\| \leq \|\widetilde{\epsilon^n}\|$. We use this to deduce from (39) the following estimate:

$$\|e^n\| = (1-\chi)\|\epsilon^n\| + \chi\|F(w^{n+1})\epsilon^n\| + \chi\|u(t_n) - F(w^{n+1})u(t_n)\|$$
$$\leq \|\widetilde{\epsilon^n}\| + \chi\|u(t_n) - F(w^{n+1})u(t_n)\|.$$

Now we apply (28) and square the resulting inequality to get (for the sake of convenience we assume $\triangle t \leq C$ and recall $\chi = \chi_0\triangle t$):

$$\|e^n\|^2 \leq (1+\triangle t)\|\widetilde{\epsilon^n}\|^2 + C\triangle t\delta_{\max}^2. \tag{40}$$

We substitute (40) to the left-hand side of (38) for $\|e^n\|$, use $\|\epsilon^n\| \leq \|\widetilde{\epsilon^n}\|$ and arrive at

$$\|\widetilde{\epsilon^{n+1}}\|^2 - \|\widetilde{\epsilon^n}\|^2 + \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \frac{1}{2}\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \nu\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2$$
$$\leq 2(\triangle t)^2\|\nabla p(t_{n+1})\|^2 + C\triangle t\|\widetilde{\epsilon^n}\|^2 + C(\triangle t)^2\left(\int_{t_n}^{t_{n+1}} t\|u_{tt}\|_{-1}^2 dt + \int_{t_n}^{t_{n+1}}\|u_t\|^2 dt\right) + C\triangle t\delta_{\max}^2. \tag{41}$$

Summing up (41) from $n = 0$ to $n = l - 1$, assuming that $\widetilde{w^0} = w^0 = u_0$ (this implies $\|e^0\| = \|\epsilon^0\| = 0$), we obtain

$$\|\widetilde{\epsilon^l}\|^2 + \sum_{n=0}^{l-1} \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \frac{1}{2}\sum_{n=0}^{l-1} \|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1} \nu\Delta t\|\nabla\widetilde{\epsilon^{n+1}}\|^2$$

$$\leq \sum_{n=0}^{l-l} C\Delta t\|\widetilde{\epsilon^n}\|^2 + 2(\Delta t)^2\sum_{n=0}^{l-1}\|\nabla p(t_{n+1})\|^2 + C(\Delta t)^2\left(\int_{t_0}^{t_l} t\|u_{tt}\|_{-1}^2 dt + \int_{t_0}^{t_l}\|u_t\|^2 dt\right) + C\delta_{\max}^2$$

$$\leq \sum_{n=0}^{l-1} C\Delta t\|\widetilde{\epsilon^n}\|^2 + C\Delta t + C\delta_{\max}^2.$$

Applying the discrete Gronwall inequality yields (31). ■

Now, we will use the result of the lemma and improve the predicted order of convergence for the velocity. The main result in this section is the following theorem, stating that all $\widetilde{w^{n+1}}$, w^{n+1} and, u^{n+1} are first-order approximations to the Navier–Stokes solution.

Theorem 3. *Assume the solution to the Navier–Stokes system satisfies* (25) *and* $\chi = \chi_0\Delta t$. *Suppose* $\partial\Omega \in C^{1,1}$ *or* Ω *is convex. It holds*

$$\Delta t\sum_{n=1}^{l}(\|\widetilde{\epsilon^n}\|^2 + \|\epsilon^n\|^2 + \|e^n\|^2) \leq C((\Delta t)^2 + \delta_{\max}^4). \tag{42}$$

Additionally assume $\int_0^T \|\nabla p_t\|^2 \leq C$ *and the filtering radius is bounded as* $\delta_{\max}^4 \leq C\,\Delta t$, *then* p^n *is an approximation to* $p(t_n)$ *in* $L^2(\Omega)/R$ *in the following sense:*

$$\Delta t\sum_{n=1}^{l}\|p^n - p(t_n)\|^2 \leq C(\Delta t + \delta_{\max}^2). \tag{43}$$

Proof. Literally reaping the arguments from Ref. [22], pp. 66–69, one shows the estimate

$$\|\epsilon^{n+1}\|_{V'}^2 - \|e^n\|_{V'}^2 + \|\epsilon^{n+1} - e^n\|_{V'}^2 + \nu\Delta t\|\epsilon^{n+1}\|^2$$

$$\leq C\left(\Delta t\|\epsilon^{n+1}\|_{V'}^2(\Delta t)^2\int_{t_n}^{t_{n+1}}(t\|u_{tt}\|_{-1}^2 + \|u_t\|^2)dt + (\Delta t)^2\|\nabla\widetilde{\epsilon^{n+1}}\|^2\right.$$

$$\left. + \Delta t\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \Delta t\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2\right). \tag{44}$$

The estimate (29) gives $\|F\epsilon^n\|_{V'} \leq \|\epsilon^n\|_{V'} + \delta_{\max}^2\|\nabla\epsilon^n\|$. Here and in the rest of the proof, the filtering is based on the w^{n+1} velocity, that is $F\cdot := F(w^{n+1})\cdot$. Due to the assumption $\partial\Omega \in C^{1,1}$ or Ω is convex, the L^2 projection on H is H^1 stable, that is, $\|\nabla\epsilon^n\| \leq C\|\nabla\widetilde{\epsilon^n}\|$ and therefore we conclude

$$\|F\epsilon^n\|_{V'} \leq \|\epsilon^n\|_{V'} + C\delta_{\max}^2\|\nabla\widetilde{\epsilon^n}\|.$$

Using this and (29), we get from (39) for $\chi = \chi_0 \triangle t$

$$\begin{aligned}\|e^n\|_{V'} &= (1-\chi)\|\epsilon^n\|_{V'} + \chi\|F\epsilon^n\|_{V'} + \chi\|u(t_n) - Fu(t_n)\|_{V'} \\ &\le \|\epsilon^n\|_{V'} + C\triangle t\left(\delta_{\max}^2\|\nabla\widetilde{\epsilon^n}\| + \|u(t_n) - Fu(t_n)\|_{V'}\right) \\ &\le \|\epsilon^n\|_{V'} + C\triangle t\delta_{\max}^2\left(\|\nabla\widetilde{\epsilon^n}\| + 1\right).\end{aligned}$$

Squaring the inequality, we get after elementary calculations

$$\|e^n\|_{V'}^2 \le (1+\triangle t)\|\epsilon^n\|_{V'}^2 + C\triangle t\delta_{\max}^4\left(\|\nabla\widetilde{\epsilon^n}\|^2 + 1\right).$$

We substitute the above estimate to the left-hand side of (44) and arrive at

$$\begin{aligned}&\|\epsilon^{n+1}\|_{V'}^2 - \|\epsilon^n\|_{V'}^2 + \|\epsilon^{n+1} - e^n\|_{V'}^2 + \nu\triangle t\|\epsilon^{n+1}\|^2 \\ &\le C\Bigg(\triangle t(\|\epsilon^{n+1}\|_{V'}^2 + \|\epsilon^n\|_{V'}^2) + (\triangle t)^2\int_{t_n}^{t_{n+1}}(t\|u_{tt}\|_{-1}^2 + \|u_t\|^2)dt + (\triangle t)^2\|\nabla\widetilde{\epsilon^{n+1}}\|^2 \\ &\qquad + \triangle t(\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2) + \triangle t\delta_{\max}^4(1 + \|\nabla\widetilde{\epsilon^n}\|^2)\Bigg).\end{aligned}$$

Assume for the sake of convenience $\delta_{\max} \le C$. Summing up the inequalities for $n = 0, \dots, l-1$, we get

$$\begin{aligned}&\|\epsilon^l\|_{V'}^2 + \sum_{n=0}^{l-1}\|\epsilon^{n+1} - e^n\|_{V'}^2 + \sum_{n=0}^{l-1}\nu\triangle t\|\epsilon^{n+1}\|^2 \\ &\le C\Bigg(\sum_{n=0}^{l-1}\triangle t\|\epsilon^{n+1}\|_{V'}^2 + (\triangle t)^2\int_{t_0}^{t_l}(\|u_{tt}\|_{V'}^2 + \|u_t\|^2)dt + \delta_{\max}^4\sum_{n=0}^{l-1}\triangle t\|\nabla\widetilde{\epsilon^n}\|^2 \\ &\qquad + \sum_{n=0}^{l-1}\triangle t\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1}\triangle t\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 + \triangle t\delta_{\max}^4\Bigg). \qquad (45)\end{aligned}$$

Now we use the result of the Lemma 3 to bound

$$\begin{aligned}&\triangle t\|\epsilon^l\|_{V'}^2 + \delta_{\max}^4\sum_{n=0}^{l-1}\triangle t\|\nabla\widetilde{\epsilon^{n+1}}\|^2 + \sum_{n=0}^{l-1}\triangle t\|\widetilde{\epsilon^{n+1}} - e^n\|^2 + \sum_{n=0}^{l-1}\triangle t\|\epsilon^{n+1} - \widetilde{\epsilon^{n+1}}\|^2 \\ &\le C((\triangle t)^2 + \triangle t\delta_{\max}^2 + \delta_{\max}^4).\end{aligned}$$

Thus, applying the Gronwall inequality to (45) yields

$$\|\epsilon^l\|_{V'}^2 + \sum_{n=0}^{l-1}\|\epsilon^{n+1} - e^n\|_{V'}^2 + \sum_{n=0}^{l-1}\nu\triangle t\|\epsilon^{n+1}\|^2 \le C((\triangle t)^2 + \delta_{\max}^4). \qquad (46)$$

Here, we also used $\triangle t\delta_{\max}^2 \le (\triangle t)^2 + \delta_{\max}^4$. Finally, the Lemma 3 helps us to estimate

$$\triangle t\sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}}\|^2 \le \triangle t\sum_{n=0}^{l-1}\|\epsilon^{n+1}-\widetilde{\epsilon^{n+1}}\|^2 + \triangle t\sum_{n=0}^{l-1}\|\epsilon^{n+1}\|^2 \le C((\triangle t)^2+\delta_{\max}^4).$$

$$\triangle t\sum_{n=0}^{l}\|e^n\|^2 \le \triangle t\sum_{n=0}^{l-1}\|\epsilon^{n+1}-e^n\|^2 + \triangle t\sum_{n=0}^{l-1}\|\epsilon^{n+1}\|^2 \le C((\triangle t)^2+\delta_{\max}^4).$$

These estimates together with (46) proves the velocity error estimate of the theorem.

Further, we show that the pressure is weakly $\frac{1}{2}$ order convergent to the true solution. Denote the pressure error as $q^n = p^n - p(t_n)$. We may assume $(q^n, 1) = 0$. It holds

$$-\nabla q^{n+1} = -\frac{1}{\triangle t}(\epsilon^{n+1}-e^n) + \nu\triangle\widetilde{\epsilon^{n+1}} + (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) + R^n. \quad (47)$$

Repeating the arguments from Ref. [22] and using the Nečas inequality, see Ref. [27], one deduces from (47)

$$\begin{aligned}\|q^{n+1}\| &\le c\sup_{v\in H_0^1(\Omega)^3}\frac{(\nabla q^{n+1}, v)}{\|\nabla v\|}\\ &\le \frac{1}{\triangle t}\|\epsilon^{n+1}-e^n\|_{-1} + C(\|R^n\|_{-1} + \|\nabla\widetilde{\epsilon^{n+1}}\| + \|\nabla\epsilon^{n+1}\| + \|u(t_{n+1})-u(t_n)\|).\end{aligned}$$

Therefore, by using (31), we get

$$\triangle t\sum_{n=0}^{l-1}\|q^{n+1}\|^2 \le \frac{1}{\triangle t}\sum_{n=0}^{l-1}\|\nabla(\epsilon^{n+1}-e^n)\|_{-1}^2 + C(\triangle t+\delta_{max}^2). \quad (48)$$

To bound the first term on the right-hand side of (48), one estimates:

$$\|\epsilon^{n+1}-e^n\|_{-1} \le c\|\epsilon^{n+1}-e^n\| \le c(\|\epsilon^{n+1}-\epsilon^n\| + \|\epsilon^n-e^n\|) \le c(\|\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}\| + \|\epsilon^n-e^n\|). \quad (49)$$

The estimate for the second term on the right-hand side of (49) follows from (39):

$$\begin{aligned}\|\epsilon^n-e^n\| &\le \chi_0\triangle t(\|\epsilon^n-F\epsilon^n\| + \|u(t_n)-Fu(t_n)\|)\\ &\le \chi_0\triangle t(\|\epsilon^n\| + \|F\epsilon^n\| + \|u(t_n)-Fu(t_n)\|).\end{aligned}$$

Thanks to (28), (31), and $\|F\epsilon^n\| \le \|\epsilon^n\|$, we continue the above estimate as

$$\|\epsilon^n-e^n\| \le C((\triangle t)^{\frac{3}{2}} + \triangle t\delta_{\max}). \quad (50)$$

Below we shall prove the bound

$$\sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}}-\widetilde{\epsilon^n}\|^2 \le C((\triangle t)^2 + \triangle t\delta_{\max}^2).$$

From (19) and (21), we get

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}} - e^n) - \nu\Delta\widetilde{\epsilon^{n+1}} + \nabla p(t_{n+1}) + (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) = R^n. \quad (51)$$

The projection step (20) gives $\epsilon^n = \widetilde{\epsilon^n} + \triangle t\nabla p^n$, so (39) yields

$$e^n = (1-\chi)(\widetilde{\epsilon^n} + \triangle t\nabla p^n) + \chi F\epsilon^n + \chi(u(t_n) - Fu(t_n)).$$

Substituting this in (51) implies

$$\frac{1}{\triangle t}(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) - \nu\Delta\widetilde{\epsilon^{n+1}} + (1-\chi)\nabla(p(t_{n+1}) - p^n) + \chi\nabla p(t_{n+1}) - \frac{\chi}{\triangle t}(F\epsilon^n - \widetilde{\epsilon^n})$$
$$- \frac{\chi}{\triangle t}(u(t_n) - Fu(t_n)) + (w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}) = R^n. \quad (52)$$

The inner product of (52) with $\triangle t(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})$ gives

$$\begin{aligned}
&\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\triangle t}{2}(\|\nabla\widetilde{\epsilon^{n+1}}\|^2 - \|\nabla\widetilde{\epsilon^n}\|^2 + \|\nabla(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\|^2)\\
&\quad = \triangle t(R^n, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) + (1-\chi)\triangle t(p(t_{n+1}) - p^n, \operatorname{div}(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}))\\
&\qquad + \triangle t((w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\\
&\qquad - \chi\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) + \chi(F\epsilon^n - \widetilde{\epsilon^n}, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) + \chi(u(t_n) - Fu(t_n), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\\
&\quad = \triangle t(R^n, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) + (1-\chi)\triangle t\Big[(q^n, \operatorname{div}(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})) + (p(t_{n+1}) - p(t_n), \operatorname{div}(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}))\Big]\\
&\qquad - \chi\Big[\triangle t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) - (F\epsilon^n - \widetilde{\epsilon^n}, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) - (u(t_n) - Fu(t_n), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\Big]\\
&\qquad + \triangle t((w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \qquad (53)
\end{aligned}$$

The last term I_7 is estimated in Ref. [26]:

$$\begin{aligned}
&\triangle t|((w^n\cdot\nabla)\widetilde{w^{n+1}} - (u(t_{n+1})\cdot\nabla)u(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})|\\
&\quad \le \sigma\|\widetilde{\epsilon}^{n+1} - \widetilde{\epsilon}^n\|^2 + C\Big((\triangle t)^2\|\widetilde{\epsilon}^{n+1}\|^2 + (\triangle t)^2\|\epsilon^{n+1}\|^2 + \triangle t^{\frac{3}{2}}\|\nabla\epsilon^n\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2\\
&\quad + \frac{\nu\triangle t}{2}\|\nabla(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\|^2 + (\triangle t)^3\Big)
\end{aligned}$$

for some $\sigma > 0$, which can be taken sufficiently small. Applying (31) and $\|\nabla\epsilon^n\| \le C\|\widetilde{\nabla\epsilon^n}\|$ leads to

$$I_7 \le \sigma\|\widetilde{\epsilon}^{n+1} - \widetilde{\epsilon}^n\|^2 + C((\Delta t)^3 + (\Delta t)^2\delta_{\max}^2) + (\Delta t)^{\frac{3}{2}}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2 + \frac{\nu\Delta t}{2}\|\nabla(\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n})\|^2. \tag{54}$$

For I_4, I_5, and I_6 one has

$$I_4 = -\chi\Delta t(\nabla p(t_{n+1}), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C\chi^2(\Delta t)^2\|\nabla p(t_{n+1})\|^2 + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2, \tag{55}$$

$$\begin{aligned} I_5 &= \chi(F\epsilon^n - \widetilde{\epsilon^n}, \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C\chi^2(\|F\epsilon^n\|^2 + \|\widetilde{\epsilon^n}\|^2) + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 \\ &\le C((\Delta t)^3 + (\Delta t)^2\delta_{\max}^2) + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2, \end{aligned} \tag{56}$$

$$I_6 = \chi(u(t_n) - Fu(t_n), \widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}) \le C(\Delta t)^2\delta_{\max}^4 + \sigma\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2. \tag{57}$$

The terms I_1, I_2 and I_3 are estimated in Ref. [22]. Using those estimates and (54)–(57) in (53) yields for sufficiently small $\sigma > 0$:

$$\begin{aligned} &\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\Delta t}{2}(\|\nabla\widetilde{\epsilon^{n+1}}\|^2 - \|\nabla\widetilde{\epsilon^n}\|^2) + (1-\chi)(\Delta t)^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) \\ &\quad \le C\left\{(\Delta t)^2\int_{t_n}^{t_{n+1}}\|u_{tt}\|^2dt + (\Delta t)^2\int_{t_n}^{t_{n+1}}\|\nabla p_t\|^2dt + (\Delta t)^4\|\nabla p(t_{n+1})\|^2\right. \\ &\quad\quad \left. + (\Delta t)^3 + (\Delta t)^2\delta_{\max}^2 + \Delta t^{\frac{3}{2}}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2\right\}. \end{aligned} \tag{58}$$

We sum up the estimate for $n = 0, \ldots, l-1$ and apply our assumptions for the solution to Navier–Stokes solution. This leads to the bound

$$\sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\Delta t}{2}\|\nabla\widetilde{\epsilon^l}\|^2 \le C\left((\Delta t)^2 + \Delta t\delta_{\max}^4 + (\Delta t)^{\frac{3}{2}}\sum_{n=0}^{l-1}\|\nabla\widetilde{\epsilon^n}\|^2\|\nabla\widetilde{\epsilon}^{n+1}\|^2\right).$$

The application of the discrete Gronwall inequality, (31) and the assumption $\delta_{\max}^4 \le C\Delta t$ yields

$$\begin{aligned} \sum_{n=0}^{l-1}\|\widetilde{\epsilon^{n+1}} - \widetilde{\epsilon^n}\|^2 + \frac{\nu\Delta t}{2}\|\nabla\widetilde{\epsilon^l}\|^2 &\le C\left((\Delta t)^2 + \Delta t\delta_{\max}^2\right)\exp\left\{(\Delta t)^{\frac{1}{2}}\sum_{n=0}^{l-1}\|\nabla\widetilde{\epsilon^{n+1}}\|^2\right\} \\ &\le C\left((\Delta t)^2 + \Delta t\delta_{\max}^2\right)\exp\left\{C((\Delta t)^{\frac{1}{2}} + (\Delta t)^{-\frac{1}{2}}\delta_{\max}^4)\right\} \\ &\le C((\Delta t)^2 + \Delta t\delta_{\max}^4). \end{aligned}$$

Therefore, (48)–(50) yield the desired bound:

$$\Delta t\sum_{n=0}^{l-1}\|q^{n+1}\|^2 \le C(\Delta t + \delta_{\max}^2).$$

■

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