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Effective preconditioning of Uzawa type schemes for a generalized Stokes problem*

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Summary. The Schur complement of a model problem is considered as a preconditioner for the Uzawa type schemes for the generalized Stokes problem (the Stokes problem with the additional term $\alpha \boldsymbol{u}$ in the motion equation). The implementation of the preconditioned method requires for each iteration only one extra solution of the Poisson equation with Neumann boundary conditions. For a wide class of 2D and 3D domains a theorem on its convergence is proved. In particular, it is established that the method converges with a rate that is bounded by some constant independent of α. Some finite difference and finite element methods are discussed. Numerical results for finite difference MAC scheme are provided.

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Introduction

The numerical solution of the generalized Stokes problem plays a fundamental role in the simulation of viscous incompressible flows (laminar and turbulent). Although plenty of iterative algorithms are available for solving the classical Stokes problem, their direct application to the generalized Stokes problem leads, as a rule, to the growth of the convergence factor when a certain parameter associated with the problem tends to zero or infinity.

Thus, we need efficient iterative methods for the generalized Stokes problem, whose rates of convergence would be at least not worse than for the

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well-known algorithms for the classical Stokes problem. Recently, several ways to develop efficient iterative solution techniques have been proposed. Cahouet and Chabard (1988) and Olshanskii (1995), (1996) improved the Uzawa scheme; Pal'tsev (1995a) and (1995b) constructed algorithms based on complete and incomplete splittings of boundary conditions; Bakhvalov (1995) considered the fictitious domain method. All these papers deal with the pressure-velocity formulation of the problem.

The motivation of this work is to develop a general mathematical theory that underlies the approaches cited above for the preconditioning of Uzawa type schemes and to demonstrate advantages of these considerations. It means, in particular, possibility to extend ideas developed in Olshanskii (1995) to a wider class of domains and the rigorous proofs of convergence theorems. The important result (see Theorem 2.1) relates the method presented to ideas of Cahouet and Chabard and establishes the equivalence of these approaches in the continuous case. Two key points of the paper should be emphasized. These are the preconditioning of the Uzawa scheme with a Schur operator for the model problem (this operator is shown to be equivalent to some pseudo-differential operator) and sufficient conditions for convergence in the form of inequalities of Ladyzhenskaya-Babuška-Brezzi type. Their original proof in domains with regular boundaries is presented in Sect. 4, and some results for Lipschitz domains can be found in the Appendix.

Basic considerations are presented for the continuous case although their application for finite differences and some remarks on finite elements can be found in Sect. 5. Some numerical results we present in Sect. 6. For further discussions on the numerical performance we refer to Cahouet, Chabard (1988) and Bramble, Pasciak (1997) for finite element calculations with different parameters, domains, elements.

1. Generalized Stokes problem, Uzawa algorithm and its preconditioning

Let Ω be a domain in $\mathbb{R}^n,\ n = 2, 3$, with Lipschitz-continuous boundary $\partial\Omega$. Consider in Ω the system of partial differential equations

$$\begin{aligned} &-\Delta \boldsymbol{u} + \alpha \boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div} \boldsymbol{u} = 0, \\ &\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \\ &\textstyle\int_\Omega p\, dx = 0 \end{aligned} \tag{1.1}$$

where $\boldsymbol{u} = (u_1(x), \ldots, u_n(x))$ is the velocity vector, $p = p(x)$ is the pressure function, $\boldsymbol{f} = (f_1(x), \ldots, f_n(x))$ is the field of external forces, and $\alpha = \text{const} \geq 0$ is an arbitrary real parameter. If $\alpha = 0$, then (1.1) becomes the classical Stokes system. In unsteady Navier-Stokes calculations typically $\alpha \sim (\bar{\nu}\,\delta t)^{-1}$, where $\bar{\nu}$ is a kinematic viscosity and δt is a time step. Hence, as a rule, $\alpha \gg 1$.

Later on we need the following function spaces:

$$\boldsymbol{H}_0^1 \equiv \{\boldsymbol{u} \in W_2^1(\Omega)^n :\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

with the energy scalar product $(\boldsymbol{u},\boldsymbol{v})_1 = (\nabla\boldsymbol{u},\nabla\boldsymbol{v}),\ \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0^1$,

$$L_2/\mathbb{R} \equiv \{p \in L_2(\Omega) :\ \int_\Omega p\,dx = 0\}$$

with the L_2-scalar product. Let $\boldsymbol{H}^{-1}$ be a dual, with respect to L_2-duality, space to $\boldsymbol{H}_0^1$ with the obvious norm:

$$\|\boldsymbol{f}\|_{-1} = \sup_{0\neq\boldsymbol{u}\in\boldsymbol{H}_0^1} \frac{<\boldsymbol{f},\boldsymbol{u}>}{\|\boldsymbol{u}\|_1},\ \ \boldsymbol{f} \in \boldsymbol{H}^{-1}.$$

The solution $\{\boldsymbol{u}, p\}$ of the *generalized Stokes problem* (1.1) exists and is unique in $\boldsymbol{H}_0^1 \times L_2/\mathbb{R}$ for any given $\boldsymbol{f} \in \boldsymbol{H}^{-1}$. The case of Dirichlet boundary conditions for the velocity is of fundamental interest both in theory and applications although some other boundary conditions can be considered as well (see, e.g., Sani, Gresho (1994)). Non-homogeneous conditions can be also considered without loss of generality (see Girault, Raviart (1986)).

Probably the simplest (but surprisingly effective Elman (1996), Turek (1999)) method to solve the *classical* ($\alpha = 0$) Stokes problem is the iterative Uzawa algorithm (see Arrow, Hurwicz, Uzawa(1958)). For the generalized Stokes problem the Uzawa algorithm is described as follows: start with an arbitrary initial guess $p_0 \in L_2/R$ and for $i = 0, 1, \ldots$ do until convergence:

$$\begin{aligned} &\text{Step1. Compute } \boldsymbol{u}^{\mathrm{i}+1} \text{ from} \\ &\qquad -\Delta\boldsymbol{u}^{\mathrm{i}+1} + \alpha\boldsymbol{u}^{\mathrm{i}+1} = \boldsymbol{f} - \nabla p^i \\ &\qquad \text{with } \boldsymbol{u}^{\mathrm{i}+1} = 0 \text{ on } \partial\Omega. \\ &\text{Step2. Define the new pressure } p^{\mathrm{i}+1} \text{ as} \\ &\qquad p^{\mathrm{i}+1} = p^i - \tau_0 \mathrm{div}\, \boldsymbol{u}^{\mathrm{i}+1}. \end{aligned} \tag{1.2}$$

The algorithm converges for sufficiently small values of $\tau_0 (> 0)$.

Consider the Schur complement for system (1.1):

$$A_0(\alpha) = \mathrm{div}\,(\Delta - \alpha I)_0^{-1}\nabla,$$

where $(\Delta - \alpha I)_0^{-1} : \boldsymbol{H}^{-1} \to \boldsymbol{H}_0^1$ denotes the solution operator for the Helmholtz problem

$$(1.3) \qquad \begin{aligned} \Delta \boldsymbol{u} - \alpha \boldsymbol{u} &= \boldsymbol{g}, \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{0} \end{aligned}$$

i.e., for a given $\boldsymbol{g} \in \boldsymbol{H}^{-1}$ the vector function $\boldsymbol{u} = (\Delta - \alpha I)_0^{-1}\boldsymbol{g}$ is the solution of (1.3).

The operator $A_0(\alpha)$ is self-adjoint and positive definite from $L_2/\mathbb{R}$ onto $L_2/\mathbb{R}$. Now the Uzawa algorithm can be considered as a first order Richardson iteration method with a fixed iterative parameter applied to the equation

$$(1.4) \qquad A_0(\alpha)p = \operatorname{div}(\Delta - \alpha I)_0^{-1}\boldsymbol{f}.$$

This simple observation provides us with the convergence rate of (1.2), i.e., $\rho \sim 1 - O(\alpha^{-1}),\ \alpha \to \infty$ (see Remark 3.2 in Sect. 3). Therefore, $\rho \to 1$ for $\alpha \to \infty$. However, the same observation allows us to improve the classical Uzawa scheme by using various Krylov subspace methods (e.g., conjugate gradient or conjugate residual ones) for the system (1.4), and provides us with natural and fruitful ideas for preconditioning.

Indeed, consider for simplicity the first order preconditioned iterative method

$$(1.5) \qquad B\frac{p^{i+1} - p^i}{\tau_i} = -A_0(\alpha)p^i + \boldsymbol{f},\ \ i = 0, 1, \dots$$

where $B = B^* > 0$ is a preconditioner depending in general on α and acting from $L_2(\Omega)/\mathbb{R}$ onto $L_2(\Omega)/\mathbb{R}$. The natural requirements are the 'easy' solvability of the equation $Bp = q$ for $q \in L_2(\Omega)/\mathbb{R}$ and the validity of the estimate $\operatorname{cond}(B^{-1}A_0(\alpha)) \le c$, where c is some constant independent of α and the mesh size.

As far as we know, at least two ways of preconditioning considered in literature can be candidates to satisfy these two requirements. To begin with, note that in each step of method (1.5) we have to compute $q = B^{-1}\bar{p}$ for some $\bar{p} \in L_2/\mathbb{R}$.

Let us set

$$B^{-1}\bar{p} = \bar{p} - \alpha r,$$

where r is a solution of the following boundary value problem

$$\Delta r = \bar{p},\ \ \left.\frac{\partial r}{\partial \boldsymbol{\nu}}\right|_{\partial\Omega} = 0,\ \ \bar{p} \in L_2/\mathbb{R}.$$

Denote by Δ_N^{-1} the solution operator for the above Poisson equation with Neumann boundary conditions. Then set formally $B_1 = (\mathrm{I} - \alpha\Delta_N^{-1})^{-1}$

and consider it as a preconditioner. This approach of Cahouet–Chabard certainly satisfies the condition of 'easy solvability' (one step of the algorithm requires only one extra solution of the Neumann problem that is standard). Numerous finite element calculations presented in Cahouet, Chabard (1988) show the efficiency of such preconditioning, however without estimates on the condition number for $B^{-1}A_0(\alpha)$ or appropriate convergence theorem.

To outline the second approach, let us consider in $\Omega = (0,1) \times (0,1)$ the generalized Stokes problem with *different boundary conditions*:

$$\begin{aligned} &-\Delta \boldsymbol{u} + \alpha \boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div} \boldsymbol{u} = 0, \\ &\boldsymbol{u}{\cdot}\boldsymbol{\nu}|_{\partial\Omega} = \left.\frac{\partial(\boldsymbol{u}{\cdot}\boldsymbol{\tau})}{\partial\boldsymbol{\nu}}\right|_{\partial\Omega} = 0. \end{aligned} \tag{1.6}$$

Hereafter $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are outer normal and tangential unit vectors to $\partial\Omega$.

This problem is well-posed and the solution $\{\boldsymbol{u}, p\}$ of (1.6) can be *explicitly* found via Fourier series:

$$\begin{aligned} u_1(x,y) &= \sum_{m,k=0}^{\infty} a_{m,k} \sin m\pi x \, \cos k\pi y, \\ u_2(x,y) &= \sum_{m,k=0}^{\infty} b_{m,k} \cos m\pi x \, \sin k\pi y, \\ p(x,y) &= \sum_{\substack{m,k=0 \\ m+n>0}}^{\infty} c_{m,k} \cos m\pi x \, \cos k\pi y. \end{aligned}$$

Now let us consider the Schur complement of (1.6) as a preconditioner

$$B_2 = \operatorname{div} (\Delta - \alpha I)_p^{-1} \nabla p,$$

where the boundary conditions from (1.6) are 'built in' into B_2 in the same way as the Dirichlet conditions in $A_0(\alpha)$. It is easy to check that $B_2 = I$ for $\alpha = 0$; hereafter I denotes the identity operator.

In Olshanskii (1995), the inequalities

$$cB_2 \le A_0(\alpha) \le B_2 \tag{1.7}$$

were proved with c independent of α. Numerical experiments with finite difference schemes demonstrate a very good convergence of (1.5) with such preconditioning. However, the estimate (1.7) was proved only for rectangular domains. Moreover, since we use Fast Fourier Transform for 'easy solvability' of the equation $B_2 p = q$, it requires a rectangular domain and a uniform grid at least in one direction.

Recently new boundary conditions were suggested for the operator B_2 for a wider class of domains (see next section). These boundary conditions generalize the previous one (1.6).

In the present paper we use these results to prove a uniform convergence of the method with respect to α for this class of domains. Moreover, we show that in the continuous case the preconditioner with these new boundary conditions appears to be the same as the one of Cahouet – Chabart. We shall clarify corresponding details in the next section.

2. Model boundary conditions

In this section we give some auxiliary results which will be used later. We assume Ω to be a bounded Lipschitz continuous domain in $\mathbb{R}^n$, $n = 2, 3$.

The traces of vector functions from $W_1^2(\Omega)^n$ induce on $\partial\Omega$ a function space denoted by $H^{\frac{1}{2}}(\partial\Omega)^n$ and equipped with the norm

$$\|\boldsymbol{\mu}\|_{\frac{1}{2}} = \inf_{\substack{\boldsymbol{v}\in W_2^1(\Omega)^n \\ \boldsymbol{v}=\boldsymbol{\mu} \text{ on } \partial\Omega}} \|\boldsymbol{v}\|_{W_2^1}, \qquad \boldsymbol{\mu} \in H^{\frac{1}{2}}(\partial\Omega)^n.$$

Let $H^{-\frac{1}{2}}(\partial\Omega)^n$ be the dual space to $H^{\frac{1}{2}}(\partial\Omega)^n$ with the norm

$$\|\boldsymbol{\xi}\|_{-\frac{1}{2}} = \sup_{0\neq\boldsymbol{\mu}\in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)^n} \frac{<\boldsymbol{\xi}, \boldsymbol{\mu}>}{\|\boldsymbol{\mu}\|_{\frac{1}{2}}}, \qquad \boldsymbol{\xi} \in H^{-\frac{1}{2}}(\Omega)^n.$$

For any vector function $\boldsymbol{u} \in L_2(\Omega)^n$ such that $\operatorname{div} \boldsymbol{u} \in L_2(\Omega)$, $\operatorname{curl} \boldsymbol{u} \in L_2(\Omega)^{2n-3}$, its normal and tangential components on $\partial\Omega$ ($\boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\gamma_\tau \boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{\tau}$ for $n = 2$, $\gamma_\tau \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\nu}$ for $n = 3$) can be considered as elements of $H^{-\frac{1}{2}}(\Omega)^r, r = 1, 3$. Thus, the definition of the following function space is correct:

$$\boldsymbol{U} \equiv \{\boldsymbol{u} \in L_2(\Omega)^n : \operatorname{div} \boldsymbol{u} \in L_2(\Omega), \operatorname{curl} \boldsymbol{u} \in L_2(\Omega)^{2n-3},$$

$$\boldsymbol{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\};$$

$\boldsymbol{U}$ is a Hilbert space with respect to the scalar product $(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{U}} = (\boldsymbol{u}, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) + (\operatorname{curl} \boldsymbol{u}, \operatorname{curl} \boldsymbol{v})$, $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{U}$. By $\boldsymbol{U}^{-1}$ we denote the dual space to $\boldsymbol{U}$ with respect to the L_2 duality.

Remark 2.1 If $\boldsymbol{u} \in \boldsymbol{U}$ and $\gamma_\tau \boldsymbol{u} = 0$, then $\boldsymbol{u} \in \boldsymbol{H}_0^1$ (see Lemma 2.5 in Girault, Raviart (1986)).

Remark 2.2 Further we need the following estimates (Girault, Raviart (1986)):

$$\begin{aligned} \|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2}} &\leq \|\boldsymbol{u}\| + \|\operatorname{div}\boldsymbol{u}\|, \\ \|\gamma_\tau \boldsymbol{u}\|_{-\frac{1}{2}} &\leq \|\boldsymbol{u}\| + \|\operatorname{curl}\boldsymbol{u}\|. \end{aligned} \tag{2.1}$$

Here and in what follows, $||\cdot||$ always denotes the L_2 norm.

Consider now the problem: find $\{\boldsymbol{u}, p\}$ from $\boldsymbol{U} \times L_2/\mathbb{R}$ for given $\{\boldsymbol{f}, g\} \in \boldsymbol{U}^{-1} \times L_2/\mathbb{R}$ such that

$$\begin{aligned} &(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) + (\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{v}) + \alpha(\boldsymbol{u}, \boldsymbol{v}) - (p, \operatorname{div}\boldsymbol{v}) = <\boldsymbol{f}, \boldsymbol{v}>, \\ &(\operatorname{div}\boldsymbol{u}, q) = (g, q), \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{U},\ q \in L_2/\mathbb{R}. \end{aligned} \tag{2.2}$$

Problem (2.2) is well posed for all $\alpha \geq 0$.

Remark 2.3 For $\alpha = 0$, we should in addition require Ω to be a simply connected domain. In this case, the bilinear form $(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) + (\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{v})$ is coercive on $\boldsymbol{U}$.

It is easy to check that (2.2) is a *weak formulation* of the following generalized Stokes problem with the *model boundary conditions:*

$$\begin{aligned} &-\Delta\boldsymbol{u} + \alpha\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ &\operatorname{div}\boldsymbol{u} = 0, \\ &\boldsymbol{u}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \mathcal{R}\boldsymbol{u}|_{\partial\Omega} = 0, \end{aligned} \tag{2.3}$$

where

$$\mathcal{R}\boldsymbol{u} = \begin{cases} \operatorname{curl}\boldsymbol{u}, & n = 2; \\ (\operatorname{curl}\boldsymbol{u}) \times \boldsymbol{\nu}, & n = 3. \end{cases}$$

We will also refer to this problem as a *model* problem.

By $A_\nu(\alpha)$ we denote the Schur complement for system (2.3):

$$A_\nu(\alpha) \equiv \operatorname{div}(\Delta - \alpha I)_\nu^{-1}\nabla,$$

where $(\Delta - \alpha I)_\nu^{-1} : \boldsymbol{U}^{-1} \to \boldsymbol{U}$ denotes the solution operator for the problem

$$\begin{aligned} \Delta\boldsymbol{u} - \alpha\boldsymbol{u} &= \boldsymbol{g}, \\ \boldsymbol{u}\cdot\boldsymbol{\nu}|_{\partial\Omega} &= \mathcal{R}\boldsymbol{u}|_{\partial\Omega} = 0, \end{aligned}$$

and ∇ acts from $L_2/\mathbb{R}$ into $\boldsymbol{U}^{-1}$, i.e., $\boldsymbol{w} = (\Delta - \alpha I)_\nu^{-1}\nabla p$ and $\boldsymbol{w} \in \boldsymbol{U}$ is a solution of the problem

$$(\operatorname{div}\boldsymbol{w}, \operatorname{div}\boldsymbol{v}) + (\operatorname{curl}\boldsymbol{w}, \operatorname{curl}\boldsymbol{v}) + \alpha(\boldsymbol{w}, \boldsymbol{v}) = (p, \operatorname{div}\boldsymbol{v}). \qquad \forall \boldsymbol{v} \in \boldsymbol{U}$$

The operator $A_\nu(\alpha)$ is self-adjoint and positive definite on $L_2/\mathbb{R}$.

The following result concludes this section.

Theorem 2.1 (Olshanskii (1997)) *For arbitrary* $\alpha \in [0, \infty)$ *and* $p \in L_2/\mathbb{R}$, *set* $q = A_\nu(\alpha)p$, $q \in L_2/\mathbb{R}$. *Then the following equality holds:* $p = q - \alpha\, r$, *where* $r \in W_2^1(\Omega)/\mathbb{R}$ *is a solution to the Neumann problem*

$$\Delta r = q, \quad \left.\frac{\partial r}{\partial \boldsymbol{\nu}}\right|_{\partial\Omega} = 0.$$

Note that in the case $\alpha = 0$ the theorem states equality of the Schur complement of the model Stokes problem to the identity operator on $L_2/\mathbb{R}$. In the general case, the model boundary conditions provide decoupling of pressure and velocity in the generalized Stokes problem.

3. Iterative method and uniform estimate of the convergence rate

For the sake of convenience, rewrite the iterative algorithm (1.5) by taking the Schur complement of the model problem from Sect. 2 as a preconditioner:

$$A_\nu(\alpha)\frac{p^{\mathrm{i}+1} - p^i}{\tau_i} = -A_0(\alpha)p^i + \boldsymbol{f}, \quad i = 0, 1, \ldots . \tag{3.1}$$

By virtue of Theorem 2.1, method (3.1) is equivalent to (1.5) with B_1 as a preconditioner, and the equation $A_\nu(\alpha)p = q$ is ‘easily’ solved. In rectangular domains we have $A_\nu(\alpha) = B_2$, so we expect to prove uniform convergence estimates in the general case.

Operators $A_\nu(\alpha)$ and $A_0(\alpha)$ differ only by the implicitly involved boundary conditions for the velocity. Therefore, these operators are expected to be rather close to each other. Indeed, the following theorem is valid.

Theorem 3.1 *There exists a constant* $c(\Omega) > 0$ *independent of* $\alpha \geq 0$ *such that*

$$c(\Omega)A_\nu(\alpha) \leq A_0(\alpha) \leq A_\nu(\alpha).$$

Proof. For convenience, introduce the following scalar product in $\boldsymbol{H}_0^1$ depending on α :

$$(\boldsymbol{u}, \boldsymbol{v})_\alpha = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \alpha(\boldsymbol{u}, \boldsymbol{v}).$$

For any $p \in L_2/\mathbb{R}$, the following equalities are valid

$$\begin{aligned}
(A_0(\alpha)\, p, p) &= (\mathrm{div}\,(\Delta - \alpha I)_0^{-1}\nabla p, p) = - < (\Delta - \alpha I)_0^{-1}\nabla p, \nabla p > \\
&= - < (\Delta - \alpha I)_0^{-1}\nabla p, (\Delta - \alpha I)(\Delta - \alpha I)_0^{-1}\nabla p > \\
&= \|(\Delta - \alpha I)_0^{-1}\nabla p\|_\alpha^2 \\
&= \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{((\Delta - \alpha I)_0^{-1}\nabla p, \boldsymbol{u})_\alpha^2}{\|\boldsymbol{u}\|_\alpha^2}
\end{aligned}$$

$$= \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{< \nabla p, \boldsymbol{u} >^2}{\|\boldsymbol{u}\|_1^2 + \alpha \|\boldsymbol{u}\|^2}$$

$$= \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\boldsymbol{u}\|_1^2 + \alpha \|\boldsymbol{u}\|^2}$$

$$= \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2}. \tag{3.2}$$

In a similar way we get

$$(A_\nu(\alpha)\, p, p) = \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2}.$$

Thus, to prove the theorem it is necessary and sufficient to check the validity of the inequalities:

$$\begin{aligned} &\sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2} \\ &\qquad \leq \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &c(\Omega) \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2} \\ &\qquad \leq \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2} \end{aligned} \tag{3.4}$$

with some $c(\Omega) > 0$ depending only on Ω.

Equality (3.3) is trivial since $\boldsymbol{H}_0^1 \subset \boldsymbol{U}$. The proof of (3.4) is a subject of Sect. 4 (see Lemma 4.3). Checking (3.3) and (3.4), we prove the theorem. □

From Theorem 3.1 and the general theory of iterative methods it immediately follows

Corollary 3.1 *For an appropriate set of* τ_i *(e.g.* $\tau_i = 1, i = 0, 1, \ldots$*), the method (3.1) converges like a geometric progression with a factor* q *such that* $0 < q < c < 1$, *where* c *is independent of* α.

Remark 3.1 Let us consider $\hat{\boldsymbol{u}} = (\Delta - \alpha I)_0^{-1} \nabla p$ for some $p \in L_2/\mathbb{R}$. Then from (3.2) it follows that

$$(A_0(\alpha)\, p, p) = \|\hat{\boldsymbol{u}}\|_\alpha^2 = \|\operatorname{div} \hat{\boldsymbol{u}}\|^2 + \|\operatorname{curl} \hat{\boldsymbol{u}}\|^2 + \alpha \|\hat{\boldsymbol{u}}\|^2.$$

At the same time we have $(A_0(\alpha)\,p,p) = (\operatorname{div}\hat{\boldsymbol{u}},p)$. Thus, the supremum in the expression

$$\sup_{0\neq\boldsymbol{u}\in\boldsymbol{H}_0^1}\frac{(p,\operatorname{div}\boldsymbol{u})^2}{\|\operatorname{div}\boldsymbol{u}\|^2+\|\operatorname{curl}\boldsymbol{u}\|^2+\alpha\|\boldsymbol{u}\|^2}$$

is attained for the function $\hat{\boldsymbol{u}} \in \boldsymbol{H}_0^1$. The same arguments are true for $\tilde{\boldsymbol{u}} = (\Delta-\alpha I)_\nu^{-1}\nabla p$ with respect to the supremum over $\boldsymbol{U}$.

Remark 3.2 Since $A_\nu(0) = I$ and $\|\boldsymbol{u}\|_1^2 = \|\operatorname{div}\boldsymbol{u}\|^2+\|\operatorname{curl}\boldsymbol{u}\|^2$ for $\boldsymbol{u} \in \boldsymbol{H}_0^1$, inequality (3.4) with $\alpha = 0$ becomes the well known Ladyzhenskaya-Babuška-Brezzi (LBB) inequality.

Remark 3.3 Let c_0 be a constant from the LBB inequality ($c_0\|p\| \le \|\nabla p\|_{-1}$) and c_1 be a constant from the Fridrichs inequality ($\|\boldsymbol{u}\| \le c_1\|\boldsymbol{u}\|_1,\ \boldsymbol{u} \in \boldsymbol{H}_0^1$), then from the above arguments it follows that the inequality

$$c_2 I \le A_0(\alpha) \le c_3 I$$

holds with $c_2 = c_0^2(1+c_1^2\alpha)^{-1},\ c_3 = 1.$

A simple observation shows that the asymptotics $c_2 = O(\alpha^{-1})$, $\alpha \to \infty$, and $c_3 = O(1)$ cannot be improved. Indeed, consider in a unit square the function $\bar{p} = \cos\pi x$, $\bar{p} \in L_2((0,1)\times(0,1))/\mathbb{R}$. Then $A_\nu(\alpha)\bar{p} = (1+\pi^{-2}\alpha)^{-1}\bar{p}$, but $A_0(\alpha) \le A_\nu(\alpha)$. Therefore, $c(1+\pi^{-2}\alpha)^{-1}\|\bar{p}\|^2 = (A_0(\alpha)\bar{p},\bar{p})$ with some $c, 0 < c \le 1$. Similar arguments with $\bar{p} = \cos m\pi x$ and $m \to \infty$ show the optimality of the asymptotics $c_3 = O(1)$.

These asymptotics for c_2 and c_3 explain the deterioration of the classical Uzawa algorithm for the generalized Stokes problem with $\alpha \gg 1$.

Remark 3.4 Sometimes the necessity of the computation of the $(\Delta-\alpha I)_0^{-1}$ at each step of (3.1) is considered as a drawback of the Uzawa type method. Therefore often an *inexact* version of the Uzawa algorithm is considered in the literature (cf. Elman, Golub (1994), Elman (1996), Bramble, etc. (1997)). The convergence estimates for preconditioned inexact algorithms also heavily depend on the condition $\operatorname{cond}(A_\nu^{-1}(\alpha)A_0(\alpha))$.

Remark 3.5 There is a variety of other preconditioned iterative methods for solving saddle point problems of type (1.1) (see, e.g., Bramble, Pasciak (1988), Rusten, Winther (1992), Silvester, Wathen (1994), Elman(1999)). Their application to (1.1) requires an appropriate preconditioner for $A_0(\alpha)$ to insure good convergence.

4. Proof of the main inequality

In this section inequality (3.4) is proved for domains with rather regular boundary (detailed below), a more technical proof of (3.4) for Lipschitz domains is presented in the Appendix.

Lemma 4.1 *Fix an arbitrary* $\alpha \geq 0$. *Let* $\boldsymbol{u}$ *be any function from* $\boldsymbol{U}$. *Then* $\boldsymbol{u} = \nabla\psi$ *for some* $\psi \in W_2^1(\Omega)/\mathbb{R}$ *if and only if*

$$\boldsymbol{u} = (\Delta - \alpha I)_\nu^{-1}\nabla p \tag{4.1}$$

for some $p \in L_2/\mathbb{R}$.

Proof.

1. Assume $\boldsymbol{u} \in \boldsymbol{U}$ and $\boldsymbol{u} = \nabla\psi$ for some $\psi \in W_2^1(\Omega)/\mathbb{R}$. It is easy to see that for such $\boldsymbol{u}$ the relation

$$\Delta\boldsymbol{u} - \alpha\boldsymbol{u} = \nabla p$$

holds (in a weak sense) with $p = \operatorname{div}\boldsymbol{u} - \alpha\psi,\ p \in L_2/\mathbb{R}$; since $\boldsymbol{u}\cdot\boldsymbol{\nu} = \mathcal{R}\boldsymbol{u} = 0$, equality (4.1) is valid.

2. Consider an arbitrary $p \in L_2/\mathbb{R}$ and $\boldsymbol{u} \in \boldsymbol{U}$ such that relation (4.1) holds. Then, according to Theorem 2.1, for $\boldsymbol{u}$ and p one has the following equality:

$$p = \operatorname{div}\boldsymbol{u} - \alpha\Delta_N^{-1}\operatorname{div}\boldsymbol{u}.$$

Recall that Δ_N^{-1} was defined in Sect. 1 as a solution operator for the scalar Poisson problem with Neumann boundary conditions.

Set now $\psi = \Delta_N^{-1}\operatorname{div}\boldsymbol{u},\ \psi \in W_2^1(\Omega)/\mathbb{R}$, and $\boldsymbol{v} = \nabla\psi$. We readily get $\boldsymbol{v}\cdot\boldsymbol{\nu} = \frac{\partial\psi}{\partial\boldsymbol{\nu}} = 0$, $\mathcal{R}\boldsymbol{v} = \mathcal{R}\nabla\psi = 0$ on $\partial\Omega$ and $\operatorname{div}\boldsymbol{v} = \Delta\psi = \operatorname{div}\boldsymbol{u}$. Moreover,

$$\Delta\boldsymbol{v} - \alpha\boldsymbol{v} = \nabla\operatorname{div}\boldsymbol{v} - \alpha\nabla\Delta_N^{-1}\operatorname{div}\boldsymbol{u} = \nabla(\operatorname{div}\boldsymbol{u} - \alpha\Delta_N^{-1}\operatorname{div}\boldsymbol{u}) = \nabla p.$$

Hence, the function $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{U}$ satisfies the equations

$$\begin{aligned} &\Delta\boldsymbol{w} - \alpha\boldsymbol{w} = 0, \\ &\boldsymbol{w}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \mathcal{R}\boldsymbol{w}|_{\partial\Omega} = 0 \end{aligned}$$

which imply $\boldsymbol{w} = \boldsymbol{0}$. Thus, we have $\boldsymbol{u} = \boldsymbol{v} = \nabla\psi$. The Lemma is proved. $\square$

In the following two lemmas we suppose that the domain Ω is such that the solution of the classical Stokes problem is regular. In particular, we suppose that for $\boldsymbol{f} \in L_2(\Omega)^n$ and $g \in W_2^1(\Omega)/R$, the solution $\{\boldsymbol{u}, p\}$ of the problem

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f},$$

$$\operatorname{div}\boldsymbol{u} = g,$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}$$

belongs to $W_2^2(\Omega)^n \times W_2^1(\Omega)/\mathbb{R}$. For example, domains with $\partial\Omega \in C^2$ or convex ones satisfy this requirement (cf. Temam (1977), Dauge (1989)).

Lemma 4.2 *Let $\boldsymbol{v}$ be any function from $\boldsymbol{U}$ and $\boldsymbol{u}, p \in \boldsymbol{U} \times L_2/R$ be a solution to the problem*

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= \mathbf{0}, \\
\operatorname{div} \boldsymbol{u} &= 0, \qquad (4.2) \\
\boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{v}|_{\partial\Omega}\,.
\end{aligned}
$$

Then the following estimates hold:

$$
\|\operatorname{div} \boldsymbol{u}\| + \|\operatorname{curl} \boldsymbol{u}\| \le c(\Omega)(\|\operatorname{div} \boldsymbol{v}\| + \|\operatorname{curl} \boldsymbol{v}\|), \qquad (4.3a)
$$

$$
\|\boldsymbol{u}\|_{0,\Omega} \le c(\Omega)\|\gamma_\tau \boldsymbol{u}\|_{-\frac{1}{2},\partial\Omega}. \qquad (4.3b)
$$

Proof. The solution $\boldsymbol{u}$ of (4.2) can be defined as $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$, where $\boldsymbol{w} \in \boldsymbol{H}_0^1$ is a solution (together with p) of the problem

$$
\begin{aligned}
&(\nabla \boldsymbol{w}, \nabla \boldsymbol{\xi}) - (p, \operatorname{div} \boldsymbol{\xi}) = (\operatorname{div} \boldsymbol{v}, \operatorname{div} \boldsymbol{\xi}) + (\operatorname{curl} \boldsymbol{v}, \operatorname{curl} \boldsymbol{\xi}), \\
&(\operatorname{div} \boldsymbol{w}, \eta) = (\operatorname{div} \boldsymbol{v}, \eta) \qquad \forall \boldsymbol{\xi} \in \boldsymbol{H}_0^1,\ \eta \in L_2/\mathbb{R}.
\end{aligned} \qquad (4.4)
$$

From (4.4) and with standard arguments (see, e.g., Girault, Raviart (1986)) we obtain for $\boldsymbol{w}$ the estimate

$$
\|\boldsymbol{w}\|_1 \le c(\|\operatorname{div} \boldsymbol{v}\| + \|\operatorname{curl} \boldsymbol{v}\|).
$$

Now from the last inequality and the relations $\|\operatorname{div} \boldsymbol{w}\|^2 + \|\operatorname{curl} \boldsymbol{w}\|^2 = \|\boldsymbol{w}\|_1^2$, $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w}$, we get (4.3a).

To prove (4.3b), assume that $\boldsymbol{v}$ is an arbitrary function from $W_2^2(\Omega)^n$. The above regularity assumptions ensure that the smoothness of $\boldsymbol{v}$ implies that $\boldsymbol{w}$ from (4.4) belongs to $W_2^2(\Omega)^n$. Thus, $\boldsymbol{u} \in W_2^2(\Omega)^n$ and $p \in W_2^1(\Omega)/\mathbb{R}$. Therefore, the relations (4.2) hold in $L_2(\Omega)^n$. Let us multiply both parts of the first equality (4.2) by an arbitrary function $\boldsymbol{\psi}$ from $W_2^2(\Omega)^n \cap \boldsymbol{H}_0^1$ satisfying $\operatorname{div} \boldsymbol{\psi} = 0$, and integrate over Ω. After integration by parts we get

$$
-(\boldsymbol{u}, \Delta \boldsymbol{\psi}) = < \gamma_\tau \boldsymbol{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega}\,.
$$

Assuming $\boldsymbol{\psi} \ne \mathbf{0}$, we get from the last equality

$$
\frac{-(\boldsymbol{u}, \Delta \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_2} = \frac{< \gamma_\tau \boldsymbol{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega}}{\|\boldsymbol{\psi}\|_2}. \qquad (4.5)
$$

To obtain the estimate for the right-hand side of (4.5), note that

$$
\|\operatorname{curl} \boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega} \le \|\operatorname{curl} \boldsymbol{\psi}\|_{W_2^1(\Omega)} \le c_1 \|\boldsymbol{\psi}\|_2
$$

and, thus,

$$\frac{|<\gamma_\tau \boldsymbol{v}, \operatorname{curl} \boldsymbol{\psi}>_{\partial\Omega}|}{\|\boldsymbol{\psi}\|_2} \le c \frac{|<\gamma_\tau \boldsymbol{v}, \operatorname{curl} \boldsymbol{\psi}>_{\partial\Omega}|}{\|\operatorname{curl} \boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}}$$
$$\le c \sup_{\boldsymbol{\xi} \in H^{\frac{1}{2}}(\partial\Omega)^{2n-3}} \frac{<\gamma_\tau \boldsymbol{v}, \boldsymbol{\xi}>_{\partial\Omega}}{\|\boldsymbol{\xi}\|_{\frac{1}{2},\partial\Omega}} \equiv c\|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2}}.$$

Now from (4.5) we get

$$\frac{|(\boldsymbol{u}, \Delta\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_2} \le c\|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2}}. \tag{4.6}$$

As far as (4.6) holds for any $\mathbf{0} \neq \boldsymbol{\psi} \in W_2^2(\Omega)^n \cap \boldsymbol{H}_0^1$ with $\operatorname{div} \boldsymbol{\psi} = 0$, choose $\boldsymbol{\psi}$ as a solution (together with q) to the problem

$$\begin{aligned} -\Delta\boldsymbol{\psi} + \nabla q &= \boldsymbol{u}, \\ \operatorname{div} \boldsymbol{\psi} &= 0, \\ \boldsymbol{\psi}|_{\partial\Omega} &= \mathbf{0} \end{aligned} \tag{4.7}$$

Since $\boldsymbol{u} \in L_2(\Omega)^n$ and Ω is assumed to be rather regular, it follows from standard regularity results for the Stokes problem that $\{\boldsymbol{\psi}, q\} \in W_2^2(\Omega)^n \times W_2^1(\Omega)/\mathbb{R}$ and

$$\|q\|_{W_2^1} + \|\boldsymbol{\psi}\|_2 \le c\|\boldsymbol{u}\|. \tag{4.8}$$

To obtain an estimate on $\|\boldsymbol{u}\|$, multiply the first equality (4.7) by$\boldsymbol{u}$ and integrate over Ω. We get

$$\|\boldsymbol{u}\|^2 = (\nabla q, \boldsymbol{u}) - (\Delta\boldsymbol{\psi}, \boldsymbol{u}).$$

From this relation, using estimates (4.6), (4.8), and equality

$$(\nabla q, \boldsymbol{u}) =< q, \boldsymbol{u}\cdot\boldsymbol{\nu} >_{\partial\Omega},$$

we deduce

$$\begin{aligned} \|\boldsymbol{u}\|^2 &\le |<q, \boldsymbol{u}\cdot\boldsymbol{\nu}>_{\partial\Omega}| + c\|\boldsymbol{\psi}\|_2 \, \|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2},\partial\Omega} \\ &\le \|q\|_{\frac{1}{2},\partial\Omega} \|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + c\|\boldsymbol{u}\| \, \|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2},\partial\Omega} \\ &\le \|q\|_{W_2^1} \|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + c\|\boldsymbol{u}\| \, \|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2},\partial\Omega} \\ &\le c_1\|\boldsymbol{u}\| \, (\|\boldsymbol{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + \|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2},\partial\Omega}). \end{aligned}$$

From the last estimate it follows that

$$\|\boldsymbol{u}\|_{0,\Omega} \le c(\|\boldsymbol{v}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + \|\gamma_\tau \boldsymbol{v}\|_{-\frac{1}{2},\partial\Omega}). \tag{4.9}$$

Assume now that $\boldsymbol{v}$ is an arbitrary function from $\boldsymbol{U}$. Then $\boldsymbol{v}$ can be approximated in the $\boldsymbol{U}$-norm by functions from $W_2^2(\Omega)^n$. Hence, using inequalities (2.1), (4.3a), and passing to the limit in $\boldsymbol{U}$, we deduce from (4.9) the estimate (4.3b). The Lemma is proved. □

Lemma 4.3 *For arbitrary $\alpha \in [0, \infty)$ and $p \in L_2/\mathbb{R}$, the inequality*

$$
\begin{aligned}
&c(\Omega) \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2} \\
&\qquad \leq \sup_{0 \neq \boldsymbol{u} \in \boldsymbol{H}_0^1} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2}
\end{aligned}
\tag{4.10}
$$

holds with $c(\Omega) > 0$ independent of α and p.
Proof. Let p be an arbitrary function from $L_2/\mathbb{R}$. Then from Remark 3.1 it follows that the supremum on the left-hand side of (4.10) is attained on the function

$$
\boldsymbol{u} = (\Delta - \alpha I)_\nu^{-1} \nabla p, \quad \boldsymbol{u} \in \boldsymbol{U}. \tag{4.11}
$$

From Lemma 4.1 and relation (4.11) it follows that $\operatorname{curl} \boldsymbol{u} = 0$. So from (2.1) we get

$$
\|\gamma_\tau \boldsymbol{u}\|_{-\frac{1}{2}, \partial\Omega} \leq \|\boldsymbol{u}\|. \tag{4.12}
$$

To prove the Lemma it is sufficient to find for given $\boldsymbol{u}$ from (4.11) such a function $\tilde{\boldsymbol{u}} \in \boldsymbol{H}_0^1$ that
(4.13)

$$
\begin{aligned}
&(p, \operatorname{div} \tilde{\boldsymbol{u}}) = (p, \operatorname{div} \boldsymbol{u}), \\
&\|\operatorname{div} \tilde{\boldsymbol{u}}\|^2 + \|\operatorname{curl} \tilde{\boldsymbol{u}}\|^2 + \alpha \|\tilde{\boldsymbol{u}}\|^2 \leq c \left(\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2 + \alpha \|\boldsymbol{u}\|^2 \right)
\end{aligned}
$$

with c independent of $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$, and α.

Let $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{w}$, where $\boldsymbol{w} \in \boldsymbol{U}$ is a solution to the problem

$$
\begin{aligned}
-\Delta \boldsymbol{w} + \nabla q &= \mathbf{0}, \\
\operatorname{div} \boldsymbol{w} &= 0, \\
\boldsymbol{w}|_{\partial\Omega} &= \boldsymbol{u}|_{\partial\Omega}.
\end{aligned}
\tag{4.14}
$$

It is easy to see that $(p, \operatorname{div} \tilde{\boldsymbol{u}}) = (p, \operatorname{div} \boldsymbol{u})$ and, in virtue of Remark 2.1, $\tilde{\boldsymbol{u}} \in \boldsymbol{H}_0^1$.

Now we check the second condition in (4.13). Using a priori estimates from Lemma 4.2 and relation (4.12), we obtain for the solution of (4.14)

$$
\begin{aligned}
&\|\operatorname{div} \boldsymbol{w}\|^2 + \|\operatorname{curl} \boldsymbol{w}\|^2 \leq c(\|\operatorname{div} \boldsymbol{u}\|^2 + \|\operatorname{curl} \boldsymbol{u}\|^2), \\
&\|\boldsymbol{w}\|^2 \leq c \|\gamma_\tau \boldsymbol{u}\|_{-\frac{1}{2}, \partial\Omega} \leq c \|\boldsymbol{u}\|^2.
\end{aligned}
\tag{4.15}
$$

Inequalities (4.15) immediately give

$$\begin{aligned}&\|\operatorname{div}\boldsymbol{w}\|^2+\|\operatorname{curl}\boldsymbol{w}\|^2+\alpha\|\boldsymbol{w}\|^2\\ &\qquad\le c\left(\|\operatorname{div}\boldsymbol{u}\|^2+\|\operatorname{curl}\boldsymbol{u}\|^2+\alpha\|\boldsymbol{u}\|^2\right)\end{aligned}\tag{4.16}$$

with c independent of α and $\boldsymbol{u}$.

In virtue of $\tilde{\boldsymbol{u}}=\boldsymbol{u}-\boldsymbol{w}$ and (4.16), the second condition in (4.13) is also valid. So the Lemma is proved. □

5. Some remarks on finite differences and finite elements

The above considerations for differential problems encourage us to expect a success of the method for discrete problems as well. Indeed, extension of the 'main' inequality (4.10) to a discrete case is similar to checking the LBB (infsup) condition that is satisfied, as a rule, for pressure-velocity finite difference (FD) and finite element (FE) approximations. Moreover, most of the approximations admit well-posed pressure Poisson problem with Neumann boundary conditions. However, while FD or FE formulations of the model problem are rather obvious, validity of Theorem 2.1 in a discrete case is vague in general. To clarify the situation we prove below the FD analogue of Theorem 2.1.

Let us consider MAC scheme with a staggered grids and central FD approximation of Δ, div, and grad operators (see, e.g., Kobelkov (1994)). Further we shall use the following notations: $\boldsymbol{U}^h$ and L^h are FD velocity and pressure spaces of functions defined in interior nodes of a grid domain. Let $\boldsymbol{U}_0^h$, $\boldsymbol{U}_\nu^h$, and L_0^h be their extensions such that discrete analogues of boundary conditions $\boldsymbol{u}=\boldsymbol{0}$, $\boldsymbol{u}\cdot\boldsymbol{\nu}=\mathcal{R}\boldsymbol{u}=0$, and $\frac{\partial p}{\partial\boldsymbol{\nu}}=0$ are satisfied for functions from $\boldsymbol{U}_0^h$, $\boldsymbol{U}_\nu^h$, and L_0^h, respectively. By Δ_0^h, Δ_ν^h, ∇^h, div^h, and Δ_N^h we denote grid approximations of the corresponding differential operators.

Assume that the following compatibility conditions are satisfied:

a) $\operatorname{div}^h\nabla^h=\Delta_N^h$ on L_0^h,
b) $\Delta_\nu^h\nabla^h=(\nabla^h\operatorname{div}^h)\nabla^h$ on L_0^h.

(This assumption is valid, for example, for a domain, which is a union of rectangulars.)

Then for this approximation the following theorem holds.

Theorem 5.1 *For any $\alpha\in[0,\infty)$ and $p\in L^h$, the equality $p=q-\alpha\,(\Delta_N^h)^{-1}q$ holds with $q\in L^h$ such that*

$$q=\operatorname{div}^h(\Delta_\nu^h-\alpha I)^{-1}\nabla^h p.\tag{5.1}$$

Proof. Fix some $p \in L^h$ and $\alpha \in [0, \infty)$. We can rewrite relation (5.1) as follows

$$\begin{aligned} -\Delta_\nu^h \boldsymbol{u} + \alpha \boldsymbol{u} + \nabla^h p &= \mathbf{0} \text{ in } \boldsymbol{U}^h, \\ \operatorname{div}^h \boldsymbol{u} &= q \text{ in } L^h \end{aligned} \tag{5.2}$$

with $\boldsymbol{u} \in \boldsymbol{U}_\nu^h$.

Assume $\alpha > 0$ and consider the functions $p_1 = -\alpha(\Delta_N^h)^{-1} q$ and $\tilde{\boldsymbol{u}} = \alpha^{-1} \nabla^h p_1$, $\tilde{\boldsymbol{u}} \in \boldsymbol{U}_\nu^h$. From our definitions and compatibility conditions we have $\operatorname{div}^h \tilde{\boldsymbol{u}} = q$ and $-\Delta_\nu^h \tilde{\boldsymbol{u}} = -\nabla^h \operatorname{div}^h \tilde{\boldsymbol{u}} = \nabla^h q$.

Set $\tilde{p} = q + p_1$. Then for $\tilde{\boldsymbol{u}} \in \boldsymbol{U}_\nu^h$ and $\tilde{p} \in L^h$ the following equalities hold,

$$\begin{aligned} -\Delta_\nu^h \tilde{\boldsymbol{u}} + \alpha \tilde{\boldsymbol{u}} + \nabla \tilde{p} &= \mathbf{0}, \\ \operatorname{div}^h \tilde{\boldsymbol{u}} &= q. \end{aligned} \tag{5.3}$$

From the well-posedness of the Stokes problem and relations (5.2), (5.3) we get $p = \tilde{p} = q - \alpha(\Delta_N^h)^{-1} q$. The case $\alpha = 0$ is treated in a similar manner. The Theorem is proved. $\square$

There is one more reason for considering the model Schur complement as a preconditioner in (1.5). The theorem below shows a specific discrete reflection of the fact that $A_0(\alpha)$ and $A_\nu(\alpha)$ in (4.1) differ only up to the boundary conditions involved implicitly.

Theorem 5.2 *Under the assumptions of Theorem 5.1, the eigenvalue problem*

$$A_0^h(\alpha) p = \lambda A_\nu^h(\alpha) p$$

with $A_0^h(\alpha) = \operatorname{div}^h (\Delta_0^h - \alpha I)^{-1} \nabla^h$, $A_\nu^h(\alpha) = \operatorname{div}^h (\Delta_\nu^h - \alpha I)^{-1} \nabla^h$ *has* $\lambda = 1$ *as an eigenvalue of* $O(h^{-n})$ *multiplicity, and the number of all the other eigenvalues is* $O(h^{-(n-1)})$, *where* h *is the mesh size-parameter.*

Proof. The proof of this theorem is similar to the proof of Theorem 1.4 from Kobelkov (1994). $\square$

From Theorem 5.2 it follows that in the first step of the method (3.1) we can take $\tau_0 = 1$. Then in the following steps with arbitrary τ_i the error function $r^i = p^i - p$ belongs to a subspace of dimension $O(h^{-(n-1)})$. This property ensures an extra convergence of method (1.5) for the finite difference approximations considered above. In this case the discrete preconditioner is equal to the Schur complement of the discrete model problem.

The lack of appropriate compatibility conditions for FE approximations, as well as troubles with smoothness requirements on the trial functions in the proof of Theorem 5.1 make the above analysis more complicated for finite elements. However, let us make the following remarks.

Consider the matrix form of the FE discretization of the model problem (2.2):

$$\begin{bmatrix} -\mathrm{L}_\nu(\alpha) & \mathrm{D}^* \\ \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix}.$$

Let I_ν^h and I_p^h be the mass matrices of the FE velocity and pressure spaces. Then the above analysis suggests the choice of $B^{-1} = (I_p^h)^{-1} - \alpha(\mathrm{D}(I_\nu^h)^{-1}\mathrm{D}^*)^{-1}$ in (1.4), which, we expect, will be better than $B^{-1} = I - \alpha(\Delta_N^h)^{-1}$ (numerical results from Cahouet, Chabard (1988) and Turek (1999) confirm this hypothesis). Also in Bramble, Pasciak (1997) the convergence theorem was proved for the FE case with certain coarse-mesh approximation of the pressure Poisson problem.

6. Numerical experiments

In this section we present results of numerical experiments for the equation

$$A_0^h(\alpha)p = F,$$

where $A_0^h(\alpha)$ is the Schur complement for a FD approximation of the generalized Stokes problem (1.1). We use the MAC scheme defined in Sect. 5 and take $\Omega = (0,1) \times (0,1)$.

Preconditioned and non preconditioned versions of conjugate gradient (CG) and minimal residual (MINRES) methods were tested.

Let B denote a preconditioner, p_0 an initial guess ($p_0 \equiv 0$ in all experiments), let $s_i = A_0^h(\alpha)p_i - F$ denote the residual for p_i defined via iterations for $i = 1, 2, \dots$, and $r_i = B^{-1}s_i$. The preconditioned version of CG was then standard.

The preconditioned MINRES method, which we used, coincides with (1.4) for $\tau_i = (B^{-1}A_0^h(\alpha)r_i, r_i)/(B^{-1}A_0^h(\alpha)r_i, B^{-1}A_0^h(\alpha)r_i)$, $i = 0, 1, \dots$.

We refer to these algorithms as preconditioned in the case $B = A_\nu^h(\alpha)$ and as nonpreconditioned in the case $B = I^h$. The stopping criterion was

$$\|r_i\|_2/\|r_0\|_2 < 10^{-9}.$$

We refer to the value of $(\|r_i\|_2/\|r_0\|_2)^{1/i}$ as to *the average convergence factor.*

Remark 6.1 If $\boldsymbol{u}_i$ is a velocity vector field corresponding to the pressure p_i via the Helmholtz equation $-\Delta^h \boldsymbol{u}_i + \alpha \boldsymbol{u}_i = \boldsymbol{f} - \nabla p_i$, then $s_i = \operatorname{div} \boldsymbol{u}_i$.

Remark 6.2 Along with the CG and MINRES methods, we will consider these methods on a subspace (see Theorem 5.2), i.e., we make one step of (3.1) with $\tau_0 = 1$ and then continue calculations according to the above algorithms.

Table Ia. Smooth test. Average convergence factor for conjugate gradients

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.099	0.139	0.169	0.189	0.204	0.223
16	0.134	0.175	0.207	0.225	0.245	0.244
32	0.098	0.151	0.190	0.208	0.224	0.238
64	0.083	0.121	0.166	0.188	0.200	0.217
128	0.066	0.109	0.143	0.185	0.212	0.234
256	0.049	0.096	0.131	0.171	0.215	0.244
512	0.033	0.072	0.111	0.146	0.190	0.222
1024	0.023	0.048	0.083	0.119	0.148	0.176
2048	0.012	0.035	0.063	0.095	0.133	0.163

Table Ib. Smooth test. Average convergence factor for conjugate gradients on subspace

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.106	0.146	0.176	0.196	0.217	0.229
16	0.114	0.166	0.197	0.225	0.244	0.257
32	0.107	0.149	0.176	0.204	0.226	0.245
64	0.097	0.125	0.154	0.191	0.222	0.239
128	0.084	0.112	0.159	0.198	0.235	0.262
256	0.050	0.103	0.143	0.189	0.222	0.260
512	0.039	0.080	0.121	0.160	0.194	0.234
1024	0.026	0.058	0.092	0.130	0.162	0.192
2048	0.015	0.041	0.073	0.104	0.134	0.161

1. *Smooth test*

For the first test we choose the smooth pressure function $p^s = x - y$ as an 'exact' solution of (6.1). The function $F = A_0^h(\alpha)p^s$ was computed and considered as the right-hand side of (6.1). Setting $p_0 = 0$, we examine the convergence of the conjugate gradient method to this smooth solution. The results are presented in Table Ia–c. For example, for $h = (512)^{-1}$ and $\alpha = 512$, the convergence factor is equal to 0.222; i.e. the residual becomes approximately 100 times less during every 3 steps.

2. *Random test*

The exact solution p^r was chosen as follows. In every grid point a random number generated with the uniform distribution over $[-1, 1]$ was taken as a value of p^r. Further p^r was normalized to ensure $\int_\Omega p^r\,dx = 0$. The values of convergence factors in Tables II, III were averaged over three random runs of the program with different initializations of the random generator. There were no pronounced differences in convergence rates observed for these substantially nonsmooth solutions in comparison with the results of the smooth test. While the averaged convergence factors for the CG method on the subspace were very close to those without $\tau_0 = 1$, the MINRES method on the subspace was evidently superior to the usual one. At any time,

Table Ic. Smooth test. Average convergence factor for conjugate gradients without preconditioning

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.099	0.139	0.169	0.189	0.204	0.223
16	0.150	0.201	0.226	0.243	0.263	0.271
32	0.176	0.220	0.244	0.269	0.280	0.289
64	0.199	0.250	0.271	0.290	0.309	0.318
128	0.250	0.290	0.319	0.329	0.347	0.354
256	0.313	0.353	0.380	0.397	0.406	0.417
512	0.389	0.429	0.462	0.475	0.473	0.481
1024	0.457	0.524	0.552	0.551	0.569	0.564
2048	0.512	0.605	0.631	0.649	0.651	0.649
4096	0.545	0.682	0.755	0.776	0.783	0.780

setting $\tau_0 = 1$ saves some computations; this can be especially appreciable in unsteady simulations when only few iterations on each time step are needed to achieve a good approximation.

In all tests, the convergence of the preconditioned methods improved when the parameter α increased and the mesh size h was fixed. The case $\alpha = 0$ corresponds to the Uzawa algorithm for the classical Stokes problem $(A^h_\nu(0) = I^h)$. The preconditioned algorithm for the generalized Stokes problem for any $\alpha \geq 0$ demonstrated convergence at least not worse than the Uzawa algorithm for the classical Stokes problem.

If we consider a typical situation in nonstationary high Reynolds simulations when $(h\,Re)(h/\delta t) \leq c < \infty$ with some absolute constant $c > 0$, then we have the following relation for $\alpha : \alpha = O(h^{-2})$. In this particular case, the preconditioned method demonstrates even an improvement of the convergence rate with $h \to 0$.

Tables Ic and IIc show the growth of convergence factor for the nonpreconditioned CG method for $\alpha \to \infty$, due to the growth of the condition number of $A^h_0(\alpha)$.

7. Appendix

As was demonstrated above, the inequality (3.4) plays an important role in the proof of the convergence of the method. This inequality was proved (Sect. 4 of the paper) for domains with sufficiently smooth boundary ore convex ones. Below we show that this inequality with a constant $c(\Omega) > 0$ independent of parameter α, is valid for the wider class of domains. Namely, we prove its validity for a curvilinear trapezoid with Lipschitz boundary $y = g(x)$, when $|g'|$ is not too large. Unfortunatly, we could not prove it for all Lipschitz domains, but we believe this hypothesis to be true.

Table IIa. Random test. Average convergence factor for conjugate gradients

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.165	0.201	0.225	0.242	0.240	0.242
16	0.150	0.188	0.202	0.234	0.221	0.222
32	0.125	0.177	0.197	0.220	0.204	0.212
64	0.106	0.153	0.176	0.205	0.197	0.204
128	0.086	0.119	0.173	0.183	0.188	0.203
256	0.065	0.100	0.147	0.164	0.186	0.189
512	0.053	0.084	0.122	0.155	0.178	0.180
1024	0.034	0.062	0.096	0.136	0.161	0.187
2048	0.019	0.043	0.083	0.115	0.145	0.178

Table IIb. Random test. Average convergence factor for conjugate gradients on subspace

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.166	0.200	0.225	0.236	0.237	0.242
16	0.140	0.170	0.191	0.223	0.213	0.215
32	0.114	0.155	0.177	0.209	0.203	0.204
64	0.096	0.136	0.160	0.188	0.187	0.196
128	0.079	0.114	0.157	0.170	0.182	0.194
256	0.063	0.095	0.139	0.159	0.180	0.190
512	0.049	0.083	0.119	0.149	0.174	0.184
1024	0.032	0.063	0.099	0.137	0.159	0.189
2048	0.019	0.044	0.084	0.117	0.146	0.179

Table IIc. Random test. Average convergence factor for conjugate gradients without preconditioning

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.168	0.201	0.225	0.241	0.245	0.242
16	0.250	0.274	0.298	0.309	0.310	0.308
32	0.288	0.308	0.320	0.335	0.333	0.334
64	0.330	0.356	0.362	0.364	0.364	0.371
128	0.398	0.415	0.425	0.418	0.418	0.413
256	0.474	0.491	0.500	0.488	0.497	0.479
512	0.551	0.584	0.581	0.570	0.587	0.562
1024	0.613	0.663	0.678	0.649	0.665	0.648
2048	0.667	0.731	0.744	0.733	0.739	0.741
4096	0.709	0.795	0.790	0.763	0.787	0.769

Further we use the notations from Sect. 2. We shall also use the notations $(\boldsymbol{v}, \boldsymbol{w})_\alpha \equiv (\mathrm{div}\, \boldsymbol{v}, \mathrm{div}\, \boldsymbol{w}) + (\mathrm{curl}\, \boldsymbol{v}, \mathrm{curl}\, \boldsymbol{w}) + \alpha(\boldsymbol{v}, \boldsymbol{w})$, $\|\boldsymbol{v}\|_\alpha^2 \equiv (\boldsymbol{v}, \boldsymbol{v})_\alpha$,

$$\|w_i\|_\alpha^2 \equiv \|\nabla w_i\|^2 + \alpha\|w_i\|^2, \quad \|w_i\|_{\alpha,x_j}^2 \equiv \left\|\frac{\partial w_i}{\partial x_j}\right\|^2 + \alpha\|w_i\|^2.$$

Later on, we shall denote independent variables either by (x_1, x_2) or (x, y).

Table IIIa. Random test. Average convergence factor for minimal residuales

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.426	0.455	0.483	0.504	0.521	0.522
16	0.312	0.368	0.404	0.449	0.470	0.487
32	0.271	0.339	0.386	0.431	0.454	0.474
64	0.227	0.301	0.362	0.410	0.433	0.459
128	0.182	0.259	0.329	0.383	0.413	0.442
256	0.136	0.214	0.292	0.351	0.388	0.420
512	0.095	0.168	0.250	0.320	0.365	0.400
1024	0.061	0.122	0.208	0.281	0.334	0.378
2048	0.037	0.085	0.162	0.239	0.303	0.348

Table IIIb. Random test. Average convergence factor for minimal residuales on subspace

$\alpha \backslash h$	$1/16$	$1/32$	$1/64$	$1/128$	$1/256$	$1/512$
0	0.363	0.396	0.428	0.448	0.471	0.474
16	0.245	0.307	0.347	0.397	0.416	0.435
32	0.200	0.273	0.328	0.377	0.399	0.420
64	0.173	0.244	0.306	0.348	0.381	0.399
128	0.143	0.215	0.284	0.324	0.364	0.372
256	0.106	0.183	0.258	0.303	0.348	0.362
512	0.074	0.143	0.222	0.283	0.331	0.358
1024	0.048	0.084	0.184	0.255	0.311	0.347
2048	0.027	0.070	0.140	0.217	0.282	0.326

Let

$$\Phi(p, \boldsymbol{v}) \equiv \frac{(p, \operatorname{div} \boldsymbol{v})^2}{\|\boldsymbol{v}\|_\alpha^2} \tag{1}$$

The aim of this Appendix is to prove for any function $p \in L_2/\mathbb{R}$ the validity of the following inequality:

$$\sup_{\boldsymbol{v} \in \boldsymbol{U}} \Phi(p, \boldsymbol{v}) \leq c_0 \sup_{\boldsymbol{w} \in \boldsymbol{H}_0^1} \Phi(p, \boldsymbol{w}) \tag{2}$$

with some constant c_0 that does not depend on $\alpha \geq 0$.

Further, we assume $\alpha \geq 1$.

At first, let $\bar{\Omega} = [0, \pi] \times [0, \pi]$. We recall that in this case $\boldsymbol{U}$ defined in Sect. 2 can be represented as

$$\boldsymbol{U} = \left\{ \boldsymbol{u} = (u^1, u^2) : \quad u^i \in H^1(\Omega),\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \right\}$$

and

$$(\boldsymbol{v}, \boldsymbol{w})_\alpha = (\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) + \alpha(\boldsymbol{v}, \boldsymbol{w}).$$

Lemma 1. *Let*

$$p(x,y)=\sum_{i,j=0}^{n} p_{ij}\cos ix\cos jy,\quad p_{00}=0. \tag{3}$$

Then

$$\arg\sup_{\boldsymbol{v}\in\boldsymbol{U}} \Phi(p,\boldsymbol{v})=\hat{\boldsymbol{u}}, \tag{4}$$

where $\hat{\boldsymbol{u}}=(\Delta-\alpha I)_\nu^{-1}\nabla p$.

The proof directly follows from Remark 3.1.

Lemma 2. *For every vector function* $\hat{\mathbf{u}}$ *that provides a maximum of* $\Phi(p,\boldsymbol{v})$ *over* $\boldsymbol{U}$, *where* p *is a trigonometric polynomial of the form (3), there exists a vector function* $\boldsymbol{u}\in\boldsymbol{H}_0^1$ *satisfying the following inequality*

$$\Phi(p,\hat{\boldsymbol{u}})\le c_0\Phi(p,\boldsymbol{u}),$$

where c_0 *does not depend on* $n,\ p$, *and* $\alpha\ge 0$.

This result follows from Lemma 4.3 of this paper (Ω satisfies here the requirements from Sect. 4).

Corollary 1. *For any* $p\in L_2/\mathbb{R}$, *estimate* (2) *is valid with the constant* c_0 *that does not depend on* α.

Proof. The set of trigonometric polynomials of the form (3) is dense in L_2/R. Take the sequence of trigonometric polynomials p_n that converges to p. In virtue of Lemma 2, estimate (2) is valid for every p_n with the constant independent of α and n. Passing to the limit with $n\to\infty$, we obtain the statement required. $\square$

Let us now proceed to the case when Ω is a curvilinear trapezoid, i.e.,

$$\bar{\Omega}=\left\{x=(x_1,x_2):\ 0\le x_1\le\pi,\ 0\le x_2\le g(x_1)\right\},$$

where

$$\pi\le g\le M_1,\quad |g'|\le M_2. \tag{5}$$

After changing variables

$$x_1=x,\ x_2=yg(x),$$

our domain Ω is mapped onto the square $D=(0,\pi)\times(0,\pi)$. Since the change of variables does not depend on α, then for any function $\boldsymbol{v}(x_1,x_2)=\tilde{\boldsymbol{v}}(x,y)$ we have

$$\gamma_1\|\tilde{\boldsymbol{v}}\|_{\alpha,D}\le\|\boldsymbol{v}\|_{\alpha,\Omega}\le\gamma_2\|\tilde{\boldsymbol{v}}\|_{\alpha,D}, \tag{6}$$

where γ_i do not depend on α.

The expression $\operatorname{div} \boldsymbol{v}$ after this change is transformed in the following way:

$$\operatorname{div} \boldsymbol{v}(x_1, x_2) = \frac{\partial \tilde{v}_1}{\partial x} + \frac{1}{g} \frac{\partial \tilde{v}_2}{\partial y} - \frac{yg'(x)}{g(x)} \frac{\partial \tilde{v}_1}{\partial y} \equiv \text{DIV}\, \tilde{\boldsymbol{v}}(x, y)$$

which implies

$$(\operatorname{div} \boldsymbol{v}, p)_\Omega = (\text{DIV}\, \tilde{\boldsymbol{v}}, q)_D,$$

where $q = g\tilde{p}$ and $(q, 1)_D = 0$, since $p \in L_2(\Omega)/\mathbb{R}$.

Along with the functional $\Phi(p, \boldsymbol{u})$, introduce the functional $\Psi(p, \boldsymbol{u})$:

$$\Psi(p, \boldsymbol{u}) \equiv \frac{(p, \text{DIV}\, \boldsymbol{u})^2}{\|\boldsymbol{u}\|_\alpha^2}$$

Then the following statement holds.

Lemma 3. *Let*

$$p(x, y) = \sum_{|k|=1}^{n} p_k \cos k_1 x \cos k_2 y$$

and $\hat{\boldsymbol{u}} \in \boldsymbol{U}$ *be the function that gives the supremum of* $\Phi(p, \boldsymbol{u})$ *over* $\boldsymbol{U}$*. Then there exists a constant* c *that does not depend on* α *and* n *such that the following inequality is valid:*

$$\sup_{\boldsymbol{u} \in \boldsymbol{U}} \Psi(p, \boldsymbol{u}) \le c\Phi(p, \hat{\boldsymbol{u}}). \tag{7}$$

Proof. From the definition of DIV $\boldsymbol{u}$ we have the trivial estimate

$$\sup_{\boldsymbol{u} \in \boldsymbol{U}} \Psi(p, \boldsymbol{u}) \le 3 \left(\sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \frac{\partial u^1}{\partial x})^2}{\|\boldsymbol{u}\|_\alpha^2} + \sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{(\frac{p}{g}, \frac{\partial u^2}{\partial y})^2}{\|\boldsymbol{u}\|_\alpha^2} + \sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{(p, h\frac{\partial u^1}{\partial y})^2}{\|\boldsymbol{u}\|_\alpha^2} \right), \tag{8}$$

where $h = \dfrac{yg'(x)}{g(x)} \equiv y\bar{g}(x)$ and $|h| \le M = \pi M_2$. Note that the functions g, $\bar{g}$ are functions of the variable x only. We shall use this property later.

Let us estimate every term in the right-hand side of (8) separately:

$$\sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial u^1}{\partial x}\right)^2}{\|\boldsymbol{u}\|_\alpha^2} \le \sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial u^1}{\partial x}\right)^2}{\|u^1\|_\alpha^2} \le \sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{(p, \operatorname{div} \boldsymbol{u})^2}{\|\boldsymbol{u}\|_\alpha^2} \le \Phi(p, \hat{\boldsymbol{u}}),$$

$$\sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{\left(\dfrac{p}{g}, \dfrac{\partial u^2}{\partial y}\right)^2}{\|\boldsymbol{u}\|_\alpha^2} = \sup_{u^2 : \boldsymbol{u} \in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (u^2/g)}{\partial y}\right)^2}{\|u^2\|_\alpha^2} \le \sup_{w : (0, w) \in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial w}{\partial y}\right)^2}{\|gw\|_{\alpha,y}^2}$$

$$\leq c \sup_{w:(0,w)\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial w}{\partial y}\right)^2}{\|w\|^2_{\alpha,y}} = c\sum_{|k|=1}^{n} \frac{k_2^2}{k_2^2+\alpha} p_k^2 \leq c\sum_{|k|=1}^{n} \gamma_k p_k^2 = c\Phi(p,\hat{\boldsymbol{u}}),$$

(9)

where $\gamma_k = |k|^2(|k|^2+\alpha)^{-1}$.

As for the third term, let us transform it before estimating. Integrating by parts, we obtain

$$\left(p, h\frac{\partial u^1}{\partial y}\right) = \left(p, \frac{\partial (hu^1)}{\partial y}\right) - \left(p, u^1\frac{\partial h}{\partial y}\right) = \left(p, \frac{\partial (hu^1)}{\partial y}\right) - (\bar{g}p, u^1).$$

Then

$$\sup_{\boldsymbol{u}\in \boldsymbol{U}} \frac{(\bar{g}p, u^1)^2}{\|\boldsymbol{u}\|^2_\alpha} \leq \sup_{w:(w,0)\in \boldsymbol{U}} \frac{(p,\bar{g}w)^2}{\|w\|^2_\alpha} \leq \sup_{w\in L_2} \frac{(p,\bar{g}w)^2}{\alpha\|w\|^2} = \sup_{w\in L_2} \frac{(\bar{g}p,w)^2}{\alpha\|w\|^2}$$

$$= \frac{1}{\alpha}\|\bar{g}p\|^2 \leq \frac{M_2^2}{\alpha}\|p\|^2 \leq \frac{M_2^2}{\alpha}\sum_{|k|=1}^{n}\frac{1}{\gamma_k}\gamma_k p_k^2 \leq 2\Phi(p,\hat{\boldsymbol{u}}) \tag{10}$$

Introduce the function space $H = \left\{w:\ w\in L_2,\ \dfrac{\partial w}{\partial y}\in L_2\right\}$. Since

$$\|hw\|^2_{\alpha,y} = \left\|\frac{\partial (hw)}{\partial y}\right\|^2 + \alpha\|hw\|^2$$

$$\leq \left\|h\frac{\partial w}{\partial y}\right\|^2 + \|\bar{g}w\|^2 + \alpha\|hw\|^2$$

$$\leq M_2^2\left\|h\frac{\partial w}{\partial y}\right\|^2 + (\tfrac{M_2^2}{\pi^2} + \alpha M_2^2)\|w\|^2 \leq 2M_2^2\|w\|^2_{\alpha,y},$$

then

$$\sup_{\boldsymbol{u}\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (hu^1)}{\partial y}\right)^2}{\|\boldsymbol{u}\|^2_\alpha} \leq \sup_{u^1:\boldsymbol{u}\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|^2_\alpha} \leq \sup_{u^1:(u^1,0)\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|^2_{\alpha,y}}$$

$$\leq 2M_2^2 \sup_{u^1:(u^1,0)\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (hu^1)}{\partial y}\right)^2}{\|hu^1\|^2_{\alpha,y}} = 2M_2^2 \sup_{w^1:(w^1,0)\in \boldsymbol{U}} \frac{\left(p, \dfrac{\partial (w^1)}{\partial y}\right)^2}{\|w^1\|^2_{\alpha,y}}$$

$$\leq c_2 M_2^2 \sup_{w\in H} \frac{\left(p, \dfrac{\partial w}{\partial y}\right)^2}{\|w\|^2_{\alpha,y}} \leq c_2 M_2^2 \sum_{|k|=1}^{n} \frac{k_2^2}{k_2^2+\alpha}p_k^2$$

$$\text{(11)} \qquad \leq c_2 M_2^2 \sum_{|k|=1}^{n} \frac{|k|^2}{|k|^2+\alpha} p_k^2 = c_2 M_2^2 \Phi(p, \hat{\boldsymbol{u}}).$$

From (8)–(11) we get (7). Thus, the Lemma is proved. □

Since the set of trigonometric polynomials is dense in $L_2/\mathbb{R}$, the estimate (7) holds for every function from $L_2/\mathbb{R}$.

Lemma 4. *For any trigonometric polynomial*

$$p(x, y) = \sum_{|k|=1}^{n} p_k \cos k_1 x \cos k_2 y$$

and sufficiently small M_2 *from* (5)*, there exists a constant* c *that does not depend on* α *and* n *such that the following inequality is valid:*

$$\Phi(p, \hat{\boldsymbol{u}}) \leq c \sup_{\boldsymbol{v} \in \boldsymbol{H}_0^1} \Psi(p, \boldsymbol{v}). \tag{12}$$

Proof. In virtue of Lemma 2, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ such that

$$\Phi(p, \hat{\boldsymbol{u}}) \leq c_0 \Phi(p, \boldsymbol{u}).$$

Consider two cases: $(p^2, q^2) \geq 0.5\Phi(p, \hat{\boldsymbol{u}})$ and $(p^1, q^1) \geq 0.5\Phi(p, \hat{\boldsymbol{u}})$, where $q = \operatorname{div} \hat{\boldsymbol{u}}$. We use the following technical result (see Olshanskii (1995), p.85): In the first case there exists a function $\boldsymbol{u} \in \boldsymbol{H}_0^1 : \ u^1 = 0,\ u^2 = v - r$ such that

$$\Phi(p, \hat{\boldsymbol{u}}) \leq c\Phi(p, \boldsymbol{u}) = c \frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|u^2\|_\alpha^2}.$$

Now we set $\boldsymbol{v} = (v^1, v^2)$, where $v^1 = 0$ and $v^2 = gu^2$. Then due to the construction $\boldsymbol{v} \in \boldsymbol{H}_0^1$. Since

$$\|gu^2\|_\alpha^2 = \|\nabla(gu^2)\|^2 + \alpha\|gu^2\|^2 \leq \left\|\frac{\partial(gu^2)}{\partial x}\right\|^2 + M_1^2 \left\|\frac{\partial u^2}{\partial y}\right\|^2$$

$$+\alpha M_1^2 \|u^2\|^2 \leq c\|u^2\|_\alpha^2, \quad \text{where } c = 2(M_1^2 + M_2^2),$$

we have

$$\Psi(p, \boldsymbol{v}) = \frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|gu^2\|_\alpha^2} \geq c\frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|u^2\|_\alpha^2} = c\,\Phi(p, \boldsymbol{u}) \geq c\Phi(p, \hat{\boldsymbol{u}})$$

So for the first case Lemma 4 is proved.

Consider the second case, i.e. $(p^1, q^1) \geq 0.5\Phi(p, \hat{\boldsymbol{u}})$. Similarly there exists the function $\boldsymbol{u} \in \boldsymbol{H}_0^1 :\ u^1 = v - r,\ u^2 = 0$ such that

$$\Phi(p, \hat{\boldsymbol{u}}) \leq c_3 \Phi(p, \boldsymbol{u}) = c_3 \, \frac{\left(p, \frac{\partial u^1}{\partial x}\right)^2}{\|u^1\|_\alpha^2}.$$

Set $\boldsymbol{v} = \boldsymbol{u}$. We have to prove that

$$\Psi(p, \boldsymbol{v}) = \frac{\left(p, \frac{\partial v^1}{\partial x} - h\frac{\partial v^1}{\partial y}\right)^2}{\|v^1\|_\alpha^2} \geq c\Phi(p, \hat{\boldsymbol{u}}).$$

From the proof of Lemma 3 we have

$$\Psi(p, \boldsymbol{v}) = \frac{\left(p, \frac{\partial u^1}{\partial x} + \bar{g}u^1 - \frac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2},$$

which implies

$$\Psi(p, \boldsymbol{u}) \geq \frac{0.25\left(p, \frac{\partial u^1}{\partial x}\right)^2 - 4\left(p, \bar{g}u^1\right)^2 - 4\left(p, \frac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2}.$$

Estimating all terms on the right-hand side of the last inequality as was done in the proof of Lemma 3, we have

$$\frac{\left(p, \frac{\partial u^1}{\partial x}\right)^2}{\|u^1\|_\alpha^2} \geq c_4 \Phi(p, \hat{\boldsymbol{u}}), \qquad \frac{(p, \bar{g}u^1)^2}{\|u^1\|_\alpha^2} \leq cM_2^2 \Phi(p, \hat{\boldsymbol{u}}),$$
$$\frac{\left(p, \frac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2} \leq cM_2^2 \Phi(p, \hat{\boldsymbol{u}}),$$

yielding

$$\Psi(p, \boldsymbol{u}) \geq c_5(1 - c_6 M_2^2)\Phi(p, \hat{\boldsymbol{u}}).$$

Thus, in the case $1 - c_6 M_2^2 > 0$ the assertion of Lemma 4 is proved. □

Corollary 2. *Let Ω be a curvilinear trapezoid*

$$\bar{\Omega} = \{x = (x_1, x_2) :\ 0 \leq x_1 \leq l,\ 0 \leq x_2 \leq g(x_1)\},$$

where g is a Lipschitz function. Then for sufficiently small l, the estimate (12) is valid for any $p \in L_2(\Omega)/\mathbb{R}$.

Proof. From Lemma 4 it follows that (12) is true for any $p \in L_2(\Omega)/\mathbb{R}$ if $l = \pi$ and $\max |g'|$ is sufficiently small. Making the change of variables

$$x = \frac{\pi}{l} x_1, \quad y = \frac{\pi}{l} x_2,$$

we obtain the domain Ω satisfying the conditions of Lemma 4, and $\frac{\pi}{l} g'_x = g'_{x_1}$. Hence, for sufficiently small l the derivative g'_x satisfies the condition of Lemma 4, whence the assertion of Corollary 2 follows. $\square$

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References

1. Arrow K., Hurwicz L., Uzawa H. (1958): Studies in Nonlinear Programming. Stanford University Press, Stanford
2. Bakhvalov N.S. (1995): Solution of the Stokes nonstationary problems by the fictitious domain method. Russ. J. Numer. Anal. Math. Modelling **10**, 163–172
3. Bramble J.H., Pasciak J.E. (1988): A preconditioning technique for indefinite systems resulting from mixed approximation of elliptic problems Math. Comp. **50**, 1–17
4. Bramble J.H., Pasciak J.E. (1997): Iterative techniques for time-dependent Stokes problem, Comput. Math. Appl. **33**, 13–30
5. Bramble J.H., Pasciak J.E., Vassilev A.T. (1997): Analysis of the inexact Uzawa algorithm for saddle point problems, SIAM J. Numer. Anal. **33**, 1072–1092
6. Cahouet J., Chabard J.P. (1988): Some fast 3D finite element solvers for the generalized Stokes problem Int. J. Numer. Methods Fluids. **8**, 869–895
7. Dauge M. (1989): Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. Part I: Linearized Equations. SIAM J. Math. Anal. **20**, 74–97
8. Elman H. C. (1996): Multigrid and Krylov subspace methods for the discrete Stokes equations Int. J. Numer. Methods Fluids **22**, 755–770
9. Elman H.C., Golub G.H. (1994): Inexact and preconditioned Uzawa algorithms for saddle point problems SIAM J. Numer. Anal. **31**, 1645–1661
10. Elman H.C. (1999): Preconditioning for the steady-state Navier-Stokes equations with low viscosity, SIAM J. Sci. Comput. **20**, 1299–1316
11. Girault V., Raviart P.A. (1986): Finite element methods for Navier-Stokes equations. Springer, Berlin
12. Kobelkov G.M. (1994): On Numerical Methods of Solving the Navier-Stokes Equations in “Velocity–Pressure” Variables. In: Marchuk, C. (ed.) Numerical methods and applications, pp. 85–115. CRC Press, New York
13. Olshanskii M.A. (1995): On numerical solution of nonstationary Stokes equations. Russ. J. Numer. Anal. Math. Modelling. **10**, 81–92
14. Olshanskii M.A. (1996): On one Stokes type problem with a parameter. Comp. Maths. Math. Phys. **36**. 193–202 (transl. from Zh. Vychisl. Mat. Mat. Fiz. **36**, 75-86)
15. Olshanskii M.A. (1997): Stokes problem with model boundary conditions. Sbornik: Mathematics **188**, 603–620 (transl. from Matematicheskii Sbornik **188**)
16. Pal’tsev B.V. (1995): On rapidly converging iterative methods with incomplete splitting of boundary conditions for a multidimensional singularly perturbed system of Stokes type. Russian Acad. Sci. Sb. Math **81**, 487–531
17. Pal’tsev B.V. (1995): On rapidly converging iterative methods with complete boundary conditions splitting for a multidimensional singularly perturbed system of Stokes type. Russian Acad. Sci. Sb. Math **83**, 93–117
18. Rusten T., Winther R. (1992): A preconditioned iterative methods for saddle point problems. SIAM J. Matrix Anal. Appl. **13**, 887–904

19. Sani R.L., Gresho P.M. (1994): Resume and Remarks on the open boundary condition minisymposium. Int. J. Numer. Methods Fluids **18**, 983–1008
20. Silvester D., Wathen A. (1994): Fast iterative solution of stabilized Stokes systems part II: Using block preconditioners, SIAM J. Numer. Anal. **31**, 1352–1367
21. Temam R. (1977): Navier-Stokes Equations. Theory and Numerical Analysis. North-Holland, Amsterdam
22. Turek S.(1999): Efficient solvers for incompressible flow problems: An algorithmic approach in view of computational aspects. LNCSE, vol. 6, Springer, Heidelberg

Note added in proof. The method presented in the paper was partially extended to non-symmetric problem of Oseen type in [1].

[1] Olshanskii, M.A. (1999): An iterative solver for the Oseen problem and the numerical solution of incompressible Navier-Stokes equations. Numer. Linear Algebra App. **6**, 353–378