An Iterative Solver for the Oseen Problem and Numerical Solution of Incompressible Navier–Stokes Equations

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Incompressible unsteady Navier–Stokes equations in pressure – velocity variables are considered. By use of the implicit and semi-implicit schemes presented the resulting system of linear equations can be solved by a robust and efficient iterative method. This iterative solver is constructed for the system of linearized Navier–Stokes equations. The Schur complement technique is used. We present a new approach of building a non-symmetric preconditioner to solve a non-symmetric problem of convection–diffusion and saddle-point type. It is shown that handling the differential equations properly results in constructing efficient solvers for the corresponding finite linear algebra systems. The method has good performance for various ranges of viscosity and can be used both for 2D and 3D problems. The analysis of the method is still partly heuristic, however, the mathematically rigorous results are proved for certain cases. The proof is based on energy estimates and basic properties of the underlying partial differential equations. Numerical results are provided. Additionally, a multigrid method for the auxiliary convection–diffusion problem is briefly discussed. Copyright © 1999 John Wiley & Sons, Ltd.

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1. Introduction

The construction of efficient numerical solvers for laminar incompressible flow problems is of vital importance not only for the numerical simulations of incompressible Navier–Stokes flows for low – moderate Reynolds numbers, but also for certain algorithms for compressible, turbulent, and other CFD problems, where the incompressible Navier–Stokes equations

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(non-linear or linearized) serve as auxiliary problems. Although a lot of contributions have been made by many scientists in establishing suitable methods to solve the problem (see, e.g., monographs [33,29,16]), there is still the crucial problem of building a robust, flexible, optimal (in some sense) and efficient algorithm. By this we mean in particular that an ideal method should be robust with respect to viscosity, time step and spatial mesh parameters, it should be readily implemented for 3D, complex geometries, efficiently parallelized (vectorized), and finally it should provide sufficient convergence and be used as a 'black box' solver in appropriate applications.

The ultimate (and probably still not reached) goal of constructing such a method requires many different tools of numerical analysis and fluid dynamics to be put together in a proper manner. In this paper we concentrate on an iterative method for linearized incompressible Navier–Stokes equations (Oseen problem). The problem has the form:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^{\mathrm{T}} \\ \mathbf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$
(1.1)

where the unknown $\{\mathbf{u}, p\}$ corresponds to discrete velocity vector function and pressure scalar function, **B** and **B**^T are due to the discrete operators -div and ∇ , **A** results after some time-discretization and linearization of convection-diffusion terms:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \tag{1.2}$$

plus boundary conditions for \mathbf{u} and some finite elements or finite differences in space. The problem (1.1) is indefinite and non-symmetric, if \mathbf{A} is non-symmetric.

The common assumptions to ensure (1.1) to be non-singular are $\mathbf{A} > 0$ and $\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}} > 0$. The operator $\mathbf{S} = \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}}$ is a Schur complement for system (1.1). This operator is in general non-symmetric, if \mathbf{A} is non-symmetric. The obvious observation is that p satisfies the equation:

$$\mathbf{S} \, p = -g + \mathbf{B} \mathbf{A}^{-1} f \tag{1.3}$$

The simple method to solve (1.1) is to iterate (1.3) for p, and than to recover **u** from (1.1), if p is obtained with the desired accuracy. However, such iterations require a proper preconditioner for **S**. The same requirement holds for many other methods to solve (1.1), although some of them do not require the exact evaluation of A^{-1} , details can be found in Section 4. The matrix **S** is not sparse and has rather complicated structure, so standard preconditioning techniques, e.g. ILU, Gauss–Seidel, are not effective.

To build an effective preconditioner for **S** one has to exploit the special differential properties of corresponding PDE systems. To be precise, if **Au** corresponds to $\frac{1}{\delta t}\mathbf{u} - \nu\Delta\mathbf{u}$, then an effective preconditioner for **S** is *known*. If **Au** corresponds to

$$\frac{1}{\delta t}\mathbf{u} - \nu\Delta\mathbf{u} + (U\cdot\nabla)\mathbf{u}$$

then an effective and robust (with respect to v, δt and mesh parameters) preconditioner for **S** is *not* known (at least to the best of our knowledge). However, the latter operator appears in implicit and robust time-stepping schemes for unsteady Navier–Stokes equations.

The idea was to linearize (1.2) in such a way that, on the one hand, the implicit nature of the time-stepping scheme is preserved, on the other hand, **S** admits an effective and robust preconditioning, although it is still non-symmetric. It appears that the well-known equality

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for **u**

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \nabla \left(\frac{\mathbf{u}^2}{2}\right)$$

helps us to linearize the convection term in such a way that **A** involves a zero order term curl $U \times \mathbf{u}$ for **u** instead of a first order term $(U \cdot \nabla)\mathbf{u}$. Further in the paper we will benefit from this fact. Details are presented.

Two key points of the method are the following: the distinguishing of a new pressure variable (Bernoulli pressure) as the basic iterated unknown; construction of a non-symmetric preconditioner for the Schur complement of the linearized Navier–Stokes problem. For the auxiliary problem of convection–diffusion type we consider a multigrid method as an inner iterator in our approach.

The present research is an extension of the one carried out for the symmetric case in [27, 5, 23], where convergence theorems are provided together with numerical results for the generalized Stokes problem. From the computational point of view the approach based on preconditioning of Schur complement for Navier–Stokes type problems is extensively studied in [35] with numerical evidence of its efficiency. As discussed in [35] this approach is closely related and can be viewed as a generalization of many schemes for the incompressible Navier–Stokes problem known as projection, pressure-correction, fractional-step, SIMPLE (with modifications), Vanka, etc.

2. Governing equations and definitions

We consider in a bounded 2D or 3D domain Ω the system of equations

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}$$

in $\Omega \times (0, T]$ (2.1)
div $\mathbf{u} = 0$

with given force field **f** and kinematic viscosity v > 0. The vector function $\mathbf{u}(t, \mathbf{x})$ (velocity) and the scalar function $p(t, \mathbf{x})$ (kinematic pressure) are to be found, subject to some conditions. The classical cases are the Dirichlet boundary conditions for velocity:

$$\mathbf{u} = \varphi \quad \text{on} \quad \partial \Omega \times [0, T] \tag{2.2}$$

(further, we take $\varphi \equiv 0$) initial condition for velocity at t = 0

$$\mathbf{u} = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \quad \overline{\Omega} \tag{2.3}$$

and integral condition $\int_{\Omega} p(\mathbf{x}, t) d\mathbf{x} = 0 \forall \in (0, T]$ to ensure the unique choice of pressure. The common assumption is also div $\mathbf{u}_0 = 0$, however, the latter is not fundamental for further considerations.

Further, we use the following notation. By $\mathbf{H}_0^1(\Omega)$ we denote the usual Sobolev space with functions vanishing on the boundary,

$$L_2^0(\Omega) = \{q \in L_2(\Omega) : (q, 1) = 0\}$$

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 $\mathbf{H}^{-1}(\Omega)$ is a space dual to $\mathbf{H}^{1}_{0}(\Omega)$. We need also the following space of vector functions:

$$\mathbf{H}_0(\operatorname{div}) \equiv \{ \mathbf{u} \in L_2(\Omega)^N : \operatorname{div} \mathbf{u} \in L_2(\Omega), \, \mathbf{u} \cdot \mathbf{n} |_{\partial \Omega} = 0 \}$$

where $\mathbf{n} = (n_1, ..., n_N)$ is an outward unit normal on $\partial \Omega$. $\mathbf{H}_0(\text{div})$ is provided with the norm

$$||\mathbf{u}||_{\mathbf{H}_0(\mathrm{div})}^2 = ||\mathbf{u}||_0^2 + ||\mathrm{div}\,\mathbf{u}||_0^2$$

The outline of the remainder of the paper is as follows. In Section 3 we consider two possibilities of time stepping for Navier–Stokes equations (2.1)–(2.3), they lead to solution of the linear Oseen system on every time step. The Oseen problem is studied in Section 4 together with two iterative algorithms to solve it. These algorithms require a proper preconditioner for the Schur complement of the Oseen problem. The Schur operator is non-symmetric. However, first, preconditioners from symmetric theory are tried. For this case some estimates for the Schur operator and convergence results are given in Section 5. Further, we obtain theoretically and numerically that the symmetric preconditioners do not work well for convection dominant problems. In Section 6, a new preconditioner is constructed. Since the problem to be solved is strongly non-symmetric for convection-dominated flows, it is natural that the appropriate preconditioner is also non-symmetric. The new preconditioning leads to a strongly elliptic problem of diffusive type for the pressure. This problem is also studied. In Section 7 we give some heuristic arguments based on Fourier analysis that predict convergence behavior of the method. In Section 8 numerical results are presented. Additionally, a multigrid method for the auxiliary convection-diffusion problem is briefly discussed. The Appendix collects technical details of proofs.

In Sections 3–6 the considerations in the paper are done for continuous problems. Here we let the reader be free to choose his particular favorite discretization method, since the basic properties of the differential problems used are still valid in some sense for 'reasonable' finite methods. Moreover, such 'continuous' treatment of the problem helps us to construct preconditioners that provide good convergence for arbitrary fine grids.

3. Time-stepping schemes

A common way of treating the problem (2.1)–(2.3) numerically is the following. First apply some finite difference discretization in time or some methods from ODE theory, say, explicit or implicit Euler or Runge–Kutta method. Further discretize in space by some finite difference, finite element or other method and solve the resulting finite system by an appropriate iterative algorithm.

Following here the same way, we note that explicit schemes for (2.1)-(2.3) cause severe stability problems for fine spatial grids and/or small viscosity, since for a small viscosity system (2.1)-(2.3) becomes stiff. Hence, the effective time-stepping techniques have an implicit nature and require a non-linear or a linearized problem to be solved on each time step. For details corresponding to stability and error control, see, e.g., [19, 20]).

Consider the following fully implicit scheme. Given \mathbf{u}^n and \mathbf{f}^{n+1} find \mathbf{u}^{n+1} and p^{n+1}

from

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \nu \Delta \mathbf{u}^{n+1} + (\mathbf{u}^{n+1} \cdot \nabla) \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1}$$

div $\mathbf{u}^{n+1} = 0$
 $\mathbf{u}^{n+1}|_{\partial \Omega} = 0$ (3.1)

where $\mathbf{u}^n = \mathbf{u}(n\delta t)$, $p^n = p(n\delta t)$ and δt is a time step. The scheme is known to be very robust and stable. However, one has to solve a non-linear problem on each time step. Below we consider two possibilities of further calculations.

Considerations are heavily based on the following formal equality for arbitrary vector functions \mathbf{u} and \mathbf{v} :

$$(\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} = (\operatorname{curl} \mathbf{v}) \times \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{v} + \nabla(\mathbf{v}, \mathbf{u}), \quad (3.2)$$

where $(\mathbf{v}, \mathbf{u}) = v_1 u_1 + \ldots + v_N u_N$ is a scalar function, \times stands for vector product, curl $\mathbf{u} = (\nabla \times \mathbf{u})$ is a vorticity function. For two dimensions we define curl $\mathbf{u} = -\partial u_1 / \partial x_2 + \partial u_2 / \partial x_1$ and

$$a \times \mathbf{u} = -\mathbf{u} \times a = \begin{cases} -a \, u_2 \\ a \, u_1 \end{cases}$$

for a scalar *a* and vector **u**.

If one takes in (3.2) $\mathbf{u} = \mathbf{v}$ it results into a well-known equality:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \nabla\left(\frac{\mathbf{u}^2}{2}\right)$$
 (3.3)

3.1. Semi-implicit scheme

The first scheme is related to a widely used linearization of convection terms. In order to avoid the solution of non-linear equations on every time step we simply linearize the convection by taking velocity from the previous time steps. To this end consider convection in the form written on the right-hand side of (3.3)

Further denote by *P* the new pressure $P = p + \frac{\mathbf{u}^2}{2}$ sometimes referred to as the Bernoulli pressure. Replace (curl \mathbf{u}) × \mathbf{u} by (curl *U*) × \mathbf{u} , were *U* is given and corresponds to some extrapolation of velocity from previous times (e.g. constant $U^n = \mathbf{u}(n\delta t)$ or linear $U^n = 2\mathbf{u}(n\delta t) - \mathbf{u}((n-1)\delta t)$ for equidistant time steps).

The resulting problem on every time step finally reads

$$\frac{1}{\delta t} \mathbf{u}^{n+1} - \nu \Delta \mathbf{u}^{n+1} + (\operatorname{curl} U^n) \times \mathbf{u}^{n+1} + \nabla P^{n+1} = \mathbf{f}^{n+1} + \frac{1}{\delta t} \mathbf{u}^n$$

div $\mathbf{u}^{n+1} = 0$
 $\mathbf{u}^{n+1}|_{\partial \Omega} = 0$ (3.4)

Note that (curl $U \times \mathbf{u}, \mathbf{v}$) = $-(\text{curl } U \times \mathbf{v}, \mathbf{u})$ due to the properties of the vector product, hence the corresponding bilinear form is skew-symmetric. Therefore, the solution of resulting scheme (3.4) satisfies the discrete analogue of the following basic energy estimate

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for (2.1)–(2.3)

$$||\mathbf{u}(t)||_{0}^{2} + \nu \int_{0}^{t} ||\mathbf{u}(s)||_{1}^{2} ds \leq ||\mathbf{u}_{0}||_{0}^{2} + \nu^{-1} \int_{0}^{t} ||\mathbf{f}(s)||_{-1}^{2} ds.$$

3.2. Fully implicit scheme

The second possibility is to apply several non-linear iterations for the direct solution of nonlinear problem (3.1). This will result in a fully implicit scheme for (2.1)–(2.3). Non-linear iterations can be performed as: given $\{\mathbf{u}_0, p_0\}$ (e.g. $\mathbf{u}_0 = \mathbf{u}^n, p_0 = p^n$) iterate k = 1, 2, ...

$$\begin{pmatrix} \mathbf{u}_k \\ p_k \end{pmatrix} = \begin{pmatrix} \mathbf{u}_{k-1} \\ p_{k-1} \end{pmatrix} - \kappa_{k-1} F(\mathbf{u}_{k-1})^{-1} \begin{pmatrix} \operatorname{res}(\mathbf{u}_{k-1}) \\ \operatorname{div} \mathbf{u}_{k-1} \end{pmatrix}$$
(3.5)

where $F(\mathbf{u}_{k-1})$ is the Frechet derivative in \mathbf{u}_{k-1} and $res(\mathbf{u}_{k-1})$ is the non-linear residual for \mathbf{u}_{k-1} ; κ_{k-1} is a relaxation parameter that can be chosen, for example, as in the adaptive fixed point defect correction method. If the desired convergence in (3.5) is achieved for some *k* set $\mathbf{u}^{n+1} = \mathbf{u}_k$, $p^{n+1} = p_k$.

The crucial point in method (3.5) is applying $F(\mathbf{u}_{k-1})^{-1}$. The common way is to replace $F(\mathbf{u}_{k-1})$ by an approximate $\tilde{F}(\mathbf{u}_{k-1})$, which is easily inverted. To be precise, consider the contribution of convection terms in $F(\mathbf{u}_{k-1})\mathbf{u}$, that is,

$$(\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_{k-1} \tag{3.6}$$

The first term in (3.6) is skew-symmetric and is preserved in $\tilde{F}(\mathbf{u}_{k-1})\mathbf{u}$, the second one is reactive and usually is not included in $\tilde{F}(\mathbf{u}_{k-1})\mathbf{u}$ to ensure good numerical properties of $\tilde{F}(\mathbf{u}_{k-1})$.

However, as was explained in Section 1 we want to avoid the first-order term $(\mathbf{u}_{k-1} \cdot \nabla)\mathbf{u}$ for velocity \mathbf{u} . To do this let us act as follows. Due to (3.2) observe the relations

$$(\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_{k-1} = (\operatorname{curl}\mathbf{u}_{k-1}) \times \mathbf{u} + (\operatorname{curl}\mathbf{u}) \times \mathbf{u}_{k-1} + \nabla(\mathbf{u}_{k-1},\mathbf{u})$$
(3.7)

Now, the first term in the right-hand side of (3.7) is skew-symmetric and is retained, the second one is dropped and the third one is also retained and added to a new pressure. Finally, the problem to be solved on each step of scheme (3.5) is of the same type as (3.4). Similar to (3.4) we have the 'convection' term of the zero order for velocity.

Remark 3.1.

In [34] it is stated that (3.4) may not be so advantageous in a specific case where the spatial meshes have high aspect ratio and if a certain adaptive time-step control is used. However, a fully implicit scheme works satisfactorily in various situations.

4. The Oseen problem and iterative methods

The problem to be solved on each time step of the semi-implicit scheme and on each inner iteration of the fully implicit scheme from Section 3 reads: given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $g \in L_2^0(\Omega)$,

and $\mathbf{w} \in L_2(\Omega)^{2N-3}$ find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $p \in L_2^0(\Omega)$ from

$$\alpha \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega$$

div $\mathbf{u} = g \quad \text{in } \Omega$
 $\mathbf{u} = 0 \quad \text{on } \partial \Omega$ (4.1)

with $\alpha > 0$, $\nu > 0$. We recall that $\Omega \in \mathbb{R}^N$, N = 2, 3, and boundary $\partial \Omega$ is assumed to be sufficiently smooth. Problem (4.1) is linear, non-symmetric and of saddle point type. The weak formulation of (4.1) is straightforward. It can be readily checked that the problem is well posed (see, e.g., [15] Chapter I for a general framework).

We intend to solve problem (4.1) iteratively. A variety of iterative techniques to solve saddle point problems are known (see, e.g., [1, 4, 7, 3, 11, 13, 30–32, 37]), however most of them were established and analyzed only for symmetric problems. Although there are important and quite recent results in papers ([17, 12, 14, 21, 22]) which deal with non-symmetric saddle-point problems, there is still a lack of theory and robust implementations in this case. At least, the robustness with respect to small parameter ν was not obtained in these papers.

To treat problem (4.1) we use the classic Uzawa approach, which includes variants of exact and inexact Uzawa algorithms. To this end, consider the Schur operator

$$\mathbf{S} = -\mathrm{div} \left(\alpha \mathbf{I} - \nu \Delta + \mathbf{w} \times\right)_0^{-1} \nabla \tag{4.2}$$

where $(\alpha \mathbf{I} - \nu \Delta + \mathbf{w} \times)_0^{-1}$ is the solution operator for the convection–diffusion problem with the Dirichlet homogeneous boundary conditions¹:

$$\begin{aligned} \alpha \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{w} \times \mathbf{u} &= \mathbf{g} \quad \text{in } \Omega \\ \mathbf{u}|_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \end{aligned}$$

S is a linear operator on $L_2^0(\Omega)$ (the space of pressure functions), it is positive (Theorem 5.1 below) and non-symmetric for $\mathbf{w} \neq 0$. Pressure *p* satisfies equation

$$\mathbf{S}p = F$$
with given $F = g - \operatorname{div} (\alpha \mathbf{I} - \nu \Delta + \mathbf{w} \times)_0^{-1} \mathbf{f}$
(4.3)

We note that equation (4.3) readily follows from (4.1) after the elimination of velocity and does not require any boundary conditions or extra regularity for pressure. Equation (4.3) can be effectively solved by iterations if some 'good' preconditioner for **S** is available.

As a simple possibility, consider the following preconditioned iterations to solve (4.3): given p^0 find p^{k+1} , k = 0, 1, ..., from

$$p^{k+1} = p^k - \tau \mathbf{Q}^{-1} (\mathbf{S} p^k - F)$$
(4.4)

This algorithm is often referred to as the (exact) Uzawa algorithm and it is very popular for symmetric problems (in combination with conjugate gradient methods). In the next section some convergence results for (4.4) are given.

Method (4.4) requires the solution of the convection–diffusion equation on every iterative

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¹ It is not a convection in a physical sense, however, we will refer to this problem as 'convection-diffusion'.

step when **S** is applied to p^k . To avoid this generally expensive operation, one can use the so-called inexact Uzawa algorithm (closely related to the Arrow–Hurwitz algorithm [33]) that iterates both pressure and velocity (see more details in [7, 13]). Let us assume that **D** is some preconditioner to the convection–diffusion operator $(\alpha I - \nu \Delta + \mathbf{w} \times)_0$ (for the particular choices of **D** refer, e.g., to [14, 21], and Section 8 of this paper). Then the inexact Uzawa algorithm can be written as follows: given \mathbf{u}^0 , p^0 find \mathbf{u}^{k+1} , p^{k+1} , $k = 0, 1, \ldots$, from

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \beta \mathbf{D}^{-1} (\alpha \mathbf{u}^k - \nu \Delta \mathbf{u}^k + \mathbf{w} \times \mathbf{u}^k + \nabla p^k - \mathbf{f})$$

$$p^{k+1} = p^k - \tau \mathbf{Q}^{-1} (\operatorname{div} \mathbf{u}^{k+1} - g)$$
(4.5)

In the particular case of $\mathbf{D}^{-1} = (\alpha \mathbf{I} - \nu \Delta + \mathbf{w} \times)_0^{-1}$ and $\beta = 1$, method, (4.5) coincides with (4.4), otherwise the first relation in (4.5) can be interpreted as one iteration for solving convection–diffusion problem. Generally more than one iteration could be done, and therefore (4.5) is often called the *inexact* Uzawa algorithm. The optimal choice of parameters in (4.5) and convergence results will be considered elsewhere. More results can be found in [22, 6] for non-symmetric problems and additionally [10, 30] for symmetric ones. In particular, we emphasize that method (4.5), as well as (4.4), requires a proper preconditioner for **S** to ensure a 'good' convergence.

5. Convergence estimates for Oseen problem with symmetric preconditioning

Let us consider the operator **S** in more detail. First, denote by S_0 the Schur operator for the symmetric problem, i.e., $S_0 = S$ for w = 0. Note that S_0 is not a symmetric part of **S**, $S_0 \neq \frac{1}{2}(S + S^*)$ in general.

5.1. Analysis of differential problems

The following theorem holds.

Theorem 5.1. For any $\alpha > 0$, $\nu > 0$ and $\mathbf{w} \in L_2(\Omega)^{2N-3}$ the estimates

$$\gamma_1 ||p||_0^2 \le (\mathbf{S}p, p) \le \gamma_2 ||p||_0^2 \tag{5.1}$$

$$(\mathbf{S}p,q) \leq \gamma_3(\mathbf{S}p,p)^{\frac{1}{2}}(\mathbf{S}q,q)^{\frac{1}{2}}$$
 (5.2)

$$\gamma_4 ||p||_0^2 \le (\mathbf{S}^{-1}p, p) \tag{5.3}$$

$$\gamma_5(\mathbf{S}_0 p, p) \leq (\mathbf{S}_p, p) \leq (\mathbf{S}_0 p, p) \tag{5.4}$$

$$\gamma_5(\mathbf{S}_0^{-1}p, p) \leq (\mathbf{S}^{-1}p, p)$$
 (5.5)

hold for all $p, q \in L_2^0(\Omega)$ with

$$\gamma_{1} = \frac{1}{4\kappa} (\rho \alpha + \nu + K(\nu, \alpha, \mathbf{w}))^{-1}$$

$$\gamma_{2} = \nu^{-1}$$

$$\gamma_{3} = (1 + C(\nu, \alpha, \mathbf{w}))$$

$$\gamma_{4} = \nu$$

$$\gamma_{5} = (1 + C(\nu, \alpha, \mathbf{w})^{2})^{-1}$$

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Constants $K(v, \alpha, \mathbf{w})$ and $C(v, \alpha, \mathbf{w})$ can be taken as follows, depending on the actual smoothness of function \mathbf{w} :

$$K(\nu, \alpha, \mathbf{w}) = c \frac{||\mathbf{w}||_0^2}{\sqrt{\alpha\nu}} \quad or \quad K(\nu, \alpha, \mathbf{w}) = c \frac{||\mathbf{w}||_{L_3}^2}{\alpha}$$

and

$$C(\nu, \alpha, \mathbf{w}) = c \frac{||\mathbf{w}||_0}{\sqrt{\nu} (\alpha \nu)^{\frac{1}{4}}} \text{ or } C(\nu, \alpha, \mathbf{w}) = c \frac{||\mathbf{w}||_{L_{\infty}}}{\alpha}$$

 $\rho = \rho(\Omega)$ and $\kappa = \kappa(\Omega)$ are positive constants from the Poincare–Fridrichs and Nečas inequalities:

$$\begin{aligned} ||\mathbf{u}||_0 &\leq \sqrt{\rho} ||\mathbf{u}||_1 \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega) \\ ||p||_0 &\leq \sqrt{\kappa} ||\nabla p||_{-1} \quad \forall p \in L_2^0(\Omega) \end{aligned}$$

Proof

The proof follows from estimates of the skew-symmetric form $(\mathbf{w} \times \mathbf{u}, \mathbf{v})$, embedding theorems and the Nečas inequality. The details can be found in the Appendix.

Inequality (5.1) ensures the positiveness, and inequality (5.2) together with (5.1) the continuity of **S**.

In fact, in viscous flow $||\mathbf{w}||$ may also depend on ν in some implicit way, e.g., in a 2D parabolic boundary layer it is typical to have (cf. [26]) $\frac{\partial u}{\partial y} = O(\nu^{-\frac{1}{2}}), \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} = O(1)$, and hence, assuming the width of boundary layer to be equal $O(\nu^{\frac{1}{2}})$, we have

$$||\mathbf{w}||_0 \approx ||\mathbf{u}||_1 = O(\nu^{-\frac{1}{4}})$$

Another remark is that the estimates from Theorem 5.1 are valid in the general 3D case. For 2D problems the dependence of the constants K and C on v is weaker, since embedding theorems are less restrictive in this case. The interested reader can easily obtain appropriate results, following the proof in Appendix.

Below, the theorem states some convergence results for method (4.4) with symmetric preconditioning. These results are based on the estimates from Theorem 5.1. To be precise, we consider two possibilities of choosing a preconditioner in (4.4):

$$\mathbf{Q}^{-1} = I \tag{5.6}$$

$$\mathbf{Q}^{-1} = \nu I - \alpha \Delta_N^{-1} \tag{5.7}$$

Here and further on Δ_N^{-1} is a solution operator for the scalar Poisson problem with Neumann's boundary conditions. The choice (5.7) is known to be optimal for the symmetric problem. In [23] it is proved that cond($\mathbf{Q}^{-1}\mathbf{S}_0$) < *c*, with some *c* independent of ν , α . However for convection dominant problems this may not be a good choice. This is indicated in Theorem 5.2 and confirmed by numerical results in Section 8.

Theorem 5.2. For \mathbf{Q}^{-1} from (5.6) and (5.7), method (4.4) converges for sufficiently small $\tau > 0$. If $e^k = p - p^k$ is the error of the iterations (4.4) and $||e^k||_Q = (Q \ e^k, e^k)$, then the convergence factor ψ defined from

$$||e^{k+1}||_Q \le \psi ||e^k||_Q \quad \forall k \ge 0$$

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can be estimated as follows:

• For the case (5.6) and $\tau = v$

$$\psi \leq \sqrt{1 - \frac{\nu}{4\kappa} \left(\rho \alpha + \nu + K(\nu, \alpha, \mathbf{w})\right)^{-1}}$$

• For the case (5.7) and $\tau = c (1 + C(\nu, \alpha, \mathbf{w})^2)^{-1}$

$$\psi \leq \sqrt{1 - c \left(1 + C(\nu, \alpha, \mathbf{w})^2\right)^{-2}}$$

Proof

The proof is quite standard and outlined in the Appendix.

The same convergence estimates hold for the more sophisticated GCG-LS method (see [2]). However, the convergence estimates are very disadvantageous for $\nu \rightarrow 0$ and $\alpha \rightarrow \infty$ if no preconditioning is applied (case (5.6)). If some preconditioner from symmetric theory is used ($\mathbf{Q} \sim \mathbf{S}_0$), then $\alpha \rightarrow \infty$ is not a poor case any more. Now, for $\alpha \rightarrow \infty$ even some improvement of convergence can be predicted (especially in two dimensions and smooth **w**). However the case $\nu \rightarrow 0$ and/or $||\mathbf{w}|| \rightarrow \infty$ is still disadvantageous. The preconditioner **Q** that takes into account convection effects is deduced in the next section.

Remark 5.1.

The estimates of theorems 5.1 and 5.2 are not optimal for $\alpha \to 0$. Since, for unsteady flow $\alpha \sim (\delta t)^{-1} > 1$, the case of $\alpha \ll 1$ is not of particular interest here. However quite similar results for steady problem $\alpha = 0$ can be readily obtained (see Remark 9.1 in the Appendix). We only comment that this analysis for the steady Oseen problem readily gives the estimate

of convergence factor as
$$\psi \leq \sqrt{1 - \frac{\nu}{2\kappa} \left(\nu + c \frac{||\mathbf{w}||_0^2}{\nu}\right)^{-1}}$$
 or $\psi \sim 1 - O\left(\frac{\nu^2}{||\mathbf{w}||_0^2}\right)$ for

small ν and/or large $||\mathbf{w}||_0^2$. This estimate agrees with the theory and numerical results from [14, 12, 21]. At the same time the estimates of Theorems 5.1 and 5.2 are optimal for $||\mathbf{w}|| \rightarrow 0$, when the problem becomes more 'symmetric'.

5.2. Remarks for discrete problems

The results of Theorems 5.1 and 5.2 obtained for the differential operators can be transferred to the discrete case with some minor changes only. As an example consider any LBB-stable [8] finite element (FE) pair $\mathbf{U}_h \times P_h$ for velocity and pressure. The arguments for the weak formulations of differential problems from the proof of Theorem 5.1 (see Appendix) can be applied in a straightforward way to the FE formulation, since the embedding theorems are possessed by FE, additionally the Nečas inequality is replaced by the LBB condition.

Hence, for the discrete operators all the estimates from Theorem 5.1 are still valid with the constants independent of mesh size *h*, but dependent on ν , α and $||\mathbf{w}_h||$ in the same manner.

Due to these arguments for a discrete preconditioner Q_h from (5.6) we have $Q_h = M_p$, where M_p is the mass matrix of the pressure FE space. Then the discrete counterpart of (5.1) implies

$$\gamma_1 M_p \le S_h \le \gamma_2 M_p$$

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Therefore the estimates of Theorem 5.2 for $Q_h = M_p$ (the case (5.6)) still hold.

The discrete case (5.7) is more delicate. The equivalence $\mathbf{S}_0^h \sim \nu M_p + \alpha \Delta_h^{-1}$ was proved in [5] for a special approximation of Δ_h^{-1} on a course pressure grid. However, the numerical experiments show that the best choice is to use $\left(\operatorname{div}^h \tilde{M}_U^{-1} \nabla^h\right)^{-1}$ rather than Δ_h^{-1} in (5.7), where \tilde{M}_U is a velocity mass matrix with a proper treatment of boundary conditions.

One more implementation issue is that the exact solvers for the elliptic subproblems (e.g., Poisson or convective diffusion) implemented in (4.4) can make these iterations rather costly for 'real-life' problems. However, we refer to [35] for examples of very effective implementations of such iterations in a multigrid context.

A common possibility is to replace the exact solution of a subproblem with an approximate one. This can be obtained by utilizing a limited number of iterations to solve the subproblem or one can explicitly construct an operator (with nice algebraic properties) that is close to the operator of the discrete subproblem.

Thus, to make the implementation more effective, one can try to replace M_p with a diagonal matrix constructed by a diagonal lumping for M_p and spectrally equivalent to M_p [36]. The exact evaluation of $r = \Delta_h^{-1}q$ for some q can be replaced by one or few multigrid cycles to solve $\Delta_h r = q$ as often recommended in literature (e.g., [12]). The convergence estimates from Theorem 5.2 will only alter in a standard way due to the constants of equivalence between 'exact' preconditioners and 'inexact' ones.

The situation is less clear if one tries to implement an inexact convection–diffusion solver. The process in form (4.4) is not applicable now and one necessarily should deal with methods like (4.5) where velocity and pressure are iterated together. However, the theory of such methods for non-symmetric problems is far from being developed. Some relevant results can be found in recent papers [21, 6].

6. Non-symmetric preconditioning and diffusive pressure problem

In this section we concentrate on pressure equation (4.3). A robust and optimal preconditioner for **S** should take care of diffusive, reactive and convection effects in (4.1). To construct a preconditioner that works well for all types of flows, let us consider separately two extreme cases: the case of strongly viscous flow ($\nu \gg 1$) and the case of slightly viscous flow ($\nu \ll 1$).

It is well known that a strongly viscous flow is nearly symmetric and the effect of convection terms can be neglected. In this case $\mathbf{S} = \mathbf{S}_0$ and one can rewrite pressure problem (4.3) as

$$-\nu \Delta \mathbf{u} + \alpha \mathbf{u} + \nabla p = \mathbf{0}$$

div $\mathbf{u} = F$
 $\mathbf{u} = 0$ on $\partial \Omega$ (6.1)

Remark 6.1.

In (6.1) and further in this section if no misunderstanding occurs, we may use notations \mathbf{u} and p for some auxiliary functions that appear in the building of a preconditioner. However, we call them 'velocity' and 'pressure'.

For problem (6.1) the operator $\mathbf{Q}_0^{-1} = \nu \mathbf{I} - \alpha \Delta_N^{-1}$ provides the estimate

$$\operatorname{cond}(\mathbf{Q}_0^{-1}\mathbf{S}_0) \le c$$

with *c* independent of ν and α . Furthermore, we recall [28] that the operator \mathbf{Q}_0 (= $(\nu \mathbf{I} - \alpha \Delta_N^{-1})^{-1}$) can be realized as a Schur complement of the generalized Stokes problem (6.1) after relaxing tangential boundary conditions on velocity and posing natural boundary conditions on vorticity: $\mathbf{u} \cdot \mathbf{n} = 0$, (curl \mathbf{u}) × $\mathbf{n} = 0$ instead of $\mathbf{u} = 0$ in (6.1).

Below we utilize the similar ideas in another extreme case of $\nu \ll 1$.

Slightly viscous flows can be considered inviscid almost everywhere in the domain except for small regions of high velocity gradients, typically, boundary layers. Bearing this in mind, let us rewrite equation (4.3) and drop the viscous terms. We arrive at the following problem:

$$\alpha \mathbf{u} + \mathbf{w} \times \mathbf{u} + \nabla p = \mathbf{0}$$

div $\mathbf{u} = F$
 $\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega$ (6.2)

Note that due to a lack of high order derivatives for the velocity in (6.2) we preserve only normal boundary conditions.

The appropriate weak saddle-point formulation of (6.2) is the following. For given $F \in L_2^0(\Omega)$ find $\{\mathbf{u}, p\} \in \mathbf{H}_0(\operatorname{div}) \times L_2^0(\Omega)$ such that for any $\{\mathbf{v}, g\} \in \mathbf{H}_0(\operatorname{div}) \times L_2^0(\Omega)$

$$\alpha(\mathbf{u}, \mathbf{v}) + (\mathbf{w} \times \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = 0$$

(div \mathbf{u}, q) = (F, q) (6.3)

It is easy to see that the bilinear form $a(\mathbf{u}, \mathbf{v}) = \alpha(\mathbf{u}, \mathbf{v}) + (\mathbf{w} \times \mathbf{u}, \mathbf{v})$ is coercive on Ker(div) in **H**₀(div), and the infsup condition

$$\inf_{p \in L_2^0(\Omega)} \sup_{\mathbf{u} \in \mathbf{H}_0(\operatorname{div})} \frac{(p, \operatorname{div} \mathbf{u})}{||\mathbf{u}||_{H(\operatorname{div})} ||p||_0} \ge c(\Omega) > 0$$

is valid. However, since the velocity space $\mathbf{H}_0(\operatorname{div})$ is more general than $\mathbf{H}_0^1(\Omega)$, the form $a(\mathbf{u}, \mathbf{v})$ is no longer continuous. The additional assumption on smoothness of \mathbf{w} ($\mathbf{w} \in L_{\infty}(\Omega)^{2N-3}$) improves this situation. Fortunately, in most applications the velocity field is smooth or almost smooth except in the neighborhoods of some singular points of the boundary.

Now, Corollary 5.1 from [15] implies that problem (6.3) is well posed.

It is clear that (6.2) can be considered as a mixed formulation of some elliptic problem for the pressure (this is the point where we benefit from the zero order of the new convection term). Indeed, we can formally eliminate velocity from the first equality in (6.2). Further using the second equality and the boundary conditions, we get the following diffusive problem with the Neumann conditions for p,

$$-\frac{1}{\alpha} \operatorname{div} \left(\mathscr{G}(\mathbf{x}) \nabla p \right) = F$$

$$\frac{\partial p}{\partial \tilde{\mathbf{n}}} = 0, \quad \text{on } \partial \Omega$$
(6.4)

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where $\mathscr{G}(\mathbf{x}) = \{g_{ij}(\mathbf{x})\}, i, j = 1, ..., N$ is the 'diffusive' matrix detailed below and $\frac{\partial p}{\partial \mathbf{\tilde{n}}} = \mathscr{G}(\mathbf{x}) \nabla p \cdot n$.

The matrix $\mathscr{G}(\mathbf{x})$ is expressed in terms of α and \mathbf{w} as follows.

• 2D case

$$\mathscr{G}(\mathbf{x}) = \frac{\alpha^2}{\alpha^2 + \mathbf{w}^2} \mathbf{I} - \frac{\alpha}{\alpha^2 + \mathbf{w}^2} (\mathbf{w} \times)$$
(6.5)

• 3D case

$$\mathscr{G}(\mathbf{x}) = \frac{\alpha^2}{\alpha^2 + \mathbf{w}^2} (\mathbf{I} + \alpha^{-2} (\mathbf{w} \otimes \mathbf{w})) - \frac{\alpha}{\alpha^2 + \mathbf{w}^2} (\mathbf{w} \times)$$
(6.6)

Here I stands for the identity matrix, $(\mathbf{w} \otimes \mathbf{w})$ for the one with *ij*-element equals $w_i(\mathbf{x})w_j(\mathbf{x})$, and $(\mathbf{w}\times)$ stands for the matrix corresponding to the vector product with **w**:

• 2D case

$$(\mathbf{w}\times) = \left(\begin{array}{cc} 0 & -w \\ w & 0 \end{array}\right)$$

• 3D case

$$(\mathbf{w} \times) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

The first term in (6.5) or (6.6) is the symmetric part of $\mathscr{G}(\mathbf{x})$ and the second is skew-symmetric.

Let us consider problem (6.4). Note that $|g_{ij}(\mathbf{x})| \leq c < \infty$, i, j = 1, ..., N, with some constant *c* independent of $\mathbf{x}, \mathbf{w}, \alpha$, and thus $g_{ij}(\mathbf{x}) \in L_{\infty}(\Omega)$. Consider the function $\mathbf{v} = \mathscr{G}(\mathbf{x}) \nabla p$ and assume that $p \in H^1(\Omega)$. Now $g_{ij}(\mathbf{x}) \in L_{\infty}(\Omega)$ implies $\mathbf{v} \in L_2(\Omega)^N$.

Therefore, the following weak formulation of (6.4) makes sense. For given $F \in L_2^0(\Omega)$ find $p \in H^1(\Omega) \cap L_2^0(\Omega)$ such that

$$\frac{1}{\alpha}(\mathscr{G}(\mathbf{x})\nabla p, \nabla q) = (F, q), \quad \forall q \in H^1(\Omega) \cap L^0_2(\Omega)$$
(6.7)

Equality (6.7) can be a starting point for a finite element discretization of (6.4).

In almost every point **x** in the domain Ω the matrix $\mathscr{G}(\mathbf{x})$ is positive, i.e.,

$$(\mathscr{G}(\mathbf{x})\zeta,\zeta) > 0$$

for any non-zero $\zeta \in \mathbb{R}^N$. Thus the problem is strongly elliptic and has a weak solution satisfying (6.7) (see, e.g., [25]). Similar to (6.3) the additional assumption $\mathbf{w} \in L_{\infty}(\Omega)^{2N-3}$ ensures problem (6.7) to be uniformly elliptic and the weak solution to be unique.

The following lemma is valid.

Lemma 6.1. Assume $\mathbf{w} \in L_{\infty}(\Omega)^{2N-3}$. Then problems (6.2) and (6.4) have unique weak solutions in the sense of (6.3) and (6.7), respectively. Moreover, the pressure component p of the solution of (6.2) belongs to $H^1(\Omega) \cap L^0_2(\Omega)$ and solves problem (6.4).

Proof See Appendix

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Remark 6.2.

The minimal regularity of the velocity function in (3.1) $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ provides that \mathbf{w} belongs to $L_2(\Omega)^{2N-3}$. The extra regularity of given data and hence of the solution ensures $\mathbf{w} \in L_{\infty}(\Omega)^{2N-3}$, and, as was noted above, the form $(\mathscr{G}(\mathbf{x})\nabla p, \nabla q)$ is coercive (uniformly elliptic) on $H^1(\Omega) \cap L_2^0(\Omega)$, i.e.,

$$||\nabla p||^2 \le C\left(\mathscr{G}(\mathbf{x})\nabla p, \nabla p\right) \quad \forall p \in H^1(\Omega) \cap L^0_2(\Omega)$$
(6.8)

If some spatial discretization is considered, then (6.8) holds in any case, with C generally depending on mesh size h. The dependence on h is weaker for a smooth w.

Now, denote by $L(\mathbf{w})^{-1}$ ($L(\mathbf{w})^{-1}$: $L_2^0(\Omega) \to H^1(\Omega) \cap L_2^0(\Omega)$) the solution operator for problem (6.4), in brackets we emphasize the dependence of the operator on given \mathbf{w} . Consider the operator

$$\mathbf{Q}(\mathbf{w})^{-1} = \nu \mathbf{I} + \mathbf{L}(\mathbf{w})^{-1}$$
(6.9)

as a preconditioner for **S**. On the one hand, for strongly viscous flows (when we can ignore the convection effects) the new preconditioner coincides with $\nu \mathbf{I} - \alpha \Delta_N^{-1} = \mathbf{Q}(0)^{-1}$, which is known to be optimal in this case. On the other hand, for convection-dominant flows, omitting the diffusion terms in (4.1) does not alter much the global properties of the problem and $\mathbf{L}(\mathbf{w})$ is again close to **S**. Hopefully, the choice of $\mathbf{Q}(\mathbf{w})$ covers all intermediate cases as well. Heuristic analysis of the next section and numerical results from Section 8, support this conclusion.

7. Fourier analysis

In this section we give some arguments in a framework of the Fourier analysis to justify the effectiveness of the preconditioning proposed. We consider only the 2D case and constant $w(\mathbf{x}) = w$. This choice considerably simplifies further calculations.

Let us consider the periodic flow in R^2 and evaluate the operator **S** on a given harmonic. To this end, assume $p(\mathbf{x}) = \exp(i(\mathbf{a}, \mathbf{x}))$, where $\mathbf{a}, \mathbf{x} \in R^2$. Then

$$\nabla p(\mathbf{x}) = \{i a_1 \exp(i(\mathbf{a}, \mathbf{x})), i a_2 \exp(i(\mathbf{a}, \mathbf{x}))\}$$

Looking for **u** of the form $u_1 = i k_1 \exp(i(\mathbf{a}, \mathbf{x})), u_2 = i k_2 \exp(i(\mathbf{a}, \mathbf{x}))$, we find from

$$-\nu\Delta u_1 + \alpha u_1 - w u_2 = -\frac{\partial p(\mathbf{x})}{\partial x_1}$$
$$-\nu\Delta u_2 + \alpha u_2 + w u_1 = -\frac{\partial p(\mathbf{x})}{\partial x_2}$$

the coefficients

$$k_1 = \frac{-(\alpha + \nu|a|^2)a_1 + w a_2}{(\alpha + \nu|a|^2)^2 + w^2}, \quad k_2 = \frac{-(\alpha + \nu|a|^2)a_2 - w a_1}{(\alpha + \nu|a|^2)^2 + w^2}$$

Therefore, we get

Ì

$$\mathbf{S}_p \exp(i(\mathbf{a}, \mathbf{x})) = \operatorname{div} \mathbf{u} = \frac{(\alpha + \nu |a|^2)|a|^2}{(\alpha + \nu |a|^2)^2 + w^2} \exp(i(\mathbf{a}, \mathbf{x}))$$

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Note that \mathbf{S}_p is not exactly \mathbf{S} from (4.2), but an operator with periodic conditions for the convection–diffusion solver involved in \mathbf{S}_p instead of the Dirichlet boundary conditions involved in \mathbf{S} .

In a similar manner we get

$$\mathbf{Q}(\mathbf{w})^{-1}\exp(i(\mathbf{a},\mathbf{x})) = \left(\nu + \frac{\alpha}{|a|^2} + \frac{w^2}{\alpha |a|^2}\right)\exp(i(\mathbf{a},\mathbf{x}))$$

Hence, by a straightforward superposition we obtain

$$\mathbf{Q}(\mathbf{w})^{-1}\mathbf{S}_p \exp(i(\mathbf{a}, \mathbf{x})) = \left(1 + \frac{w^2 v \alpha^{-1} |a|^2}{(\alpha + v|a|^2)^2 + w^2}\right) \exp(i(\mathbf{a}, \mathbf{x}))$$

Let us denote

$$\rho(|a|^2) = \frac{w^2 v \alpha^{-1} |a|^2}{(\alpha + v|a|^2)^2 + w^2}$$

We readily get

$$|a|_m^2 = \arg\max_{|a|\ge 0} \rho(|a|^2) = \frac{\sqrt{\alpha^2 + w^2}}{\nu}$$

and

$$\rho_{\max} = \max_{|a| \ge 0} \rho(|a|^2) = \frac{w^2}{2\alpha(\sqrt{\alpha^2 + w^2} + \alpha)}$$
(7.1)

Since $\rho(|a|^2) \to 0$ for $|a|^2 \to \infty$, one gets

$$\operatorname{cond}(\mathbf{Q}(\mathbf{w})^{-1}\mathbf{S}_p) \sim 1 + \rho_{\max}$$

Therefore, the smaller the coefficient ρ_{max} is, the closer to identity the operator $\mathbf{Q}(\mathbf{w})^{-1}\mathbf{S}_p$ is.

Remark 7.1.

In this model example we see that the preconditioning gives the estimate independent of viscosity ν . It is no contradiction to the fact that for $\nu = 0$ the preconditioner is 'exact'. The explanation is that we have a non-uniform convergence with respect to mesh size for $\nu \to 0$ of the problem to the limit case of $\nu = 0$. Moreover, the above analysis predicts that the worst case occurs for mesh size $h_w \sim |a|_m^{-1} < \sqrt{\alpha^{-1}\nu}$, hence $h_w \to 0$ for $\nu \to 0$. Recall that $\alpha^{-1} \sim \delta t$ (see (3.1)).

Remark 7.2.

If other parameters, including the mesh size, are fixed, the condition number improves with $\nu \rightarrow 0$.

Remark 7.3.

Convergence improves with $\alpha \to \infty$ (the time step goes to zero).

Remark 7.4. Let us denote $\xi = w/\alpha$ and rewrite (7.1) as

$$\rho_{\max} = \frac{\xi^2}{2(\sqrt{1+\xi^2}+1)}$$

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The latter indicates some deterioration of convergence for $\xi \to \infty$.

The same analysis can be done for the preconditioner provided by the theory for symmetric problem. Indeed, one can check

$$\mathbf{Q}(0)^{-1}\mathbf{S}_p \exp(i(\mathbf{a}, \mathbf{x})) = \left(1 - \frac{w^2}{(\alpha + \nu |a|^2)^2 + w^2}\right) \exp(i(\mathbf{a}, \mathbf{x}))$$

Now, the worst convergence case is observed for low harmonics. By the same arguments we get

$$\operatorname{cond}(\mathbf{Q}(0)^{-1}\mathbf{S}_p) \sim 1 + \frac{w^2}{(\alpha + \nu)^2} = 1 + \frac{\xi^2}{(1 + \nu/\alpha)^2}$$

If we consider the worst cases with respect to v and the mesh size, it follows for $\xi \to \infty$

$$\operatorname{cond}(\mathbf{Q}(\mathbf{w})^{-1}\mathbf{S}_p) \sim 1 + O(\xi)$$

$$\operatorname{cond}(\mathbf{Q}(0)^{-1}\mathbf{S}_p) \sim 1 + O(\xi^2)$$

In this paper these asymptotics are not checked numerically. What we observe numerically is that for increasing ξ the convergence of an iterative method with $\mathbf{Q}^{-1}(\mathbf{w})$ does not deteriorate much, while the convergence of the same method with $\mathbf{Q}^{-1}(0)$ can deteriorate dramatically if ν is sufficiently small (see Section 8).

Finally, although the present Fourier analysis does not *prove* the appropriate convergence results for operator **S**, it is widely recognized to be a good predictor of a 'real' solvers' behavior, at least for symmetric problems (see [9]). For the problem with $\mathbf{w} \neq \text{const}$ numerical results from the next section support the conclusions from Remarks 7.1–7.4. However, the actual convergence rates appear not to be predicted by the above formulas. Probably, some effects induced by $\mathbf{w} \neq \text{const}$ are not recovered by Fourier analysis.

8. Numerical results and convection-diffusion solver

We consider $\Omega = (0, 1) \times (0, 1)$ and a finite difference scheme on staggered grids for velocity and pressure. This scheme is sometimes referred to as MAC and is known to be LBB stable (for details see, e.g., [22]). We set $\mathbf{w} = \nabla \times \mathbf{v}$, where $\mathbf{v} = (v_1, v_2)$,

$$v_1 = \kappa (2y - 1)x(1 - x) v_2 = \kappa (2x - 1)y(1 - y)$$
(8.1)

The convection function \mathbf{v} can be considered as the velocity field of a rotating vortex in a cavity. On a discrete level, the condition $\mathbf{v} = 0$ on $\partial \Omega$ is satisfied, so, near the boundary \mathbf{v} is not smooth. The magnitude of the convection is ruled by the parameter κ .

As an iterative method to solve (4.3) we consider the MINRES algorithm with one search direction on each iteration. As an exact solution $p(\mathbf{x})$ we take a function with a random value from [0,1] in every grid point, finally normalized to satisfy (p, 1) = 0. Hence the solution is substantially non-smooth. The initial guess is $p^0 = 0$.

The convergence criteria is $||\text{res}^n||/||\text{res}^0|| < 10^{-6}$, where res^i is the residual $\mathbf{S}p^i - F$. By the average convergence factor we call $q = (||\text{res}^n||/||\text{res}^0||)^{1/n}$.

In Tables 1 and 2 the convergence results for the slightly non-symmetric problem ($\kappa = 1$)

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	Mesh size					
Viscosity	1/32	1/64	1/128	1/256	1/512	
1	17 (0.43)	18 (0.45)	18 (0.45)	17 (0.44)	17 (0.44)	
1e-1	12 (0.30)	14(0.37)	15(0.39)	16 (0.41)	15(0.39)	
1e-2	7 (0.13)	10 (0.23)	11(0.28)	13 (0.37)	13 (0.36)	
1e-4	3 (3e-3)	3 (9e-3)	4 (0.03)	6 (0.08)	7(0.13)	
1e-6	2 (2e-5)	2 (9e-5)	2 (3e-4)	3 (1e-3)	3 (5e-3)	

Table 1. MINRES method with new non-symmetric preconditioning, $\alpha = 20, \kappa = 1$

Number of iterations and average convergence factor.

	Mesh size					
Viscosity	1/32	1/64	1/128	1/256	1/512	
1	17 (0.43)	18 (0.46)	18 (0.45)	17 (0.44)	17 (0.43)	
1e-1	12 (0.31)	14 (0.37)	15 (0.39)	16 (0.41)	15 (0.40)	
1e-2	8 (0.15)	10 (0.23)	11(0.28)	13 (0.34)	13 (0.34)	
1e-4	7 (0.13)	7(0.13)	6 (0.10)	6 (0.10)	8 (0.16)	
1e-6	7(0.13)	7(0.14)	7(0.13)	7(0.13)	7(0.13)	

Table 2. MINRES method with symmetric preconditioning, $\alpha = 20, \kappa = 1$

Number of iterations and average convergence factor.

are presented. The convergence rates are quite good for both preconditioners: the symmetric one from (5.7) and the new non-symmetric one from (6.9). The difference is seen for $\nu \rightarrow 0$. In this case the convergence rates for the method with the new preconditioning significantly improve.

Remark 8.1.

As we expect from the construction of the preconditioner and Fourier analysis the value of $cond(\mathbf{Q}^{-1}(\mathbf{w})\mathbf{S}(\mathbf{w}))$ is bounded independently of mesh size. However, if ν is small a bound for convergence factors is achieved for very fine mesh (see Section 7), hence it was expected that for sufficiently small ν the results should be mesh-dependent until the mesh is not very fine. This is observed in Tables 1 and 3.

In Tables 3 and 4 we present the convergence results for the problem with stronger convection ($\kappa = 10$). Now the convergence rates are quite good only for the algorithm with the new non-symmetric preconditioner from (6.9). The symmetric preconditioning gives poor results for $\nu \rightarrow 0$. In this case the convergence rates for the method with the new preconditioning significantly improve contrary to the symmetric case.

Note that the convergence rates with the new preconditioner do not differ much in the cases $\kappa = 1$ and $\kappa = 10$. The latter case is slightly worse. This is in agreement with the analysis of Section 7: the value of $\xi = ||\mathbf{w}||/\alpha$ is higher for $\kappa = 10$.

We complete the section with the brief description of the solver used for the 'convection-

			Mesh size		
Viscosity	1/32	1/64	1/128	1/256	1/512
1	17 (0.42)	17 (0.42)	18 (0.43)	18 (0.45)	19 (0.47)
1e-1	18 (0.45)	21 (0.52)	23(0.54)	23(0.55)	23(0.55)
1e-2	12 (0.30)	15(0.39)	17(0.44)	20 (0.50)	21 (0.51)
1e-4	4 (0.01)	5(0.05)	8(0.15)	11(0.26)	12(0.30)
1e-6	2 (1e-4)	2 (1e-4)	3 (1e-3)	3 (3e-3)	4(0.02)

Table 3. MINRES method with new non-symmetric preconditioning, $\alpha = 20, \kappa = 10$

Number of iterations and average convergence factor.

	Mesh size					
Viscosity	1/32	1/64	1/128	1/256	1/512	
1	18 (0.46)	18 (0.46)	18 (0.45)	17 (0.44)	17 (0.44)	
1e-1	29 (0.62)	20 (0.50)	22 (0.53)	21 (0.51)	19 (0.48)	
1e-2	29 (0.62)	22(0.53)	20 (0.50)	18 (0.45)	18 (0.46)	
1e-4	78 (0.84)	79 (0.84)	57 (0.78)	31 (0.63)	21 (0.52)	
1e-6	90 (0.86)	90 (0.85)	90 (0.85)	89 (0.85)	83 (0.84)	

Table 4. MINRES method with symmetric preconditioning, $\alpha = 20, \kappa = 10$

Number of iterations and average convergence factor.

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	Mesh size					
Viscosity	1/32	1/64	1/128	1/256	1/512	
∞	9 (0.10)					
1	9(0.10)	9 (0.10)	9 (0.10)	9 (0.10)	9 (0.10)	
1e-1	9(0.08)	9 (0.09)	9(0.10)	9 (0.10)	9 (0.10)	
1e-2	5(0.02)	8(0.06)	9 (0.09)	9 (0.09)	9 (0.10)	
1e-3	3 (6e-5)	4 (3e-3)	6 (0.03)	8 (0.07)	9 (0.09)	
1e-4	2 (1e-8)	2 (2e-6)	3 (3e-4)	5 (8e-3)	7(0.05)	

Table 5. Multigrid V-cycle for convection–diffusion problem, $\alpha = 20, \kappa = 1$

Number of iterations and average convergence factor.

diffusion' problem: given \mathbf{w} , \mathbf{f} , find \mathbf{u} from

$$\begin{aligned} \alpha \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{w} \times \mathbf{u} &= \mathbf{f} \\ \mathbf{u}|_{\partial \Omega} &= 0 \end{aligned} \tag{8.2}$$

As an iterative method to solve (8.2) we consider a V-cycle multigrid, as a smoother MINRES algorithm with one search direction is used with the following preconditioning. The diagonal lumping for the symmetric part of the convection–diffusion operator is done and a first-order approximation for the term $\mathbf{w} \times \mathbf{u}$ is made. Thus, on every cell we solve the problem (2D case) for the local values of \mathbf{u} and \mathbf{w} :

$$\begin{cases} \beta_1 \, u_1 - w \, u_2 \ = \ g_1 \\ \beta_2 \, u_2 + w \, u_1 \ = \ g_2 \end{cases}$$

The smoother is nearly an exact solver for (8.2) in the case $\nu = 0$ (this is another point where we benefit from the zero order of the convection term). In the literature this property is often required from robust multigrid solvers for singularly-perturbed problems (see, e.g., [18]).

We will study the convergence properties of this method elsewhere. The numerical results (see Tables 5 and 6) show that the algorithm is very efficient and robust with respect to parameter ν . Additionally, it does not require any renumbering strategy and can be very efficiently parallelized, however, one can expect the deterioration of the convergence for highly anisotropic meshes.

The numerical results below are presented for a V-cycle multigrid with two pre-smoothing and two post-smoothing steps. First order prolongation and restriction is used for the velocity function. As an exact solution we take $\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{r}(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ is a smooth function, $\mathbf{r}(x)$ is a function with a random value from [0,1] in every grid point. The boundary condition $\mathbf{u} = 0$ is imposed on the discrete level. The initial guess is $\mathbf{u}^0 = 0$. The function \mathbf{w} is taken as in (8.1). The convergence criterion is $||\mathbf{res}^n||/||\mathbf{res}^0|| < 10^{-9}$. One iteration is one sweep of the V-cycle.

In the row marked by ∞ we present the results for the Poisson problem: $\alpha = 0$, $\mathbf{w} = 0$ in (8.2).

The last remark in the section is that the one sweep of the V-cycle for the convection– diffusion problem can be used as a preconditioner \mathbf{D}^{-1} in the inexact Uzawa algorithm (4.5).

	Mesh size					
Viscosity	1/32	1/64	1/128	1/256	1/512	
1	9 (0.10)					
1e-1	10 (0.11)	10 (0.11)	10 (0.11)	10 (0.10)	9 (0.10)	
1e-2	6 (0.02)	9 (0.09)	9 (0.10)	10 (0.11)	10 (0.11)	
1e-3	3 (8e-4)	4 (3e-3)	7(0.04)	9 (0.10)	10 (0.11)	
1e-4	2 (1e-8)	2(6e-7)	3 (1e-4)	5 (9e-3)	8(0.07)	

Table 6. Multigrid V-cycle for convection–diffusion problem, $\alpha = 20, \kappa = 10$

Number of iterations and average convergence factor

9. Appendix

9.1. Proof of Theorem 5.1

First we prove estimates (5.1). To this end, let us fix some $\nu > 0$, $\alpha > 0$, $\mathbf{w} \in L_2(\Omega)^N$, $p \in L_2^0(\Omega)$ and consider an auxiliary velocity vector \mathbf{u}_1 from $\mathbf{H}_0^1(\Omega)$, which solves

$$\alpha(\mathbf{u}_1, \mathbf{v}) + \nu(\nabla \mathbf{u}_1, \nabla \mathbf{v}) + (\mathbf{w} \times \mathbf{u}_1, \mathbf{v}) = -(p, \operatorname{div} \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$
(9.1)

By the definition of **S** one has $(\mathbf{S}p, p) = -(\operatorname{div} \mathbf{u}_1, p)$, and hence, choosing $\mathbf{v} = \mathbf{u}_1$ in (9.1), one gets

$$(\mathbf{S}p, p) = \alpha ||\mathbf{u}_1||_0^2 + \nu ||\mathbf{u}_1||_1^2$$
(9.2)

Further, we use the following estimates (see, e.g., [23]):

$$c_0||p||_0^2 \le (\operatorname{div} \Delta_0^{-1} \nabla p, p) \le ||p||_0^2 \quad \forall \, p \in L_2^0(\Omega)$$
(9.3)

The first inequality in (9.3) can be observed as the continuous analogue of the LBB condition (sometimes referred to as the Nečas inequality), since one has

$$(\operatorname{div} \Delta_0^{-1} \nabla p, p) = \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(p, \operatorname{div} \mathbf{v})^2}{||\mathbf{v}||_1^2} \quad \forall \, p \in L_2^0(\Omega)$$

Consider one more velocity vector \mathbf{u}_2 from $\mathbf{H}_0^1(\Omega)$, which solves

$$(\nabla \mathbf{u}_2, \nabla \mathbf{v}) = -(p, \operatorname{div} \mathbf{v}) \quad \forall \, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \tag{9.4}$$

Similarly, one obtains

$$- (\operatorname{div} \Delta_0^{-1} \nabla p, p) = ||\mathbf{u}_2||_1^2$$
(9.5)

Take now in (9.1) and (9.4) $\mathbf{v} = \mathbf{u}_1$, subtract (9.4) from (9.1) and use ε -inequality with $\varepsilon = v$ to estimate ($\nabla \mathbf{u}_2, \nabla \mathbf{u}_1$). These result in

$$2\alpha ||\mathbf{u}_1||_0^2 + \nu ||\mathbf{u}_1||_1^2 \le \nu^{-1} ||\mathbf{u}_2||_1^2$$

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hence the estimate,

$$(\mathbf{S}p, p) \le v^{-1} ||p||_0^2$$

immediately follows thanks to (9.2), (9.3), and (9.5).

To prove the lower bound for S we will use the following estimates (they are proved in a straightforward way by applying the Hölder inequality with proper subscripts) for the convection part:

$$|(\mathbf{w} \times \mathbf{u}, \mathbf{v})| \leq c ||\mathbf{w}||_0 ||\mathbf{u}||_{L_4} ||\mathbf{v}||_{L_4} |(\mathbf{w} \times \mathbf{u}, \mathbf{v})| \leq c ||\mathbf{w}||_0 ||\mathbf{u}||_{L_3} ||\mathbf{v}||_{L_6}$$

together with (e.g., [25],)

$$||\mathbf{u}||_{L_{4}} \leq c||\mathbf{u}||_{0}^{\frac{1}{4}}||\mathbf{u}||_{1}^{\frac{3}{4}}$$
$$||\mathbf{u}||_{L_{3}} \leq c||\mathbf{u}||_{0}^{\frac{1}{2}}||\mathbf{u}||_{1}^{\frac{1}{2}}$$
$$||\mathbf{u}||_{L_{6}} \leq c||\mathbf{u}||_{1}$$

for arbitrary $\mathbf{w} \in L_2(\Omega)^N$, $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$.

Now, we choose $\mathbf{v} = \mathbf{u}_2$ in (9.1) and (9.4), further subtracting (9.4) from (9.1) we get the following chain of inequalities.

$$\begin{aligned} ||\mathbf{u}_{2}||_{1}^{2} &= \alpha(\mathbf{u}_{1},\mathbf{u}_{2}) + \nu(\nabla\mathbf{u}_{1},\nabla\mathbf{u}_{2}) + (\mathbf{w}\times\mathbf{u}_{1},\mathbf{u}_{2}) \\ &\leq \rho\alpha^{2}||\mathbf{u}_{1}||_{0}^{2} + \frac{1}{4}||\mathbf{u}_{2}||_{1}^{2} + \nu^{2}||\mathbf{u}_{1}||_{1}^{2} + \frac{1}{4}||\mathbf{u}_{2}||_{1}^{2} + c||\mathbf{w}||_{0}||\mathbf{u}_{1}||_{L_{3}}||\mathbf{u}_{2}||_{L_{6}} \\ &\leq \frac{1}{2}||\mathbf{u}_{2}||_{1}^{2} + (\rho\alpha + \nu)(\alpha||\mathbf{u}_{1}||_{0}^{2} + \nu||\mathbf{u}_{1}||_{1}^{2}) + c||\mathbf{w}||_{0}^{2}||\mathbf{u}_{1}||_{0}||\mathbf{u}_{1}||_{1} + \frac{1}{4}||\mathbf{u}_{2}||_{1}^{2} \\ &\leq \frac{3}{4}||\mathbf{u}_{2}||_{1}^{2} + (\rho\alpha + \nu)(\alpha||\mathbf{u}_{1}||_{0}^{2} + \nu||\mathbf{u}_{1}||_{1}^{2}) + c\frac{||\mathbf{w}||_{0}^{2}}{\sqrt{\nu\alpha}}(\alpha||\mathbf{u}_{1}||_{0}^{2} + \nu||\mathbf{u}_{1}||_{1}^{2}) \end{aligned}$$

Hence

$$||\mathbf{u}_{2}||_{1}^{2} \leq 4(\rho\alpha + \nu + c\frac{||\mathbf{w}||_{0}^{2}}{\sqrt{\nu\alpha}})(\alpha||\mathbf{u}_{1}||_{0}^{2} + \nu||\mathbf{u}_{1}||_{1}^{2})$$

The last inequality together with (9.2), (9.3) and (9.5) proves estimate (5.1) of the theorem with $K(\nu, \alpha, \mathbf{w}) = c \frac{\|\mathbf{w}\|_0^2}{\sqrt{\nu\alpha}}$ Equality (9.1) with $\mathbf{v} = \mathbf{u}_1$ provides

$$\nu(\operatorname{div} \mathbf{u}_1, \operatorname{div} \mathbf{u}_1) \leq -(\operatorname{div} \mathbf{u}_1, p)$$

which is nothing but

$$\nu(\mathbf{S}p,\mathbf{S}p) \le (\mathbf{S}p,p)$$

Taking $q = \mathbf{S}p$ one obtains inequality (5.3) with $\gamma_4 = \nu$.

To prove estimate (5.2) we fix some arbitrary $q \in L_2^0(\Omega)$ and consider \mathbf{u}_2 as a solution to

$$\alpha(\mathbf{u}_2, \mathbf{v}) + \nu(\nabla \mathbf{u}_2, \nabla \mathbf{v}) + (\mathbf{w} \times \mathbf{u}_2, \mathbf{v}) = -(q, \operatorname{div} \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$
(9.6)

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Similarly to (9.2) we have

$$(\mathbf{S}q, q) = \alpha ||\mathbf{u}_2||_0^2 + \nu ||\mathbf{u}_2||_1^2$$

By the definition of **S** it holds that $(\mathbf{S}p, q) = -(\operatorname{div} \mathbf{u}_1, q)$ with \mathbf{u}_1 from (9.1), and hence, taking in (9.6) $\mathbf{v} = \mathbf{u}_1$ one obtains

$$\begin{aligned} (\mathbf{S}p,q) &= \alpha(\mathbf{u}_{2},\mathbf{u}_{1}) + \nu(\nabla\mathbf{u}_{2},\nabla\mathbf{u}_{1}) + (\mathbf{w}\times\mathbf{u}_{2},\mathbf{u}_{1}) \\ &\leq (\alpha||\mathbf{u}_{1}||_{0}^{2} + \nu||\mathbf{u}_{1}||_{1}^{2})^{\frac{1}{2}}(\alpha||\mathbf{u}_{2}||_{0}^{2} + \nu||\mathbf{u}_{2}||_{1}^{2})^{\frac{1}{2}} + c||\mathbf{w}||_{0}||\mathbf{u}_{1}||_{L_{4}}||\mathbf{u}_{2}||_{L_{4}} \\ &\leq (\mathbf{S}p,p)^{\frac{1}{2}}(\mathbf{S}q,q)^{\frac{1}{2}} + c||\mathbf{w}||_{0}||\mathbf{u}_{1}||_{0}^{\frac{1}{4}}||\mathbf{u}_{2}||_{0}^{\frac{1}{4}}||\mathbf{u}_{1}||_{1}^{\frac{3}{4}}||\mathbf{u}_{2}||_{1}^{\frac{3}{4}} \\ &\leq (\mathbf{S}p,p)^{\frac{1}{2}}(\mathbf{S}q,q)^{\frac{1}{2}} + 6c||\mathbf{w}||_{0}\left(\frac{1}{2\varepsilon}||\mathbf{u}_{1}||_{0}^{\frac{1}{2}}||\mathbf{u}_{1}||_{1}^{\frac{1}{2}}||\mathbf{u}_{2}||_{0}^{\frac{1}{4}}||\mathbf{u}_{2}||_{1}^{\frac{1}{2}} \\ &\quad + \frac{\varepsilon}{2}||\mathbf{u}_{1}||_{1}||\mathbf{u}_{2}||_{1}\right) \\ &\leq (\mathbf{S}p,p)^{\frac{1}{2}}(\mathbf{S}q,q)^{\frac{1}{2}} + c||\mathbf{w}||_{0}\left(\frac{1}{4\varepsilon\delta}||\mathbf{u}_{1}||_{0}||\mathbf{u}_{2}||_{0} + \left(\frac{\delta}{4\varepsilon} + \frac{\varepsilon}{2}\right)||\mathbf{u}_{1}||_{1}||\mathbf{u}_{2}||_{1}\right) \end{aligned}$$

Since ε and δ are arbitrary positive, we choose in the last inequality $\delta = \frac{1}{2} \left(\frac{\nu}{\alpha}\right)^{\frac{1}{2}}$ and $\varepsilon = \frac{1}{2} \left(\frac{\nu}{\alpha}\right)^{\frac{1}{4}}$. Thus, it follows that

$$\begin{aligned} (\mathbf{S}p,q) &\leq (\mathbf{S}p,p)^{\frac{1}{2}} (\mathbf{S}q,q)^{\frac{1}{2}} + c \frac{||\mathbf{w}||_{0}}{\sqrt{\nu} (\alpha \nu)^{\frac{1}{4}}} (\alpha ||\mathbf{u}_{1}||_{0} ||\mathbf{u}_{2}||_{0} + \nu ||\mathbf{u}_{1}||_{1} ||\mathbf{u}_{2}||_{1}) \\ &\leq (\mathbf{S}p,p)^{\frac{1}{2}} (\mathbf{S}q,q)^{\frac{1}{2}} + c \frac{||\mathbf{w}||_{0}}{\sqrt{\nu} (\alpha \nu)^{\frac{1}{4}}} (\alpha ||\mathbf{u}_{1}||_{0}^{2} + \nu ||\mathbf{u}_{1}||_{1}^{2})^{\frac{1}{2}} (\alpha ||\mathbf{u}_{2}||_{0}^{2} + \nu ||\mathbf{u}_{2}||_{1}^{2})^{\frac{1}{2}} \\ &= (\mathbf{S}p,p)^{\frac{1}{2}} (\mathbf{S}q,q)^{\frac{1}{2}} + c \frac{||\mathbf{w}||_{0}}{\sqrt{\nu} (\alpha \nu)^{\frac{1}{4}}} (\mathbf{S}p,p)^{\frac{1}{2}} (\mathbf{S}q,q)^{\frac{1}{2}} \end{aligned}$$

Estimate (5.2) is proved with $C(\nu, \alpha \mathbf{w}) = c \frac{||\mathbf{w}||_0}{\sqrt{\nu} (\alpha \nu)^{\frac{1}{4}}}$

Estimate (5.4) is proved in a similar manner to (5.1). Instead of (9.4) we choose now \mathbf{u}_2 as a solution to

$$\alpha(\mathbf{u}_2, \mathbf{v}) + \nu(\nabla \mathbf{u}_2, \nabla \mathbf{v}) = -(p, \operatorname{div} \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

Term $(\mathbf{w} \times \mathbf{u}_1, \mathbf{u}_2)$ is estimated exactly as in the proof of (5.2).

The last estimate, (5.5), is proved as follows. Given $p \in L_2^0(\Omega)$ consider p_1 and p_2 that solve together with \mathbf{u}_1 and \mathbf{u}_2 the equations: for any $\{\mathbf{v}, q\} \in \mathbf{H}_0^1(\Omega) \times L_2^0(\Omega)$

$$\alpha(\mathbf{u}_1, \mathbf{v}) + \nu(\nabla \mathbf{u}_1, \nabla \mathbf{v}) - (p_1, \operatorname{div} \mathbf{v}) = 0,$$

(div \mathbf{u}_1, q) = (p, q) (9.7)

and for any $\{\mathbf{v}, q\} \in \mathbf{H}_0^1(\Omega) \times L_2^0(\Omega)$

$$\alpha(\mathbf{u}_2, \mathbf{v}) + \nu(\nabla \mathbf{u}_2, \nabla \mathbf{v}) + (\mathbf{w} \times \mathbf{u}_2, \mathbf{v}) - (p_2, \operatorname{div} \mathbf{v}) = 0,$$

(div $\mathbf{u}_2, q) = (p, q)$ (9.8)

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Inequality (5.5) now takes the form

$$\gamma_5(p_1, p) \le (p_2, p)$$
 (9.9)

Additionally, the following equalities are valid:

$$(p_1, p) = (p_1, \operatorname{div} \mathbf{u}_2) = (p_1, \operatorname{div} \mathbf{u}_1) = \alpha ||\mathbf{u}_1||_0^2 + \nu ||\mathbf{u}_1||_1^2, (p_2, p) = (p_2, \operatorname{div} \mathbf{u}_1) = (p_2, \operatorname{div} \mathbf{u}_2) = \alpha ||\mathbf{u}_2||_0^2 + \nu ||\mathbf{u}_2||_1^2$$
(9.10)

Let us take in (9.7) $\mathbf{v} = \mathbf{u}_2$ and in (9.8) $\mathbf{v} = \mathbf{u}_1$, subtracting one equality from the other one gets

$$(p_1, \operatorname{div} \mathbf{u}_2) = (p_2, \operatorname{div} \mathbf{u}_1) - (\mathbf{w} \times \mathbf{u}_2, \mathbf{u}_1)$$
(9.11)

In the same way as before one obtains

$$|(\mathbf{w} \times \mathbf{u}_{2}, \mathbf{u}_{1})| \leq \frac{1}{2} (\alpha ||\mathbf{u}_{1}||_{0}^{2} + \nu ||\mathbf{u}_{1}||_{1}^{2}) + c \frac{||\mathbf{w}||_{0}^{2}}{\nu (\alpha \nu)^{\frac{1}{2}}} (\alpha ||\mathbf{u}_{2}||_{0}^{2} + \nu ||\mathbf{u}_{2}||_{1}^{2})$$

The combination of the last estimate with (9.10) and (9.11) gives (9.9), hence inequality (5.5) is proved.

If the function \mathbf{w} is more smooth one can get use of the sharper estimates for convection terms:

$$|(\mathbf{w} \times \mathbf{u}, \mathbf{v})| \leq c||\mathbf{w}||_{L_3}||\mathbf{u}||_0||\mathbf{v}||_{L_6}$$
$$|(\mathbf{w} \times \mathbf{u}, \mathbf{v})| \leq c||\mathbf{w}||_{L_{\infty}}||\mathbf{u}||_0||\mathbf{v}||_0$$

The same considerations give the sharper constants γ_1 , γ_3 , and γ_5 . Theorem 5.1 is proved.

Remark 9.1.

For $\alpha = 0$ the similar estimates can be proved. The only alteration is the use of $\mathbf{H}_0^1(\Omega)$ norm $||\cdot||_1$ instead of $||\cdot||_0$ in the estimates of the convection term. This can be always done thanks to the Poincare–Fridrichs inequality.

9.2. Proof of Theorem 5.2

The error $e_k = p - p_k$ of the method (4.4) satisfies the relation $e_{k+1} = (\mathbf{I} - \tau \mathbf{Q}^{-1}\mathbf{S})e_k$ for all k = 0, 1, ... Denote by $(p, q)_{\mathbf{Q}}$ the scalar product $(\mathbf{Q} p, q)$ with a positive and self-adjoint \mathbf{Q} . One gets

$$||e_{k+1}||_{\mathbf{Q}}^{2} = ||(\mathbf{I} - \tau \mathbf{Q}^{-1}\mathbf{S})e_{k}||_{\mathbf{Q}}^{2} = ||e_{k}||_{\mathbf{Q}}^{2} - 2\tau (\mathbf{Q}^{-1}\mathbf{S}e_{k}, e_{k})\mathbf{Q} + \tau^{2}||\mathbf{S}e_{k}||_{\mathbf{Q}}^{2}$$

Let us consider the case of Q^{-1} from (5.7). Inequality (5.5) of Theorem 5.1 is equal to

$$\gamma_5 ||\mathbf{S}p||_{\mathbf{S}_0}^2 \le (\mathbf{S}_0^{-1}\mathbf{S}p, p)_{\mathbf{S}_0}$$

Since $\mathbf{Q} \sim \mathbf{S}_0$ the last estimate implies

$$c\gamma_5||\mathbf{S}p||_{\mathbf{Q}}^2 \le (\mathbf{Q}^{-1}\mathbf{S}p, p)_{\mathbf{Q}}$$

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Therefore, one obtains

$$||e_{k+1}||_{\mathbf{Q}}^2 \le ||e_k||_{\mathbf{Q}}^2 - (2\tau - c^{-1}\tau^2\gamma_5^{-1})(\mathbf{Q}^{-1}\mathbf{S}e_k, e_k)_{\mathbf{Q}}$$

from which, assuming $\tau \leq 2c\gamma_5$ and due to (5.4), one gets

$$||e_{k+1}||_{\mathbf{Q}}^2 \le ||e_k||_{\mathbf{Q}}^2 - (2\tau - c^{-1}\tau^2\gamma_5^{-1})\gamma_5||e_k||_{\mathbf{Q}}^2$$

In the last inequality let us take $\tau = c\gamma_5$, and finally obtain the desired estimate

$$||e_{k+1}||_{\mathbf{Q}} \le \sqrt{1 - c\gamma_5^2}||e_k||_{\mathbf{Q}}$$

The case of \mathbf{Q}^{-1} from (5.6) is simpler and proved in the same manner. The theorem is proved.

9.3. Proof of Lemma 6.1

Assume that p from $H^1(\Omega) \cap L_2^0(\Omega)$ is some solution to the problem (6.4) and consider $\mathbf{u} = -\alpha^{-1} \mathcal{G}(\mathbf{x}) \nabla p$. As is shown in Section 5 $\mathbf{u} \in L_2(\Omega)^N$, moreover div $\mathbf{u} = F \in L_2(\Omega)$ and $\mathbf{u} \cdot \mathbf{n} = \frac{\partial p}{\partial \tilde{\mathbf{n}}} = 0$. Thus the vector function \mathbf{u} belongs to $\mathbf{H}_0(\text{div})$. Due to the definition of \mathbf{u} and (6.6) one readily gets

$$\alpha(\mathcal{G}^{-1}(\mathbf{x})\mathbf{u}, \mathbf{v}) + (\nabla p, \mathbf{v}) = \mathbf{0} \qquad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{div}) -(\mathbf{u}, \nabla q) = (F, q), \quad \forall q \in H^1(\Omega) \cap L^0_2(\Omega)$$

Integrating the above equalities by parts, and due to the boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$ and the explicit form of $\mathcal{G}^{-1}(\mathbf{x})$ we obtain

$$\alpha(\mathbf{u}, \mathbf{v}) + (\mathbf{w} \times \mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = \mathbf{0} \qquad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{div})$$

(div \mathbf{u}, q) = (F, q), $\forall q \in H^1(\Omega) \cap L_2^0(\Omega)$ (9.12)

The second equality in (9.12) is still true for arbitrary $q \in L_2^0(\Omega)$. Hence, one gets that $\{p, \mathbf{u}\}$ is exactly the weak solution to the saddle-point problem (6.2). Both problems are well posed under the assumption of $\mathbf{w} \in L_{\infty}(\Omega)^{2N-3}$. The lemma is proved.

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