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Stable finite-element calculation of incompressible flows using the rotation form of convection

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Conforming finite-element approximations are considered for the incompressible Navier–Stokes equations with nonlinear terms written in the convection or rotation forms. Implicit time integration results in nice stability properties of auxiliary problems which can be solved by efficient numerical algorithms. The original nonlinear system admits relatively simple stabilization strategies. The paper presents in a unified form the convergence analysis, including the design of stabilization parameters, for linearized equations in both convection and rotation forms. Moreover, it is shown that a Galerkin discretization of the pressure-regularized Oseen problem with skew-symmetric terms in rotation form possesses better stability properties and, being much easier to solve, can be used as a predictor in implicit calculations.

Keywords: incompressible flow; Navier–Stokes equations; convection–diffusion; rotation form; stabilized finite-elements; Galerkin/least-squares; SUPG.

1. Introduction

The incompressible Navier–Stokes problem written in velocity–pressure variables has several equivalent forms. However, upon discretization these forms lead to finite-dimensional systems with different algebraic properties. So the discrete solutions may differ in their quality, and linear (or nonlinear) solvers may differ in their performance, depending on what particular formulation has been used.

The common choice for finite-element (FE) users is the *convection* form of the Navier–Stokes problem: find a velocity $\mathbf{u}(t, \mathbf{x})$ and a kinematic pressure $p(t, \mathbf{x})$ from

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega \times (0, T], \\ \operatorname{div} \mathbf{u} = 0 \quad & \text{in } \Omega \times [0, T] \end{aligned} \tag{1.1}$$

with given force field $\mathbf{f}$ and viscosity $\nu > 0$. Some boundary and initial conditions should be additionally supplied. FE methods for (1.1) are well studied in the literature from the

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algorithmical and mathematical points of view (see, e.g. Pironneau, 1989; Roos *et al.*, 1996; Turek, 1999).

Here we are also concerned with the *rotation* form of these equations:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{u} + \nabla P = \mathbf{f} \quad & \text{in } \Omega \times (0, T], \\ \operatorname{div} \mathbf{u} = 0 \quad & \text{in } \Omega \times [0, T], \end{aligned} \tag{1.2}$$

which results from (1.1) after replacing the kinematic pressure by the Bernoulli (or total) pressure $P = p + \frac{\mathbf{u}^2}{2}$.

There are several motivations for this research. One comes from numerical linear algebra: the recent papers of Olshanskii (1999) and Olshanskii & Reusken (2000) show that the auxiliary problems of Oseen and convection–diffusion type, which involve the proper (i.e. ellipticity preserving) linearization of nonlinear terms in the rotation form, can be solved in a fast and robust manner due to the zero order of skew-symmetric terms.

Another one is the numerical analysis for flow problems in the presence of a Coriolis force. In this case the extra term has the form $\mathbf{w} \times \mathbf{u} (\nabla \mathbf{w} = \mathbf{0})$, which is the same as for the linearized convection from (1.2), although now $\nabla \mathbf{w} \neq \mathbf{0}$. As shown in Codina & Soto (1997), the diffusion problem with such a term does not require any global stabilization for small diffusion, in contrast to the case when the $(\mathbf{a} \cdot \nabla)\mathbf{u}$ term is added. Although, in the presence of pressure, stabilization may be needed again, the numerical results from Codina (2001) encourage us to be optimistic about the flexibility in the choice of stabilization parameters and the range of Reynolds numbers when stabilization is used. The analysis of the present paper attempts to support this conclusion.

Finally, the generation of skew-symmetric terms in (1.2) requires approximately 50% (resp. 30%) less CPU time compared to (1.1) for 2D (resp. 3D) calculations. In Zang (1991) it was argued that using *spectral* methods and explicit treatment of convection in unsteady calculations, the form (1.1) is preferable due to being less prone to generation of instabilities caused by truncation errors. However, we believe that implicit and/or proper stabilized FE schemes have a rather distinct error propagation mechanism.

Summarizing all these reasons, we feel that the topic of Navier–Stokes calculations in rotation form for the convection is worth revisiting. The goal of this paper is to present, in a unified form, the convergence analysis for linearized equations originating from (1.1) and (1.2) as a basis for a fair comparison and for further numerical investigations.

The remainder of the paper is organized as follows. In Section 2 we discuss certain schemes for (1.1) and (1.2). Section 3 presents stabilized schemes for the linearized equations. The analysis for div-stable FE pairs is then given in Section 4. Section 5 treats the stabilized FE method for the linearized equations and rather general FE pairs (including equal-order elements). Finally, in Section 6 we study a pressure-regularized Galerkin scheme, using the rotation form of convection.

2. Preliminaries

We assume Ω to be a bounded domain in R^n, $n = 2, 3$. Denote $\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^n v_i u_i$ for vector functions $\mathbf{u}$ and $\mathbf{v}$, $\operatorname{curl} \mathbf{u} := (\nabla \times \mathbf{u})$, $\times$ stands for vector product. In 2D define $\operatorname{curl} \mathbf{u} := -\partial u_1/\partial x_2 + \partial u_2/\partial x_1$ and $a \times \mathbf{u} := \{-a\, u_2, a\, u_1\}$ for a scalar a and vector $\mathbf{u}$.

With these notations observe the formal equality for arbitrary vector functions $\mathbf{u}$ and $\mathbf{v}$, which will be useful in what follows:

$$(\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} = (\operatorname{curl} \mathbf{v}) \times \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{v} + \nabla(\mathbf{v} \cdot \mathbf{u}). \tag{2.1}$$

For $\mathbf{v} = \mathbf{u}$ relation (2.1) proves the equivalence of (1.1) and (1.2).

Compared to explicit methods, an implicit time integration of (1.1) or (1.2) enhances stability, accuracy and flexibility in the choice of the size of time steps. However, implicit schemes are superior to explicit ones only if the auxiliary problems on each time level can be solved efficiently (Schäfer & Turek, 1996).

Thus a scheme for (1.1) or (1.2) can include the fully implicit time integration and an iterative solver to find $\{\mathbf{u}^{k+1}, p^{k+1}\}$ from

$$\begin{aligned} \frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\delta t} - \nu \Delta \mathbf{u}^{k+1} + N(\mathbf{u}^{k+1}, \mathbf{u}^{k+1}) + \nabla p^{k+1} &= \mathbf{f}^{k+1}, \\ \operatorname{div} \mathbf{u}^{k+1} &= 0, \end{aligned} \tag{2.2}$$

with a desired tolerance. Here we set $\mathbf{u}^k = \mathbf{u}(k\,\delta t)$, $p^k = p(k\,\delta t)$, and δt is a time step. $N(\mathbf{u}^{k+1}, \mathbf{u}^{k+1})$ corresponds to the nonlinear term in any of the two forms. Boundary conditions and spatial discretization should be implemented also.

Another option is to consider a linearization at each time step. Then linear problems of Oseen type for the unknown $\{\mathbf{u}^{n+1}, p^{n+1}\}$ have to be solved on each time level, e.g. for (1.2):

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \nu \Delta \mathbf{u}^{n+1} + (\operatorname{curl} \tilde{\mathbf{u}}^n) \times \mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f}^{n+1} \\ \operatorname{div} \mathbf{u}^{n+1} &= 0, \end{aligned} \tag{2.3}$$

where $\tilde{\mathbf{u}}^n$ is some extrapolation of the velocity values from previous time levels. We remark that the above linearization, $(\operatorname{curl} \mathbf{u}) \times \mathbf{u} \simeq (\operatorname{curl} \mathbf{a}) \times \mathbf{u}$, results in a skew-symmetric term, while another choice, e.g. $(\operatorname{curl} \mathbf{u}) \times \mathbf{u} \simeq (\operatorname{curl} \mathbf{u}) \times \mathbf{a}$, may break the ellipticity of the system. For the same reason $(\mathbf{a} \cdot \nabla)\mathbf{u}$ is commonly used to linearize the convection from (1.1), but not $(\mathbf{u} \cdot \nabla)\mathbf{a}$.

Now let us consider an iterative scheme to solve (2.2). For the favored choice of some Newton-like iteration, we need an approximation to the Fréchet derivative of the nonlinear operator from (2.2). We observe that the contribution of the nonlinear terms to the Fréchet derivative in some point $\mathbf{a}$ is

$$(\mathbf{a} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{a}. \tag{2.4}$$

Therefore, one approach (see, e.g. Turek, 1999) is to drop in (2.4) the undesirable second term. Another approach (see Olshanskii, 1999) is to rewrite (2.4), thanks to (2.1), as

$$(\operatorname{curl} \mathbf{a}) \times \mathbf{u} + (\operatorname{curl} \mathbf{u}) \times \mathbf{a} + \nabla(\mathbf{a} \cdot \mathbf{u}). \tag{2.5}$$

Now the term $(\operatorname{curl} \mathbf{u}) \times \mathbf{a}$ is undesirable and will be dropped. The last term is absorbed into the new pressure term. Hence, we arrive at the problem of the same type as in (2.3):

for given $\alpha > 0$, $\nu > 0$, $\mathbf{f}$, g, and $\mathbf{w}$, find $\mathbf{u}$ and a new pressure p from

$$\begin{aligned} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= g \quad \text{in } \Omega, \end{aligned} \tag{2.6}$$

together with appropriate boundary conditions on $\partial\Omega$. As a candidate for a robust predictor in (2.2), we analyse in Section 6 the following pressure-regularized problem

$$\begin{aligned} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} - \delta\Delta p &= g \quad \text{in } \Omega, \\ \mathbf{u} = 0, \quad \frac{\partial p}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

with an appropriate parameter δ. Homogeneous Dirichlet boundary conditions for $\mathbf{u}$ and Neumann condition for p, respectively, are imposed to make the analysis clearer.

A notable implementational difference between schemes (2.2) and (2.3) appears when we apply spatial (e.g. FE) discretization. In the case of large-mesh Reynolds numbers, the scheme may need to be stabilized to produce accurate results. For the first scheme one can choose a consistent method of high accuracy (e.g. the streamline–diffusion method with well-tuned parameters, see Sections 4 and 5) for the nonlinear equations, and not care about algebraic properties of the stabilized system required for linear solvers (M-matrices, saddle-point form, appropriate preconditioners). Here problem (2.7) appears as an auxiliary one. It should be in some sense ‘close’ to the Fréchet derivative and upon discretization it should be much easier to solve. This idea is exploited in Turek (1999) for both linear and nonlinear problems in convection form with a streamline–diffusion method for the nonlinear problem, and first-order upwinding for the auxiliary linear problems. For the second scheme (2.3) the questions of discrete solution accuracy on the one hand, and the algebraic properties on the other hand should be considered together. For both schemes the analysis of linearized problems appears to be a crucial point.

3. Stable FEM for the linearized model

We start with a variational formulation of the following *generalized Oseen-type* problem with $0 < \nu \leqslant 1$, $\alpha \geqslant 0$ where we (for simplicity only) impose homogeneous Dirichlet boundary conditions

$$\begin{aligned} \mathcal{L}(U) := -\nu\Delta\mathbf{u} + \alpha\mathbf{u} + (\mathbf{a}\cdot\nabla)\mathbf{u} + \mathbf{w}\times\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= g \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here the case $\mathbf{a} \equiv \mathbf{0}$ corresponds to the rotation form of the Oseen problem whereas the case $\mathbf{w} \equiv \mathbf{0}$ is the standard convection form. The case with non-vanishing terms $\mathbf{a}$ and $\mathbf{w}$ stems from the Oseen problem with additional Coriolis forces.

Set $\mathbf{V} := H_0^1(\Omega)^n$ with the norm $\|\cdot\|_V := |\cdot|_{1,\Omega}$ and $\mathrm{Q} := L_0^2(\Omega)$ with norm $\|\cdot\|_{\mathrm{Q}} := \|\cdot\|_{0,\Omega}$. The inner product in $L^2(G)$ with $G \subseteq \Omega$ is denoted by $(\cdot,\cdot)_G$. For $G = \Omega$ we will drop the subscript and simply write $(\cdot,\cdot)$ or $\|\cdot\|$ for the L^2 norm. Then a variational formulation of (3.1) reads: *given* $\mathbf{f} \in H^{-1}(\Omega)$, $g \in \mathrm{Q}$, *find* $U := \{\mathbf{u}, p\} \in \mathbf{W} := \mathbf{V} \times \mathrm{Q}$ *such that*

$$
\begin{aligned}
a(U, V) &= f(V) \qquad \forall\, V := \{\mathbf{v}, q\} \in \mathbf{W}, \qquad (3.2)\\
a(U, V) &:= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\alpha \mathbf{u} + \mathbf{w} \times \mathbf{u} + (\mathbf{a} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u})\\
f(V) &:= (\mathbf{f}, \mathbf{v}) + (g, q).
\end{aligned}
$$

Let $\mathcal{T}_h := \{K\}$ be a regular family of simplicial triangulations of $\overline{\Omega}$. Suppose that $\mathcal{T}_h$ is shape-regular such that $h_K/\rho_K \leqslant c$ for all elements K with constant $c \neq c(h)$. Here h_K and ρ_K denote the diameter of the minimal ball circumscribed on K, respectively the maximal ball inscribed in K. Suppose (for simplicity) an exact triangulation with $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.

Let $\mathbf{V}_h \subset \mathbf{V}$ and $\mathrm{Q}_h \subset \mathrm{Q}$ be conforming FE spaces to approximate velocity and pressure, consisting of piecewise polynomials of degree $l \in \mathbf{N}$ and $k \in \mathbf{N}_0$. Later on we apply standard local inverse inequalities (Ciarlet, 1991):

$$
\|\Delta \mathbf{v}_h\|_K \leqslant \mu_u h_K^{-1} \|\nabla \mathbf{v}_h\|_K, \qquad \|\nabla q_h\|_K \leqslant \mu_p h_K^{-1} \|q_h\|_K. \tag{3.3}
$$

For any smooth functions $\mathbf{v} \in \mathbf{V}$ and $q \in \mathrm{Q}$ we assume the existence of interpolants $\hat{\mathbf{v}}_h \in \mathbf{V}_h$ and $\hat{q}_h \in \mathrm{Q}_h$ with the following local approximation properties for $i = 0, 1, 2$, $j = 0, 1$:

$$
\|\mathbf{v} - \hat{\mathbf{v}}_h\|_{H^i(K)} \leqslant c h_K^{l-i+1} \|\mathbf{v}\|_{H^{l+1}(K)}, \quad \|q - \hat{q}_h\|_{H^j(K)} \leqslant c h_K^{k-j+1} \|q\|_{H^{k+1}(K)} \tag{3.4}
$$

on each $K \in \mathcal{T}_h$, see Clement (1975). We only mention a Clement-type interpolation under reduced regularity assumptions, see also Verfürth (1999).

The basic Galerkin method for (3.2) is: *find* $U_h = \{\mathbf{u}_h, p_h\} \in \mathbf{W}_h = \mathbf{V}_h \times \mathrm{Q}_h$ *such that*

$$
a(U_h, V_h) = f(V_h) \qquad \forall\, V_h = \{\mathbf{v}_h, q_h\} \in \mathbf{W}_h. \tag{3.5}
$$

As a first class, we consider velocity/pressure approximations in $\mathbf{V}_h \times \mathrm{Q}_h$, which are *div-stable*, i.e. the following discrete Babuška–Brezzi condition is valid:

$$
\exists \beta_0 > 0,\ \beta_0 \neq \beta_0(h) \text{ s.t. } \inf_{q_h \in \mathrm{Q}_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{V}} \|q_h\|_{\mathrm{Q}}} \geqslant \beta_0. \tag{3.6}
$$

Henceforth we assume that $\sup_x$ and $\inf_x$ are taken for $x \neq 0$ if $\|x\|$ appears in the denominator. Observe also the continuous counterpart of (3.6), the Nečas inequality

$$
\bar{\beta}_0 \|p\| \leqslant \|\nabla p\|_{H^{-1}(\Omega)} \quad \forall\, p \in \mathrm{Q}.
$$

Denote $c_0 = \min\{\beta_0, \bar{\beta}_0\}$.

In the subsequent analysis we assume that

$$
\mathbf{a} \in L^\infty(\Omega)^n \cap H_0^1(\Omega)^n, \quad \operatorname{div} \mathbf{a} = 0; \qquad \mathbf{w} \in L^\infty(\Omega)^{2n-3}.
$$

In the context of linearization of the Navier–Stokes equations the smoothness assumptions on $\mathbf{a}$ are reasonable if $\mathbf{a}$ represents a FE velocity. The condition $\operatorname{div} \mathbf{a} = 0$ will be needed only to ensure that the bilinear form $(\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v})$ is skew-symmetric. Although this condition is usually valid in some weak sense only and the skew-symmetry of the bilinear form can be lost, there are several standard ways to ensure skew-symmetry in practice. One is to include $\frac{1}{2}((\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v})-(\mathbf{a}\cdot\nabla\mathbf{v},\mathbf{u}))$ into the weak formulation instead of $(\mathbf{a}\cdot\nabla\mathbf{u},\mathbf{v})$ (see e.g. Temam, 1977). We will keep in mind this possibility as it does not alter our analysis.

Fortunately, for rotation form, the skew-symmetry of the form $(\mathbf{w}\times\mathbf{u},\mathbf{v})$ and hence energy conservation of the FE solution hold for any $\mathbf{w}$ thanks to the properties of the vector product. At the same time the assumption $\mathbf{w}\in L^\infty(\Omega)^{2n-3}$ can be unrealistic when $\mathbf{w}=\operatorname{curl}\mathbf{u}$ for some 'real' velocity field $\mathbf{u}$. However, our case of interest is $\mathbf{w}=\operatorname{curl}\mathbf{u}_h$ where $\mathbf{u}_h$ is a FE velocity. Then $\mathbf{w}$ is a bounded function and $\|\mathbf{w}\|_\infty\leqslant c\,h^{-1}\|\mathbf{u}_h\|_\infty$ for fixed h.

In general, formulation (3.5) may exhibit spurious solutions for two reasons:

(i) The velocity/pressure approximation in $\mathbf{V}_h\times\mathrm{Q}_h$ is not div-stable. As a remedy, some pressure regularization is introduced below, see Sections 5 and 6.

(ii) The mesh is too coarse in order to resolve instabilities stemming from (locally) dominating advective and/or rotation terms such that

$$\mathrm{Re}_K=\nu^{-1}\|\mathbf{a}\|_{\infty,K}h_K\gg 1\quad\text{and/or}\quad \mathrm{Ek}_K^{-1}=\nu^{-1}\|\mathbf{w}\|_{\infty,K}h_K^2\gg 1.\tag{3.7}$$

Then eventually some artificial diffusion is required, see Sections 4 and 5. In three dimensions we define $\|\mathbf{w}\|_{\infty,G}=\operatorname{ess\,sup}_G\sum_{i=1}^3|w_i|$ so that $\|\mathbf{w}\times\mathbf{u}\|_G\leqslant\|\mathbf{w}\|_{\infty,G}\|\mathbf{u}\|_G$. We also introduce the dimensionless local number $\mathrm{D}_K=\nu^{-1}\alpha\,h_K^2$ measuring the impact of the reactive term.

Below we consider the following stabilized form of the Galerkin scheme (3.5): *find* $U_h=\{\mathbf{u}_h,p_h\}\in\mathbf{W}_h$ *such that*

$$a_h(U_h,V_h)\;=f_h(V_h)\qquad\forall\,V_h=\{\mathbf{v}_h,q_h\}\in\mathbf{W}_h,\tag{3.8}$$

$$\begin{aligned}a_h(U,V):=\;&a(U,V)+\sum_K(\gamma_K\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v})_K\\&+\sum_K(\mathcal{L}(U),\delta_K^{\mathbf{a}}(\mathbf{a}\cdot\nabla)\mathbf{v}+\delta_K^{\mathbf{w}}\mathbf{w}\times\mathbf{v}+\delta_K^p\nabla q)_K,\end{aligned}\tag{3.9}$$

$$f_h(V):=f(V)+\sum_K((\gamma_K g,\operatorname{div}\mathbf{v})_K+(\mathbf{f},\delta_K^{\mathbf{a}}(\mathbf{a}\cdot\nabla)\mathbf{v}+\delta_K^{\mathbf{w}}\mathbf{w}\times\mathbf{v}+\delta_K^p\nabla q)_K).$$

The Galerkin scheme (3.5) is a special case of (3.8) with $\gamma_K=\delta_K^{\mathbf{a}}=\delta_K^{\mathbf{w}}=\delta_K^p=0$.

Scheme (3.8) is built (for accuracy reasons, see Sections 4 and 5) to be of *residual type*, i.e. the sum of stabilizing terms vanishes for a smooth solution of (3.1). This implies the basic property of *generalized Galerkin orthogonality*

$$a_h(U-U_h,V_h)=0\qquad\forall\,V_h\in\mathbf{W}_h.\tag{3.10}$$

The analysis of these *consistent* schemes will be given in Sections 4 and 5. Furthermore, we address an *inconsistent* variant in Section 6.

REMARK 3.1 Another clan of stabilization methods related to (3.8), (3.9) are the so-called *sub-grid scale* or GLS methods (Codina, 2001). In these methods different test functions are taken in the second stabilizing terms in (3.9) and in $f_h(V)$, e.g. the last term in (3.9) is $-\sum_K \delta_K(\mathcal{L}(U), \mathcal{L}^*(V))_K$ with the adjoint operator $\mathcal{L}^*$. This technique was applied to the Oseen problem with additional Coriolis forces. A novelty in the present paper is that we consider linear problems in the context of the full Navier–Stokes calculations where skew-symmetric terms in (3.1) appear from linearization of convection, while the estimates in the case of the stabilized Oseen problem with additional Coriolis forces are obtained as a by-product of our analysis.

REMARK 3.2 A drawback of the scheme (3.8) is, particularly in three dimensions, that the generation of the stabilizing terms requires a lot of CPU time (Turek, 1999). Moreover, its algebraic structure can be too complicated for standard solvers, in contrast with, for example, (less accurate) simple upwinding which produces M-matrices and preserves the structure of the Galerkin scheme. So we are interested in avoiding as many of the stabilization terms as possible. Therefore, we introduced different parameters δ_K^x with $x \in \{\mathbf{a}, \mathbf{w}, p\}$. However, in our analysis we always take $\delta_K^{\mathbf{a}}$ and $\delta_K^{\mathbf{w}}$ equal. This assumption simplifies the still very technical analysis and it is not a serious restriction, since for the problem of our interest we have either $\mathbf{a} = 0$ or $\mathbf{w} = 0$.

4. Div-stable schemes with SUPG stabilization

Here we analyse the consistent scheme (3.8) in the case of div-stable elements, and with possible SUPG-type stabilization. Assumptions for this case follow.

Case A: $\delta_K^p \equiv 0, \quad \delta_K^A := \delta_K^{\mathbf{a}} = \delta_K^{\mathbf{w}} \geqslant 0, \quad \gamma_K \geqslant 0.$

4.1 *Stability of the scheme*

First we present an inf-sup stability estimate for the bilinear form $a_h(\cdot,\cdot)$ on $\mathbf{W}_h = \mathbf{V}_h \times \mathrm{Q}_h$ with respect to the norm $\|\cdot\|_A$ (with parameter $\sigma_A > 0$ to be determined below) defined as

$$\|V\|_A^2 = |[V]|_A^2 + \sigma_A \|q\|^2,$$
$$|[V]|_A^2 = \nu\|\nabla \mathbf{v}\|^2 + \alpha\|\mathbf{v}\|^2 + \sum_K (\gamma_K \|\operatorname{div} \mathbf{v}\|_K^2 + \delta_K^A \|(\mathbf{a}\cdot\nabla)\mathbf{v} + \mathbf{w}\times\mathbf{v}\|_K^2).$$

LEMMA 4.1 Assume the following conditions for the stabilization parameters:

$$0 \leqslant \delta_K^A \leqslant \frac{1}{3}\min\left\{\frac{\lambda_0\beta_0^2 h_K^2}{\mu_p^2 N_A^2}; \frac{h_K^2}{\mu_u^2 \nu}; \frac{1}{\alpha}\right\}, \qquad 0 \leqslant \gamma_K \leqslant \gamma \leqslant N_A^2 \tag{4.1}$$

with appropriate λ_0. Furthermore, set with Friedrichs' constant C_F,

$$N_A = \sqrt{\nu} + \sqrt{\alpha} C_F + (\|\mathbf{a}\|_\infty + C_F\|\mathbf{w}\|_\infty)\frac{C_F}{\sqrt{\nu + C_F^2\alpha}},$$
$$M_A = \sqrt{\delta_A}(\|\mathbf{a}\|_\infty + C_F\|\mathbf{w}\|_\infty), \quad \delta_A = \sup_K \delta_K^A. \tag{4.2}$$

Then there exist positive constants $\beta_A \neq \beta_A(h, \nu)$ and $\sigma_A = \frac{1}{27}\beta_0^2 N_A^{-2}$ such that

$$\inf_{U_h \in \mathbf{W}_h} \sup_{V_h \in \mathbf{W}_h} \frac{a_h(U_h, V_h)}{\|U_h\|_A \|V_h\|_A} \geqslant \beta_A \tag{4.3}$$

for sufficiently small $h = \sup_K h_K \colon h \leqslant C_F \mu_u$.

REMARK 4.1 (i) The stability result for div-stable elements with/without SUPG stabilization of the skew-symmetric terms is apparently new. Note that the scheme considered in Chapter IV.3 of Roos *et al.* (1996) requires $\delta_K^p = \delta_K$. The present analysis also provides a modified inf-sup condition for the Galerkin scheme with $\delta_K^A = 0$, i.e. for diffusion-dominated problems.

(ii) The first condition on δ_K^A in (4.1) is rather restrictive if $\alpha = 0$ and $\nu \ll 1$. But it disappears for the interesting case of piecewise constant pressure, thanks to $\mu_p = 0$. Moreover, regardless of pressure approximation, if $\alpha > 0$ then N_A remains bounded, and the condition on δ_K^A is not restrictive any more.

Proof. We fix an arbitrary $U_h \in \mathbf{W}_h$. Below we find $V_h \in \mathbf{W}_h$ satisfying (4.3).
(i) We use the following abbreviations:

$$\begin{aligned}
A^2 &:= \nu\|\nabla \mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2, & B^2 &:= \|p_h\|^2,\\
Y^2 &:= \sum_K \delta_K^A \| -\nu\Delta\mathbf{u}_h + \alpha\mathbf{u}_h + \nabla p_h\|_K^2, & &\\
X^2 &:= \sum_K \delta_K^A \|(\mathbf{a}\cdot\nabla)\mathbf{u}_h + \mathbf{w}\times\mathbf{u}_h\|_K^2, & Z^2 &:= \sum_K \gamma_K \|\operatorname{div}\mathbf{u}_h\|_K^2,
\end{aligned}$$

hence $|[U_h]|_A^2 = A^2 + X^2 + Z^2$. In the first step we set $V_h = U_h$ in (3.9), hence

$$a_h(U_h, U_h) \geqslant A^2 + X^2 + Z^2 - YX.$$

The main difficulty comes from the term YX. We have, via triangle inequality, inverse inequalities (3.3), and using (4.1),

$$Y^2 \leqslant \nu\|\nabla\mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2 + \lambda\|p_h\|^2 = A^2 + \lambda B^2,$$

where $\lambda := \lambda_0\beta_0^2 N_A^{-2}$. Then we obtain via Young's inequality

$$a_h(U_h, U_h) \geqslant \frac{1}{2}(A^2 + X^2 + Z^2) - \frac{\lambda}{2}B^2. \tag{4.4}$$

(ii) Consider the following form of the condition (3.6):

$$\exists \mathbf{z}_h \in \mathbf{V}_h : (\operatorname{div}\mathbf{z}_h, p_h) \geqslant \beta_0\|p_h\|_{\mathrm{Q}}\|\mathbf{z}_h\|_{\mathbf{V}} \tag{4.5}$$

with $\beta_0 \neq \beta_0(h)$. We can assume $\|\mathbf{z}_h\|_{\mathbf{V}} = \|p_h\|_{\mathrm{Q}}$. Consider now

$$a_h(U_h, (-\mathbf{z}_h, 0)) = (p_h, \operatorname{div}\mathbf{z}_h) - \sum_{i=1}^{4} T_i^A \geqslant \beta_0 B^2 - \sum_{i=1}^{4} T_i^A.$$

Standard inequalities and integration of the advective term by parts imply

$$
\begin{aligned}
T_1^A &:= \nu(\nabla \mathbf{u}_h, \nabla \mathbf{z}_h) + (\alpha \mathbf{u}_h + \mathbf{w} \times \mathbf{u}_h, \mathbf{z}_h) - (\mathbf{u}_h, (\mathbf{a} \cdot \nabla)\mathbf{z}_h) \\
&\leqslant \left(\sqrt{\nu} + \sqrt{\alpha} C_F + (\|\mathbf{a}\|_\infty + C_F \|\mathbf{w}\|_\infty) \frac{C_F}{\sqrt{\nu + C_F^2 \alpha}} \right) A \|\nabla \mathbf{z}_h\| = N_A A\, B.
\end{aligned}
$$

Similarly with γ, δ_A, and M_A given by (4.1) and (4.2)

$$
\begin{aligned}
T_2^A &:= \sum_K \gamma_K (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{z}_h)_K \leqslant \sqrt{\gamma} Z B \leqslant N_A Z B, \\
T_3^A &:= \sum_K \delta_K^A (-\nu \Delta \mathbf{u}_h + \alpha \mathbf{u}_h + \nabla p_h, \mathbf{w} \times \mathbf{z}_h + (\mathbf{a} \cdot \nabla)\mathbf{z}_h)_K \\
&\leqslant Y \sqrt{\delta_A} (\|\mathbf{a}\|_\infty + \|\mathbf{w}\|_\infty C_F) B \leqslant (A + \sqrt{\lambda} B) M_A B, \\
T_4^A &:= \sum_K \delta_K^A (\mathbf{w} \times \mathbf{u}_h + (\mathbf{a} \cdot \nabla)\mathbf{u}_h, \mathbf{w} \times \mathbf{z}_h + (\mathbf{a} \cdot \nabla)\mathbf{z}_h)_K \\
&\leqslant X \sqrt{\delta_A} (\|\mathbf{a}\|_\infty + \|\mathbf{w}\|_\infty C_F) B \leqslant M_A X B.
\end{aligned}
$$

We summarize these estimates and use Young's inequality (with $\kappa > 0$)

$$
\begin{aligned}
a_h(U_h, (-\mathbf{z}_h, 0)) &\geqslant \beta_0 B^2 - (N_A(A + Z) + M_A(A + \sqrt{\lambda} B + X)) B \\
&\geqslant \left(\beta_0 - M_A \sqrt{\lambda} - \frac{3\kappa}{2} \right) B^2 - \frac{1}{2\kappa} (N_A + M_A)^2 (A^2 + X^2 + Z^2).
\end{aligned}
$$

Using (4.1), we get for $h \leqslant C_F \mu_u$ the inequality $2M_A \leqslant N_A$; then the definition of λ (with $\lambda_0 \leqslant \frac{1}{2}$) gives $M_A \sqrt{\lambda} \leqslant \frac{1}{4} \beta_0$. Hence we obtain

$$
a_h(U_h, (-\mathbf{z}_h, 0)) \geqslant \frac{\beta_0}{2} B^2 - \frac{27}{4\beta_0} N_A^2 (A^2 + X^2 + Z^2). \tag{4.6}
$$

(iii) Define $V_h := U_h + \rho_A(-\mathbf{z}_h, 0)$ with some $\rho_A > 0$, then via (4.4), (4.6)

$$
\begin{aligned}
a_h(U_h, V_h) &= a_h(U_h, U_h) + \rho_A a_h(U_h, (-\mathbf{z}_h, 0)) \\
&\geqslant \left(\frac{1}{2} - \frac{27 \rho_A N_A^2}{4\beta_0} \right) (A^2 + X^2 + Z^2) + \left(\frac{\beta_0 \rho_A}{2} - \frac{\lambda}{2} \right) B^2.
\end{aligned}
$$

We choose

$$
\rho_A = \tfrac{1}{27} \beta_0 N_A^{-2}, \qquad \sigma_A = \beta_0 \rho_A = \tfrac{1}{27} \beta_0^2 N_A^{-2}.
$$

Then, setting $\lambda_0 = \frac{1}{54}$, we get $\lambda = \frac{1}{2} \beta_0 \rho_A$. Hence

$$
a_h(U_h, V_h) \geqslant \tfrac{1}{4} (A^2 + X^2 + Z^2 + \sigma_A B^2) \equiv \tfrac{1}{4} \|U_h\|_A^2. \tag{4.7}
$$

(iv) For h such that $2M_A \leqslant N_A$ the following estimate holds:

$$
\begin{aligned}
\|(-\mathbf{z}_h, 0)\|_A^2 &= \nu \|\nabla \mathbf{z}_h\|^2 + \alpha \|\mathbf{z}_h\|^2 + \sum_K [\delta_K^A \|(\mathbf{a} \cdot \nabla)\mathbf{z}_h + \mathbf{w} \times \mathbf{z}_h\|_K^2 + \gamma_K \|\operatorname{div} \mathbf{z}_h\|_K^2] \\
&\leqslant (\nu + \alpha C_F^2 + 2\delta_A (\|\mathbf{a}\|_\infty^2 + C_F^2 \|\mathbf{w}\|_\infty^2) + \gamma_K) \|\nabla \mathbf{z}_h\|^2 \ \leqslant\ 3 N_A^2 B^2.
\end{aligned}
$$

By the definition of ρ_A and σ_A we have $\rho_A^2 N_A^2 \leqslant \frac{1}{27}\sigma_A$, and hence

$$\begin{aligned}\|V_h\|_A^2 &\leqslant 2\|U_h\|_A^2 + 2\rho_A^2\|(-\mathbf{z}_h, 0)\|_A^2 \\ &\leqslant 2(|[U_h]|_A^2 + (\sigma_A + \rho_A^2 N^2)\|p_h\|^2) \leqslant \tfrac{20}{9}\|U_h\|_A^2,\end{aligned}$$

which together with (4.7) implies (4.3) with $\beta_A = \frac{3}{8\sqrt{5}}$. □

4.2 *Error analysis and parameter design*

Let $U = \{\mathbf{u}, p\} \in \mathbf{W}$ and $U_h = \{\mathbf{u}_h, p_h\} \in \mathbf{W}_h$ be the solutions of the continuous and of the discrete problems, respectively. Furthermore, $\hat{U} = \{\hat{\mathbf{u}}_h, \hat{p}_h\} \in \mathbf{W}_h$ denotes an appropriate interpolant for U. Then we define the error by $E_h = \{\mathbf{e}_\mathbf{u}, e_p\} = \{\mathbf{u}-\mathbf{u}_h, p-p_h\}$ and set

$$\{\eta_\mathbf{u}, \eta_p\} := \{\mathbf{u} - \hat{\mathbf{u}}_h, p - \hat{p}_h\}, \qquad \{\chi_\mathbf{u}, \chi_p\} := \{\hat{\mathbf{u}}_h - \mathbf{u}_h, \hat{p}_h - p_h\}.$$

Galerkin orthogonality (3.10) and Lemma 4.1 imply that there exists $V_h = \{\mathbf{v}_h, q_h\} \in \mathbf{W}_h$ such that

$$\beta_A\|\{\chi_\mathbf{u}, \chi_p\}\|_A\|V_h\|_A \leqslant a_h(\{\chi_\mathbf{u}, \chi_p\}, V_h) = -a_h(\{\eta_\mathbf{u}, \eta_p\}, V_h). \tag{4.8}$$

LEMMA 4.2 For arbitrary $U = \{\mathbf{u}, p\} \in \mathbf{W}$ with $\mathcal{L}U|_K \in L^2(K)\forall K \in \mathcal{T}_h$ and $V_h \in \mathbf{W}_h$ we have

$$\begin{aligned}a_h(U, V_h) \leqslant C\|V_h\|_A\Bigg\{&|[U]|_A + \left(\sum_K(\sigma_A^{-1}\|\nabla\mathbf{u}\|_K^2 + (\|\mathbf{a}\|_\infty^2 + C_F^2\|\mathbf{w}\|_\infty^2)\nu^{-1}\|\mathbf{u}\|_K^2)\right)^{\frac{1}{2}} \\ &+ \left(\sum_K 2(\nu+\gamma_K)^{-1}\|p\|_K^2\right)^{\frac{1}{2}} + \left(\sum_K \delta_K^A\|-\nu\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p\|_K^2\right)^{\frac{1}{2}}\Bigg\}.\end{aligned} \tag{4.9}$$

Proof. The symmetric terms of a_h are bounded by $|[U]|_A\,|[V_h]|_A$. Integration by parts and antisymmetry properties imply

$$|(\mathbf{a}\cdot\nabla\mathbf{u} + \mathbf{w}\times\mathbf{u}, \mathbf{v}_h)| \leqslant \frac{\|\mathbf{a}\|_\infty + C_F\|\mathbf{w}\|_\infty}{\sqrt{\nu}}\|\mathbf{u}\|\sqrt{\nu}\|\nabla\mathbf{v}_h\|, \tag{4.10}$$

$$(\operatorname{div}\mathbf{u}, q_h) \leqslant \frac{1}{\sqrt{\sigma_A}}\|\nabla\mathbf{u}\|\sqrt{\sigma_A}\|q_h\|. \tag{4.11}$$

Next we have

$$-(p, \operatorname{div}\mathbf{v}_h) \leqslant \left(\sum_K 2(\nu+\gamma_K)^{-1}\|p\|_K^2\right)^{\frac{1}{2}}\left(\nu\|\nabla\mathbf{v}_h\|^2 + \sum_K\gamma_K\|\operatorname{div}\mathbf{v}_h\|_K^2\right)^{\frac{1}{2}}.$$

Finally, consider the remaining stabilizing terms:

$$\sum_K \delta_K^A(-\nu\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p, (\mathbf{a}\cdot\nabla)\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h)_K$$

$$\leqslant \left(\sum_K \delta_K^A \| -\nu\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p\|_K^2\right)^{\frac{1}{2}} \left(\sum_K \delta_K^A \|(\mathbf{a}\cdot\nabla)\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h\|_K^2\right)^{\frac{1}{2}}.$$

This implies the assertion (4.9) via definition of $\|\cdot\|_A$ and $|[\cdot]|_A$. □

We now combine (4.8) and (4.9) with $U = \{\eta_{\mathbf{u}}, \eta_p\}$. After cancelling $\|V_h\|_A$ we get

$$\|\{\chi_{\mathbf{u}}, \chi_p\}\|_A \leqslant C\beta_A^{-1}\|\{\eta_{\mathbf{u}}, \eta_p\}\|_A.$$

Then the triangle inequality

$$\|E_h\|_A \leqslant \|\{\chi_{\mathbf{u}}, \chi_p\}\|_A + \|\{\eta_{\mathbf{u}}, \eta_p\}\|_A$$

and usual local interpolation properties with $\mathbf{a}_K = \|\mathbf{a}\|_{\infty,K}$, $\mathbf{w}_K = \|\mathbf{w}\|_{\infty,K}$ imply that

$$\begin{aligned}
\|E_h\|_A^2 \leqslant C\sum_K\{&(\delta_K^A + \sigma_A h_K^2 + h_K^2(\nu+\gamma_K)^{-1})h_K^{2k}|p|^2_{H^{k+1}(K)} \\
&+ (\nu + \sigma_A^{-1} + (\alpha C_F^2 + (\|\mathbf{a}\|_\infty^2 + C_F^2\|\mathbf{w}\|_\infty^2)\nu^{-1})h_K^2 \\
&+ \gamma_K + \delta_K^A(\mathbf{a}_K^2 + \mathbf{w}_K^2 h_K^2))h_K^{2l}|\mathbf{u}|^2_{H^{l+1}(K)}\} \\
\leqslant C\sum_K&(\delta_K^A + h_K^2 N_A^{-2} + h_K^2(\nu+\gamma_K)^{-1})h_K^{2k}|p|^2_{H^{k+1}(K)} \\
&+ C\sum_K(\nu + N_A^2 + \gamma_K + \delta_K^A(\mathbf{a}_K^2 + \mathbf{w}_K^2))h_K^{2l}|\mathbf{u}|^2_{H^{l+1}(K)}.
\end{aligned}$$

The second estimate follows thanks to (4.1) and $\sigma_A \sim N_A^{-2}$.

Note that for div-stable elements usually $k \leqslant l-1$. A reasonable choice of the stabilization parameters is now

$$0 \leqslant \delta_K^A \leqslant \frac{\beta_0^2 h_K^2}{54\mu_p^2 N_A^2}; \qquad \gamma_K \sim N_A^2 (\geqslant \nu), \tag{4.12}$$

where N_A is given in (4.2). Hence we arrive at the following theorem.

THEOREM 4.1 For div-stable velocity–pressure interpolation, the discrete problem (3.8) with SUPG-type stabilization, i.e. $\delta_K^p \equiv 0$ and $\delta_K^A = \delta_K^{\mathbf{a}} = \delta_K^{\mathbf{w}}$, satisfying (4.12), obeys the error estimate

$$\|E_h\|_A^2 \leqslant C\sum_K (h_K^{2(k+1)} N_A^{-2}|p|^2_{H^{k+1}(K)} + N_A^2 h_K^{2l}|\mathbf{u}|^2_{H^{l+1}(K)}). \tag{4.13}$$

REMARK 4.2

(i) We note that the parameter δ_K^A is included in the definition of the norm in the LHS of (4.13) and, being positive, allows us to gain additional control on the error.

(ii) Suppose that $l = k + 1 \geqslant 1$. Then the estimate (4.13) is reasonable provided that $N_A^2 = \mathcal{O}(1)$. This is valid, after proper scaling so that $\|\mathbf{a}\|_\infty + C_F\|\mathbf{w}\|_\infty \leqslant 1$, if $\alpha = \mathcal{O}(1)$. This is not a restriction within a time-dependent context for time steps satisfying $\delta t = \alpha^{-1} > 0$, but it is undesirable when $\alpha = 0$. For $\alpha \gg 1$ the second term on the RHS of the estimate (4.13) can be large; however, the velocity approximation is still controlled, since the $\|\cdot\|_A$ norm includes the α-dependent term (see (4.1)). However, (4.13) does not give a good error estimate for the pressure in the case $\alpha \gg 1$.

(iii) The analysis shows that the div-stabilization terms $\gamma_K(\operatorname{div}\cdot, \operatorname{div}\cdot)_K$ with sufficiently large γ_K are important. This kind of stabilization corresponds, in some sense, to the usual penalization technique of the continuity constraint. The numerical results reported in Codina (1993) for the case $\mathbf{w} = \mathbf{0}$ and in Olshanskii (2001) for both cases $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ underpin the role of these terms.

(iv) Estimate (4.13) is uniformly valid w.r.t. δ_K^A satisfying (4.12). The numerical results in (Codina, 2001) indicate that stabilization is necessary even in the case of $\mathbf{a} = \mathbf{0}$ if the mesh number Ek_K is large.

The result (4.13) can be improved for $\nu \ll 1$ if Q_h consists of piecewise-constant functions ($k = 0$). This is a common case for div-stable FE. As was noted in Remark 4.1, condition (4.1) is much less restrictive now. Thus modifying (4.10) as

$$|(\mathbf{a}\cdot\nabla\mathbf{u} + \mathbf{w}\times\mathbf{u}, \mathbf{v}_h)| \leqslant \left(\sum_K \delta_K^A \|\mathbf{a}\cdot\nabla\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h\|^2\right)^{\frac{1}{2}} \left(\sum_K (\delta_K^A)^{-1}\|\mathbf{u}\|_K^2\right)^{\frac{1}{2}}$$

we get, balancing the error estimate, the 'standard' choice $\delta_K^A \sim h_K^2[\nu(1 + \mathrm{Re}_K + \mathrm{Ek}_K^{-1} + \mathrm{D}_K)]^{-1}$. Although the dependence of the RHS of the error estimate on N_A remains the same (due to σ_A^{-1} in (4.9)), we gain better control on the convective terms in $\|E_h\|_A$.

Improving the error estimate further, we can get rid of σ_A^{-1} in (4.9), however sacrificing the local nature of the analysis. This is done as follows: in the proof of Lemma 4.2 we do not apply (4.11), but take $|(\operatorname{div}\mathbf{u}, q_h)|$ to the RHS of (4.9); at the same time the term $\sigma_A^{-1}\|\nabla\mathbf{u}\|_K^2$ disappears from (4.9). Moreover, for the interpolant $\{\hat{\mathbf{u}}_h, \hat{p}_h\}$ we take the solution of the discrete Stokes problem:

$$(\nabla(\hat{\mathbf{u}}_h - \mathbf{u}), \nabla\mathbf{v}_h) - (\hat{p}_h - p, \operatorname{div}\mathbf{v}_h) + (\operatorname{div}(\mathbf{u}_h - \mathbf{u}), q_h) = 0, \quad \forall\{\mathbf{v}_h, q_h\} \in \mathbf{W}_h.$$

For a sufficiently regular domain we have (Girault & Raviart, 1986; Dauge, 1989) with $h = \sup_K h_K$:

$$h^{-1}\|\hat{\mathbf{u}}_h - \mathbf{u}\| + \|\nabla(\hat{\mathbf{u}}_h - \mathbf{u})\| + \|\hat{p}_h - p\| \leqslant c\, h\,(\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}). \tag{4.14}$$

These interpolation properties and $(\operatorname{div}\eta_{\mathbf{u}}, \chi_p) = 0$ lead to the following result.

THEOREM 4.2 Assume that Q_h consists of piecewise-constant functions and $\delta_K^A \sim \delta_A = h^2[\nu(1 + \mathrm{Re}_h + \mathrm{Ek}_h^{-1} + \mathrm{D}_h)]^{-1}$ with $\mathrm{Re}_h = \frac{\|\mathbf{a}\|_\infty h}{\nu}$, $\mathrm{Ek}_h^{-1} = \frac{\|\mathbf{w}\|_\infty h^2}{\nu}$, $\mathrm{D}_h = \frac{\alpha h^2}{\nu}$, and $\gamma_K \sim \gamma = \mathcal{O}(1)$, $\sigma_K^A \sim \sigma^A \leqslant \beta_0 N_A^{-2}$. Assume also that (4.14) holds; then for div-stable velocity–pressure interpolation, the scheme (3.8) obeys the error estimate

$$\|E_h\|_A^2 \leqslant C\{1 + \nu(1 + \mathrm{Re}_h + \mathrm{Ek}_h^{-1} + \mathrm{D}_h) + \sigma^A\}h^2(\|\mathbf{u}\|_2^2 + |p|_1^2).$$

REMARK 4.3 Results for nonconforming FE schemes corresponding to Theorem 4.2 can be found in Knobloch & Tobiska (1999) in the case of $\mathbf{w} = \mathbf{0}$, $\alpha = 0$.

5. Pressure- and SUPG-stabilized schemes

Here we analyse scheme (3.8) in the case of pressure-stabilized elements, i.e. without condition (3.6), and together with SUPG-type stabilization. We assume

Case B: $\delta_K^B := \delta_K^p = \delta_K^{\mathbf{a}} = \delta_K^{\mathbf{w}} \geqslant 0, \quad \gamma_K \geqslant 0.$

For FE pressure we assume, for simplicity, that $Q_h \subset Q \cap H^1(\Omega)$. Nevertheless, the analysis can easily be extended to discontinuous pressure approximation if certain pressure jump terms are added at inter-element boundaries, see e.g. Roos *et al.* (1996, Chapter IV.3.1).

5.1 *Stability of the discrete problem*

We start with a modified inf-sup stability estimate on $\mathbf{W}_h = \mathbf{V}_h \times \mathrm{Q}_h$ with respect to the norm $\|\cdot\|_B$ (with $\sigma_B > 0$ to be determined below) defined as

$$\|V\|_B^2 = |[V]|_B^2 + \sigma_B \|q\|^2, \tag{5.1}$$

$$|[V]|_B^2 = \nu\|\nabla \mathbf{v}\|^2 + \alpha\|\mathbf{v}\|^2 + \sum_K (\gamma_K \|\operatorname{div} \mathbf{v}\|_K^2 + \delta_K^B \|(\mathbf{a}\cdot\nabla)\mathbf{v} + \mathbf{w}\times\mathbf{v} + \nabla q\|_K^2). \tag{5.2}$$

Furthermore, define with $\mathbf{a}_K = \|\mathbf{a}\|_{\infty,K}$, $\mathbf{w}_K = \|\mathbf{w}\|_{\infty,K}$ the following quantities:

$$M_B^2 = 2\max_K\{\delta_K^B(\mathbf{a}_K^2 + C_F^2\mathbf{w}_K^2)\}, \tag{5.3}$$

$$N_B = \sqrt{\nu} + \sqrt{\alpha}C_F + (\|\mathbf{a}\|_\infty + C_F\|\mathbf{w}\|_\infty)C_F\nu^{-\frac{1}{2}}. \tag{5.4}$$

LEMMA 5.1 For the case B with pressure- and SUPG-stabilization we assume

$$\mu_0 h_K^2 \leqslant \delta_K^B \leqslant \frac{1}{2}\min\left\{\frac{h_K^2}{\mu_u^2\nu}, \frac{1}{\alpha}\right\}, \quad 0 \leqslant \gamma_K \leqslant \gamma \leqslant N_B^2 \tag{5.5}$$

with appropriate $\mu_0 > 0$ defined later. Then there exist positive constants $\beta_B \neq \beta_B(h,\nu)$ and $\sigma_B = cN_B^{-2}$ such that

$$\inf_{U_h\in\mathbf{W}_h}\sup_{V_h\in\mathbf{W}_h}\frac{a_h(U_h,V_h)}{\|U_h\|_B\|V_h\|_B} \geqslant \beta_B \tag{5.6}$$

for sufficiently small $h = \sup_K h_K$: $h \leqslant C_F\min\{\mu_u, 1\}$.

REMARK 5.1 The stability result for pressure-stabilized elements with SUPG-type stabilization (case B) can be found (in modified form) with $\mathbf{w} \equiv \mathbf{0}$ in Roos *et al.* (1996). The present estimate is more precise, if $\mathrm{Re}_\Omega = C_F\|\mathbf{a}\|_\infty\nu^{-1} \leqslant 1$. Estimates with $\mathbf{w} \neq \mathbf{0}$, but with $\nabla\mathbf{w} = \mathbf{0}$, can be found in Codina & Soto (1997), and only with respect to the stabilized energy-type norm $|[\cdot]|_B$.

Proof. Later on, we follow the lines of the case A with some modifications. We fix arbitrary $U_h = \{\mathbf{u}_h, p_h\} \in \mathbf{W}_h$ and introduce the additional abbreviation:

$$\tilde{X}^2 = \sum_K \delta_K^B \|(\mathbf{a}\cdot\nabla)\mathbf{u}_h + \mathbf{w}\times\mathbf{u}_h + \nabla p_h\|_K^2.$$

As a remedy to the missing div-stability condition on $\mathbf{W}_h$, some pressure stabilization is proposed. We start with the following auxiliary result. □

LEMMA 5.2 Assume that $\mu_0 > 0$ in condition (5.5), together with $h \leqslant C_F$, N_B as in (5.4), and the other constants given in the proof, are chosen according to

$$C_I \mu_0^{-\frac{1}{2}} \leqslant \tfrac{1}{2} C_S N_B, \tag{5.7}$$

where C_I and C_S are interpolation constants defined in the proof. Then there exists a constant $C_\Omega > 0$ such that for any $p_h \in \mathrm{Q}_h$, there is an element $\mathbf{z}_h \in V_h$ such that

$$(p_h, \operatorname{div}\mathbf{z}_h) \geqslant \|p_h\|^2 - \tfrac{1}{2} C_\Omega N_B (A + \tilde{X})\|p_h\|. \tag{5.8}$$

Proof. The Nečas inequality yields the existence of $\mathbf{z} \in \mathbf{V}$ (see Corollary 2.4 in Girault & Raviart (1986)) such that $\operatorname{div}\mathbf{z} = p_h$, with $\|\nabla\mathbf{z}\| \leqslant \bar{\beta}_0\|p_h\|$. Moreover, with the local interpolation operator $I_h : \mathbf{V} \to \mathbf{V}_h$, see Section 3, we have

$$\mathbf{z}_h = I_h\mathbf{z}, \quad \|\nabla\mathbf{z}_h\| \leqslant C_S\|\nabla\mathbf{z}\| \leqslant C_S\bar{\beta}_0\|p_h\|, \quad \|\mathbf{z}-\mathbf{z}_h\|_K \leqslant C_I h_K |\mathbf{z}|_{H^1(K)}$$

for all $K \in \Omega$. Integration by parts, the triangle inequality, and (5.5) imply

$$\begin{aligned}
(p_h, \operatorname{div}\mathbf{z}_h) &= (p_h, \operatorname{div}\mathbf{z}) - (p_h, \operatorname{div}(\mathbf{z}-\mathbf{z}_h)) \geqslant \|p_h\|^2 - \left|\sum_K (\nabla p_h, \mathbf{z}-\mathbf{z}_h)_K\right| \\
&\geqslant \|p_h\|^2 - \left|\sum_K ((\mathbf{a}\cdot\nabla)\mathbf{u}_h + \mathbf{w}\times\mathbf{u}_h + \nabla p_h, \mathbf{z}-\mathbf{z}_h)_K\right| \\
&\quad - \left|\sum_K ((\mathbf{a}\cdot\nabla)\mathbf{u}_h + \mathbf{w}\times\mathbf{u}_h, \mathbf{z}-\mathbf{z}_h)_K\right| = \|p_h\|^2 - S_1 - S_2,
\end{aligned}$$

$$\begin{aligned}
S_1 &\leqslant \tilde{X}\left(\sum_K (\delta_K^B)^{-1} C_I^2 h_K^2 |\mathbf{z}|^2_{H^1(K)}\right)^{\frac{1}{2}} \leqslant \mu_0^{-1/2} C_I \bar{\beta}_0 \tilde{X}\|p_h\|, \\
S_2 &\leqslant \sum_K C_I h_K |\mathbf{z}|_{H^1(K)} \max_K \nu^{-1/2}(\mathbf{a}_K + C_F\mathbf{w}_K)\sqrt{\nu}|\mathbf{u}_h|_{H^1(K)} \\
&\leqslant C_I\bar{\beta}_0 \max_K \nu^{-1/2} h_K(\mathbf{a}_K + C_F\mathbf{w}_K) A\|p_h\| \leqslant C_I\bar{\beta}_0 N_B A\|p_h\|.
\end{aligned}$$

The last estimate holds for $h \leqslant C_F$. Now assertion (5.8) follows with $C_\Omega = C_I\bar{\beta}_0$ and thanks to (5.7). □

Lemma 5.1, on imposing a restriction on μ_0, produces a lower bound for δ_K^B:

$$\frac{C_0 h_k^2}{\nu + C_F(\alpha + \nu^{-1}(\|\mathbf{a}\|_\infty + C_S\|\mathbf{w}\|_\infty)^2)} \leqslant \delta_K^B. \tag{5.9}$$

The constant C_0 in (5.9) can be taken small enough, since we can choose constant C_S in (5.7) arbitrary large, but still independent of the problem parameters. Therefore the lower bound (5.9) does not contradict the upper bound from (5.5).

We continue the proof of Lemma 5.1:

(i) As in Section 3 we get, from local inverse inequalities and condition (5.5),

$$a_h(U_h, U_h) \geqslant \tfrac{1}{2}(A^2 + \tilde{X}^2 + Z^2). \tag{5.10}$$

(ii) In the next step we obtain via Lemma 4.2 and $\|\nabla \mathbf{z}_h\| \leqslant C_\Omega \|p_h\|$ that

$$a_h(U_h, (-\mathbf{z}_h, 0)) = (p_h, \operatorname{div} \mathbf{z}_h) - \sum_{i=1}^{4} T_i^B \geqslant B^2 - \frac{C_\Omega}{2} N_B (A + \tilde{X}) B - \sum_{i=1}^{4} T_i^B,$$

$$\begin{aligned}
T_1^B &= \nu(\nabla \mathbf{u}_h, \nabla \mathbf{z}_h) + (\alpha \mathbf{u}_h + \mathbf{w} \times \mathbf{u}_h, \mathbf{z}_h) \\
&\quad - (\mathbf{u}_h, (\mathbf{a} \cdot \nabla)\mathbf{z}_h) \leqslant C_\Omega N_B A B, \\
T_2^B &= \sum_K \gamma_K (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{z}_h)_K \leqslant \sqrt{\gamma} Z B \leqslant C_\Omega N_B Z B, \\
T_3^B &= \sum_K \delta_K (-\nu \Delta \mathbf{u}_h + \alpha \mathbf{u}_h, \mathbf{w} \times \mathbf{z}_h + (\mathbf{a} \cdot \nabla)\mathbf{z}_h)_K \leqslant C_\Omega M_B A B, \\
T_4^B &= \sum_K \delta_K (\mathbf{w} \times \mathbf{u}_h + \mathbf{a} \cdot \nabla \mathbf{u}_h + \nabla p_h, \mathbf{w} \times \mathbf{z}_h + \mathbf{a} \cdot \nabla \mathbf{z}_h)_K \\
&\leqslant C_\Omega M_B \tilde{X} B.
\end{aligned}$$

Assuming $h_K \leqslant C_F \mu_u$ to ensure $M_B \leqslant N_B$, we get, using Young's inequality with $\kappa = \frac{1}{3}$,

$$\begin{aligned}
a_h(U_h, (-\mathbf{z}_h, 0)) &\geqslant B^2 - 2C_\Omega N_B (A + Z + \tilde{X}) B \geqslant \left(1 - \frac{3\kappa}{2}\right) B^2 \\
&\quad - \frac{2C_\Omega^2 N_B^2}{\kappa} (A^2 + \tilde{X}^2 + Z^2) = \tfrac{1}{2} B^2 - 6 C_\Omega^2 N_B^2 |[U]|_B^2.
\end{aligned} \tag{5.11}$$

(iii) Setting $V_h := U_h + \rho_B(-\mathbf{z}_h, 0)$ with appropriate $\rho_B > 0$, we find via (5.10), (5.11)

$$a_h(U_h, V_h) \geqslant \left(\frac{1}{2} - 6\rho_B C_\Omega^2 N^2\right) |[U]|_B^2 + \frac{\rho_B}{2} B^2.$$

Let $\rho_B := \frac{1}{24 C_\Omega^2 N_B^2}$ and $\sigma_B := 2\rho_B = \frac{1}{12 C_\Omega^2 N_B^2}$; hence

$$a_h(U_h, V_h) \geqslant \tfrac{1}{4}(A^2 + \tilde{X}^2 + Z^2 + \sigma_B B^2) \equiv \tfrac{1}{4} \|U_h\|_B^2. \tag{5.12}$$

(iv) For sufficiently small h such that $M_B \leqslant N_B$,

$$\begin{aligned}
\|(-\mathbf{z}_h, 0)\|_B^2 &= \nu \|\nabla \mathbf{z}_h\|^2 + \alpha \|\mathbf{z}_h\|^2 \\
&\quad + \sum_K (\delta_K \|\mathbf{a} \cdot \nabla \mathbf{z}_h + \mathbf{w} \times \mathbf{z}_h\|_K^2 + \gamma_K \|\operatorname{div} \mathbf{z}_h\|_K^2) \\
&\leqslant (\nu + \alpha C_F^2 + M_B^2 + \gamma) \|\nabla \mathbf{z}_h\|^2 \leqslant 3 C_\Omega^2 N_B^2 B^2.
\end{aligned}$$

By definition of ρ_B and σ_B we have $\rho_B^2 C_\Omega^2 N_B^2 \leqslant \frac{1}{48}\sigma_B$; hence

$$\begin{aligned}\|V_h\|_B^2 &\leqslant 2\|U_h\|_B^2 + 2\rho_B^2\|(-\mathbf{z}_h, 0)\|_B^2 \\ &\leqslant 2(|[U_h]|_B^2 + (\sigma_B + 3C_\Omega^2\rho_B^2 N_B^2)\|p_h\|^2) \leqslant \tfrac{13}{6}\|U_h\|_B^2,\end{aligned}$$

which together with (5.12) implies (5.6) with $\beta_B = \frac{1}{4}\sqrt{\frac{6}{13}}$. Lemma 5.1 is proved.

5.2 *Error analysis and parameter design*

We use the notation of Section 4.2. Galerkin orthogonality (3.10) and Lemma 5.1 imply that there exists $V_h \in \mathbf{W}_h$ such that

$$\beta_B\|\{\chi_{\mathbf{u}}, \chi_p\}\|_B \ \|V_h\|_B \leqslant a_h(\{\chi_{\mathbf{u}}, \chi_p\}, V_h) = -a_h(\{\eta_{\mathbf{u}}, \eta_p\}, V_h). \tag{5.13}$$

LEMMA 5.3 For each $U = \{\mathbf{u}, p\} \in \mathbf{W}$ with $\mathcal{L}U|_K \in L^2(K)\forall K \in \mathcal{T}_h$ and $V_h = \{\mathbf{v}_h, q_h\} \in \mathbf{W}_h$ we have

$$\begin{aligned}a_h(U, V_h) \leqslant C\|V_h\|_B &\left(|[U]|_B + \left(\sum_K (\delta_K^B)^{-1}\|\mathbf{u}\|_K^2\right)^{\frac{1}{2}} \right.\\ &\left. + \left(\sum_K 2(\nu+\gamma_K)^{-1}\|p\|_K^2\right)^{\frac{1}{2}} + \left(\sum_K \delta_K\| -\nu\Delta\mathbf{u} + \alpha\mathbf{u}\|_K^2\right)^{\frac{1}{2}}\right).\end{aligned} \tag{5.14}$$

Proof. The symmetric terms of a_h are bounded by the product $|[U]|_B\ |[V]|_B$. Further we have as in Lemma 4.1

$$\begin{aligned}((\mathbf{a}\cdot\nabla)\mathbf{u} + \mathbf{w}\times\mathbf{u}, \mathbf{v}_h) + (\operatorname{div}\mathbf{u}, q_h) &\leqslant \left(\sum_K (\delta_K^B)^{-1}\|\mathbf{u}\|_K^2\right)^{\frac{1}{2}} \\ &\times \sum_K \delta_K^B\|(\mathbf{a}\cdot\nabla)\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h + \nabla q_h\|_K^2, \\ &- (p, \operatorname{div}\mathbf{v}_h) \\ &\leqslant \left(\sum_K 2(\nu+\gamma_K)^{-1}\|p\|_K^2\right)^{\frac{1}{2}} \left(\nu\|\nabla\mathbf{v}_h\|^2 + \sum_K \gamma_K\|\operatorname{div}\mathbf{v}_h\|_K^2\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, for the remaining stabilizing terms holds

$$\begin{aligned}&\sum_K \delta_K^B(-\nu\Delta\mathbf{u} + \alpha\mathbf{u}, (\mathbf{a}\cdot\nabla)\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h + \nabla q_h)_K \\ &\qquad\qquad \leqslant A\left(\sum_K \delta_K^B\|(\mathbf{a}\cdot\nabla)\mathbf{v}_h + \mathbf{w}\times\mathbf{v}_h + \nabla q_h)\|_K^2\right)^{\frac{1}{2}}.\end{aligned}$$

This implies the assertion (5.14) via the definition of $\|\cdot\|_B$ and $|[\cdot]|_B$. $\square$

Similarly as in the proof of Theorem 4.1, we set $U = \{\eta_{\mathbf{u}}, \eta_p\}$. Then (5.13), the triangle inequality and local interpolation properties imply

$$\begin{aligned}\|E_h\|_B^2 \leqslant C\Bigg\{ &\sum_K (\delta_K^B + h_K^2(\nu+\gamma_K)^{-1} + \sigma_B h_K^2) h_K^{2k} |p|^2_{H^{k+1}(K)} \\ &+ \left(\nu + \alpha C_F^2 h_K^2 + \gamma_K + \delta_K^B\left(\mathbf{a}_K^2 + \mathbf{w}_K^2 h_K^2 + \frac{\nu^2}{h_K^2} + \alpha^2 h_K^2\right) + \frac{h_K^2}{\delta_K^B}\right) \\ &\times h_K^{2l}|\mathbf{u}|^2_{H^{l+1}(K)}\Bigg\}.\end{aligned}$$

We choose δ_K^B balancing the coefficients of the $\mathbf{u}$-dependent term. This results in

$$\delta_K^B \sim h_K^2[\nu(1 + \mathrm{Re}_K + \mathrm{Ek}_K^{-1} + \mathrm{D}_K)]^{-1}. \tag{5.15}$$

To satisfy (5.5) on the one hand and to balance the p-dependent terms on the other hand we set $h_K^2/\gamma_K \sim \delta_K^B$. Hence

$$\gamma_K \sim \nu(1 + \mathrm{Re}_K + \mathrm{Ek}_K^{-1} + \mathrm{D}_K)(\geqslant \nu). \tag{5.16}$$

Using (5.15), (5.16), we summarize the result as follows.

THEOREM 5.1 The discrete problem (3.8) with pressure- and SUPG-type stabilization, i.e. $\delta_K = \delta_K^{\mathbf{a}} = \delta_K^{\mathbf{w}} = \delta_K^p$ as in (5.15), (5.16), obeys the error estimate

$$\begin{aligned}\|E_h\|_B^2 \leqslant C\Bigg\{ &\sum_K (\nu^{-1}(1 + \mathrm{Re}_K + \mathrm{Ek}_K^{-1} + \mathrm{D}_K)^{-1} + \sigma_B) h_K^{2(k+1)} |p|^2_{H^{k+1}(K)} \\ &+ (\nu(1 + \mathrm{Re}_K + \mathrm{Ek}_K^{-1} + \mathrm{D}_K)) h_K^{2l} |\mathbf{u}|^2_{H^{l+1}(K)}\Bigg\}.\end{aligned} \tag{5.17}$$

REMARK 5.2

(i) Equal-order interpolation $l = k$ of velocity and pressure is optimal if $\nu \ll 1$. For a detailed discussion of different cases, covering $\mathbf{w} \equiv \mathbf{0}$ and $\mathbf{a} \equiv \mathbf{0}$, see Codina (2001, Section 4.3). Note that the analysis of this paper does not include L^2 estimates of pressure.

(ii) The basic critical remark on case B is concerned with the bulk of stabilizing terms within (3.8), see Remark 3.2. Another critical point is the non-transparent physical interpretation of the stabilized quantity $((\mathbf{a}\cdot\nabla)\mathbf{u}_h + \mathbf{w}\times\mathbf{u}_h + \nabla p_h)$; there is no separate control of the three ingredients of this term. A separate control, say in case of $\mathbf{w} \equiv \mathbf{0}$, of the pressure gradient is restricted to the case $\max_K \mathrm{Re}_K \leqslant 1$.

6. Galerkin scheme with pressure regularization

Here we are interested in the pressure-regularized problem (2.7), or written in the weak form: *find* $U = \{\mathbf{u}, p\} \in \bar{\mathbf{W}} := \mathbf{V} \times H^1(\Omega)$ *satisfying*

$$\begin{aligned}\tilde{a}(U, V) &:= \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\alpha\mathbf{u} + \mathbf{w}\times\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) + \delta(\nabla p, \nabla q) \\ &= (\mathbf{f}, \mathbf{v}) + (g, q) \quad \forall\, V = \{\mathbf{v}, q\} \in \bar{\mathbf{W}}.\end{aligned}$$

For discrete spaces we will consider the form

$$\tilde{a}_h(U_h, V_h) = \tilde{a}(U_h, V_h) + \sum_K (\gamma_K \operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v}_h)_K \quad \forall U_h, V_h \in \mathbf{W}_h$$

with parameters γ_K which will be chosen from stability-accuracy reasons. Then the stabilized Galerkin FE scheme C reads: *find* $U^h \equiv \{\mathbf{u}^h, p^h\} \in \mathbf{W}_h$ *satisfying*

$$\tilde{a}_h(U_h, V_h) = (\mathbf{f}, \mathbf{v}_h) + (g, q_h) \quad \forall\, V^h \in \mathbf{W}_h. \tag{6.1}$$

In contrast with schemes A and B, problem (6.1) is not a consistent (or residual type) approximation of (2.6) for $\delta > 0$. Consequently it is restricted to low-order accuracy only. For the sake of clarity we consider henceforth the simplified situation with $\gamma_K = \gamma$ and global constant δ. Furthermore, we assume $\mathrm{Q}_h \subset H^1(\Omega)$ and the grid to be quasi-uniform, i.e. $h_K \sim h$. For the subsequent analysis we introduce the following mesh-dependent norm on $\mathbf{W}_h$ for any $\tau \geqslant 0, \sigma > 0$:

$$|||U_h||| = (\nu\|\nabla\mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2 + \tau\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 + \sigma\|p\|^2 + \gamma\|\operatorname{div}\mathbf{u}_h\|^2 + \delta\|\nabla p\|^2)^{\frac{1}{2}}.$$

Here $\mathrm{P_h}$ is a projector from $L_2(\Omega)^n$ on $\mathbf{V}_h$ defined by

$$(\mathrm{P_h}\psi - \psi, \mathbf{v}_h) = 0 \forall \psi \in L_2(\Omega)^n, \mathbf{v}_h \in \mathbf{V}_h.$$

6.1 *Stability of the scheme*

First we prove the stability of $a_h(\cdot,\cdot)$ on $\mathbf{W}_h$. The results below are valid regardless the div-stability of FE pair $\mathbf{V}_h \times Q_h$.

LEMMA 6.1 For any $\tau > 0, \delta > 0, \gamma > 0$ satisfying

$$0 \leqslant \tau \leqslant \min\{R1, R2, R3\}, \quad \tau^2\|\mathbf{w}\|_\infty^2 \leqslant 1, \quad \text{and} \quad \sigma = \frac{c_0^2}{16}\kappa, \tag{6.2}$$

where

$$R1 = \max\left\{\frac{h^2}{2\mu_u^2\nu}, \frac{\alpha^{\frac{1}{2}}h}{2\mu_u\nu^{\frac{1}{2}}\|\mathbf{w}\|_\infty}\right\}, \quad R2 = \max\left\{\frac{h^2}{2\mu_u^2\gamma}, \frac{\alpha^{\frac{1}{2}}h}{2\mu_u\gamma^{\frac{1}{2}}\|\mathbf{w}\|_\infty}\right\},$$

$$R3 = \max\left\{\frac{\delta}{2}, \frac{\alpha^{\frac{1}{2}}\delta^{\frac{1}{2}}}{2\|\mathbf{w}\|_\infty}\right\}, \qquad \kappa = \min\left\{\frac{\tau}{C_F^2}, \frac{1}{\nu}, \frac{1}{\gamma}, \frac{1}{C_F^2\alpha}, \frac{\delta}{c_1h^2}\right\},$$

we have

$$\inf_{U_h\in\mathbf{W}_h}\sup_{V_h\in\mathbf{W}_h}\frac{\tilde{a}_h(U_h, V_h)}{|||U_h|||\,|||V_h|||} \geqslant \frac{1}{12}. \tag{6.3}$$

Proof. Fix any $U_h = \{\mathbf{u}_h, p_h\} \in \mathbf{W}_h$ and consider $W_h = \{\mathbf{v}_h, p_h\}$ with $\mathbf{v}_h = \mathbf{u}_h + \tau\mathrm{P_h}\,(\mathbf{w}\times\mathbf{u}_h)$. Noting that

$$(\mathbf{w}\times\mathbf{u}_h, \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)) = (\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h), \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)),$$
$$\alpha(\mathbf{u}_h, \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)) = 0,$$

and thanks to (6.2) one has

$$
\begin{aligned}
\tilde{a}_h(U_h, W_h) &= \nu\|\nabla \mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2 + \tau\nu(\nabla \mathbf{u}_h, \nabla \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)) + \tau\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 \\
&\quad + \tau\gamma(\operatorname{div}\mathbf{u}_h, \operatorname{div}\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)) + \tau(\nabla p_h, \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)) \\
&\quad + \delta\|\nabla p_h\|^2 + \gamma\|\operatorname{div}\mathbf{u}_h\|^2 \\
&\geqslant \nu\|\nabla \mathbf{u}_h\|^2 - \mu_u\tau\nu\, h^{-1}\|\nabla\mathbf{u}_h\|\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\| + \tau\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 \\
&\quad + \alpha\|\mathbf{u}_h\|^2 - \mu_u\tau\gamma\, h^{-1}\|\operatorname{div}\mathbf{u}_h\|\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\| - \frac{\tau}{\varepsilon}\|\nabla p_h\|^2 \\
&\quad - \frac{\varepsilon\tau}{4}\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 + \delta\|\nabla p_h\|^2 + \gamma\|\operatorname{div}\mathbf{u}_h\|^2 \\
{\scriptstyle(\varepsilon=1)} &\geqslant \tfrac{1}{2}\nu\|\nabla\mathbf{u}_h\|^2 + \alpha\|\mathbf{u}_h\|^2 - (\tfrac{1}{2}\mu_u^2\tau^2\nu\, h^{-2} + \tfrac{1}{2}\mu_u^2\tau^2\gamma\, h^{-2})\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 \\
&\quad + \frac{3\tau}{4}\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 + \frac{\delta}{2}\|\nabla p_h\|^2 + \frac{\gamma}{2}\|\operatorname{div}\mathbf{u}_h\|^2 \geqslant \frac{1}{2}|||U_h|||^2 - \sigma\|p_h\|^2.
\end{aligned} \tag{6.4}
$$

In the above inequalities the negative $\mathbf{u}_h$-dependent terms are compensated by the $\tau\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2$ term in $|||U_h|||^2$. If $\alpha > 0$ one can overestimate $\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|$, see (6.6), and then compensate these terms in (6.4) with the $\alpha\|\mathbf{u}_h\|^2$ term in $|||U_h|||^2$. This results in different upper bounds on τ. One can take the maximum of these bounds (see condition (6.2) and definition of $R1$, $R2$, $R3$). For example, the choice of ε in (6.4) depends on which term is dominant in the definition of $R3$. If $\delta/2$ does, we set $\varepsilon = 1$, otherwise $\varepsilon = 2\tau/\delta$.

If the inf-sup condition (3.6) holds, then there exists such $\mathbf{z}_h \in \mathbf{V}_h$ that

$$
c_0^2\|p_h\|^2 = (p_h, \operatorname{div}\mathbf{z}_h), \quad \|\nabla\mathbf{z}_h\| \leqslant c_0\|p_h\|.
$$

Further, taking $Z_h = (-\mathbf{z}_h, 0)$, we find

$$
\begin{aligned}
\tilde{a}_h(U_h, Z_h) &= -\nu(\nabla\mathbf{u}_h, \nabla\mathbf{z}_h) - (\alpha\mathbf{u}_h + \mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h), \mathbf{z}_h) \\
&\quad + (p_h, \operatorname{div}\mathbf{z}_h) - \gamma(\operatorname{div}\mathbf{u}_h, \operatorname{div}\mathbf{z}_h) \\
&\geqslant c_0^2\|p_h\|^2 - 2\nu^2\|\nabla\mathbf{u}_h\|^2 - \tfrac{1}{8}\|\nabla\mathbf{z}_h\|^2 - 2\alpha^2 C_F^2\|\mathbf{u}_h\| - \tfrac{1}{8}\|\nabla\mathbf{z}_h\|^2 \\
&\quad - 2C_F^2\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 - \tfrac{1}{8}\|\nabla\mathbf{z}_h\|^2 - 2\gamma^2\|\operatorname{div}\mathbf{u}_h\|^2 - \tfrac{1}{8}\|\nabla\mathbf{z}_h\|^2,
\end{aligned}
$$

hence

$$
\begin{aligned}
\tilde{a}_h(U_h, Z_h) &\geqslant \frac{c_0^2}{2}\|p_h\|^2 - 2\nu^2\|\nabla\mathbf{u}_h\|^2 - 2\alpha^2 C_F^2\|\mathbf{u}_h\|^2 - 2C_F^2\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 \\
&\quad - 2\gamma^2\|\operatorname{div}\mathbf{u}_h\|^2.
\end{aligned} \tag{6.5}
$$

We combine this estimate and (6.4) in such a way that all negative terms from (6.5) are absorbed into $|||U_h|||^2$ in (6.4). Now the constant κ defined earlier, which measures the relation between negative terms in (6.5) and positive terms in $|||U_h|||^2$, turns out to be important. Setting $V_h = W_h + \frac{\kappa}{8}Z_h$ we get, thanks to the definition of κ and the choice of σ in (6.2),

$$
\tilde{a}_h(U_h, V_h) \geqslant \tfrac{1}{4}|||U_h|||^2.
$$

If the discrete inf-sup condition (3.6) does not hold, we adopt the same approach as in Section 5, proving that there exists $\mathbf{z}_h \in \mathbf{V}_h$, such that

$$(\operatorname{div}\mathbf{z}_h, p_h) \geqslant c_0^2\|p_h\|^2 - c_0c_1h\|p_h\|\,\|\nabla p_h\|, \quad \|\nabla\mathbf{z}_h\| \leqslant c_0c_2\|p_h\|,$$

and continue with estimating $\tilde{a}_h(U_h, Z_h)$ in the same fashion as in (6.5). This case additionally contributes to the condition on σ, namely $\sigma \leqslant \frac{\delta}{c_1h^2}$.

To complete the proof we need an upper bound for $|||V_h|||$. Using (6.2), we get

$$\begin{aligned}
|||V_h|||^2 &= \nu\left\|\nabla\left(\mathbf{u}_h + \tau\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h) - \frac{\kappa}{8}\mathbf{z}_h\right)\right\|^2 + \alpha\left\|\mathbf{u}_h + \tau\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h) - \frac{\kappa}{8}\mathbf{z}_h\right\|^2 \\
&\quad + \tau\left\|\mathrm{P_h}\left(\mathbf{w}\times\left(\mathbf{u}_h + \tau\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h) - \frac{\kappa}{8}\mathbf{z}_h\right)\right)\right\|^2 + \sigma\|p_h\|^2 \\
&\quad + \gamma\left\|\operatorname{div}\left(\mathbf{u}_h + \tau\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h) - \frac{\kappa}{8}\mathbf{z}_h\right)\right\|^2 + \delta\|\nabla p_h\|^2 \\
&\leqslant 3\bigg(\nu\|\nabla\mathbf{u}_h\|^2 + (\mu_u^2\tau^2\nu\,h^{-2} + \tau + \alpha\tau^2)\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 + \frac{\nu\kappa^2}{64}\|\nabla\mathbf{z}_h\|^2 \\
&\quad + \alpha\|\mathbf{u}_h\|^2 + \frac{\alpha\kappa^2}{64}\|\mathbf{z}_h\|^2 + \tau^3\|\mathbf{w}\times\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 + \frac{\tau\kappa^2\|\mathbf{w}\|_\infty^2}{64}\|\mathbf{z}_h\|^2\bigg) + \sigma\|p_h\|^2 \\
&\quad + 3\gamma\left(\|\operatorname{div}\mathbf{u}_h\|^2 + \frac{\kappa^2}{64}\|\nabla\mathbf{z}_h\|^2 + \mu_u^2\tau^2h^{-2}\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2\right) + \delta\|\nabla p_h\|^2 \\
&\leqslant 3\nu\|\nabla\mathbf{u}_h\|^2 + 3\alpha(1 + \tau^2\|\mathbf{w}\|_\infty^2)\|\mathbf{u}_h\|^2 + \tau(6 + 3\tau^2\|\mathbf{w}\|_\infty^2)\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\|^2 \\
&\quad + 4\sigma\|p_h\|^2 + 3\gamma\|\operatorname{div}\mathbf{u}_h\|^2 + \delta\|\nabla p_h\|^2 \\
&\leqslant 9|||U_h|||^2.
\end{aligned}$$

In order to estimate the $\mathbf{z}_h$-terms by $4\sigma\|p_h\|^2$ we used that (6.2) implies the inequality $C_F^2\|\mathbf{w}\|_\infty^2\tau + \gamma + \nu + C_F\alpha \leqslant \frac{64\sigma}{c_0^2\kappa^2}$. □

REMARK 6.1

(i) Note that $\sigma \to 0$ if $\delta \to 0$. This is not the case for div-stable elements. Indeed, the last term in the definition of κ appears only if (3.6) does not hold and the term τ/C_F^2 can be replaced by $\nu/(C_F^4\|\mathbf{w}\|_\infty^2)$ if we proceed in (6.5) with

$$\|\mathrm{P_h}(\mathbf{w}\times\mathbf{u}_h)\| \leqslant \|\mathbf{w}\|_\infty\|\mathbf{u}_h\| \leqslant C_F\|\mathbf{w}\|_\infty\|\nabla\mathbf{u}_h\|. \tag{6.6}$$

This is a standard approach; however, now σ essentially depends on ν.

(ii) Lemma 6.1 imposes restrictions on τ. This is not a parameter we have to design for implementation reasons, but it is an auxiliary one involved in stability and error estimates. The choice $\tau > 0$ allows us to demonstrate a control of the skew-symmetric terms in the discrete problem—see the examples at the end of the section.

6.2 *Error analysis and parameter design*

Below we will distinguish between two cases with respect to restrictions imposed on the given data.

Case C1. Assume that one of the following inequalities holds for the 2D problem

$$-\alpha < -c_w \leqslant \mathbf{w}(\mathbf{x}) \quad \text{a.e. in } \Omega, \tag{6.7}$$

$$\mathbf{w}(\mathbf{x}) \leqslant c_w < \alpha \quad \text{a.e. in } \Omega, \tag{6.8}$$

or $\nabla \mathbf{w} = 0$ in 3D. This is the case, for example, if the term $\mathbf{w} \times \mathbf{u}$ stems from the effect of Coriolis forces, or if the time-stepping scheme with sufficiently small time step was used.

Case C2. 2D and 3D problems without any special assumptions on data.

LEMMA 6.2 Let $\chi = \min\{\nu^{-\frac{1}{2}}, \gamma^{-\frac{1}{2}}, \mu_u h^{-1}\alpha^{-\frac{1}{2}}\}$, then for any $\tau > 0$ and $V_h \in \mathbf{W}_h,\ U \in \mathbf{W}$ the following estimates hold.

Case C1. Let $\eta = \frac{\alpha + \|\mathbf{w}\|_\infty}{\alpha - c_w}$, if (6.7) or (6.8) holds, and $\eta = 1$ in 3D, then

$$\tilde{a}_h(V_h, U) \leqslant |||V_h|||(\nu\|\nabla\mathbf{u}\|^2 + ((\tau^{-1} + \alpha)\eta^2 + \delta^{-1})\|\mathbf{u}\|^2 + \chi^2\|p\|^2)^{\frac{1}{2}}; \tag{6.9}$$

Case C2.

$$\begin{aligned}\tilde{a}_h(V_h, U) \leqslant c|||V_h|||(\nu\|\nabla\mathbf{u}\|^2 &+ (\delta^{-1} + \alpha)\|\mathbf{u}\|^2 + \chi^2\|p\|^2 \\ &+ \|\mathbf{w}\|_\infty^2 \min\{\nu^{-1}, \alpha^{-1}\} h\|\mathbf{u}\|^2)^{\frac{1}{2}}.\end{aligned} \tag{6.10}$$

Proof. For any $V_h \in \mathbf{W}_h,\ U \in \mathbf{W}$

$$\begin{aligned}\tilde{a}_h(V_h, U) = \nu(\nabla\mathbf{v}_h, \nabla\mathbf{u}) &+ \alpha(\mathbf{v}_h, \mathbf{u}) + (\mathbf{w} \times \mathbf{v}_h, \mathbf{u}) + \gamma(\operatorname{div}\mathbf{v}_h, \operatorname{div}\mathbf{u}) \\ &- (p, \operatorname{div}\mathbf{v}_h) + (q_h, \operatorname{div}\mathbf{u}) + \delta(\nabla q_h, \nabla p).\end{aligned} \tag{6.11}$$

The estimate of the first, second, fourth and seventh terms on the RHS of (6.11) is trivial. Observe also that

$$|(q_h, \operatorname{div}\mathbf{u})| \leqslant \delta^{\frac{1}{2}}\|\nabla q_h\|\delta^{-\frac{1}{2}}\|\mathbf{u}\| \tag{6.12}$$

$$|(p, \operatorname{div}\mathbf{v}_h)| \leqslant \|p\|\,\|\operatorname{div}\mathbf{v}_h\| \leqslant \chi\|p\|\big(\nu\|\nabla\mathbf{v}_h\|^2 + \gamma\|\operatorname{div}\mathbf{v}_h\|^2 + \alpha\|\mathbf{v}_h\|^2\big)^{\frac{1}{2}}. \tag{6.13}$$

One needs the estimate of the following type: $|(\mathbf{w} \times \mathbf{v}_h, \mathbf{u})| \leqslant \|\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)\|\,\|\mathbf{u}\|$. However, the latter is not straightforward, since $\mathbf{u}$ is not in a discrete space and one cannot consider $\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)$ on the LHS of the inequality. The only exception is $\nabla\mathbf{w} = 0$, then $\mathbf{w} \times \mathbf{v}_h \in \mathbf{U}_h$, hence $\mathbf{w} \times \mathbf{v}_h = \mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)$ and we are done.

First we consider case C1. Suppose that (6.7) holds, then

$$\begin{aligned}-c_w\|\mathbf{v}_h\|^2 &\leqslant (\mathbf{w}\mathbf{v}_h, \mathbf{v}_h) = (\mathbf{w} \times \mathbf{v}_h, 1 \times \mathbf{v}_h) \\ &= (\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h), 1 \times \mathbf{v}_h) \leqslant \|\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)\|\,\|\mathbf{v}_h\|. \qquad (6.14) \\ (\alpha - c_w)\|\mathbf{v}_h\| &\leqslant \alpha\|\mathbf{v}_h\| + \|\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)\|.\end{aligned}$$

On the other hand

$$|\alpha(\mathbf{v}_h, \mathbf{u})| + |(\mathbf{w} \times \mathbf{v}_h, \mathbf{u})| \leqslant (\alpha + \|\mathbf{w}\|_\infty)\|\mathbf{v}_h\|\,\|\mathbf{u}\|. \tag{6.15}$$

Now (6.14) and (6.15) give

$$|\alpha(\mathbf{v}_h, \mathbf{u})| + |(\mathbf{w} \times \mathbf{v}_h, \mathbf{u})| \leqslant \frac{\alpha + \|\mathbf{w}\|_\infty}{\alpha - c_w}(\alpha\|\mathbf{v}_h\| + \|\mathrm{P_h}(\mathbf{w} \times \mathbf{v}_h)\|)\|\mathbf{u}\|. \tag{6.16}$$

Estimates (6.12), (6.13), and (6.16) imply (6.9).

The next case C2 will differ only in handling the third term in (6.11):

$$|(\mathbf{w} \times \mathbf{v}_h, \mathbf{u})| \leqslant \|\mathbf{w}\|_\infty(|\mathbf{v}_h|, |\mathbf{u}|) \leqslant C_F\|\mathbf{w}\|_\infty\|\nabla\mathbf{v}_h\|\|\mathbf{u}\|.$$

If $\alpha > 0$ we also have

$$|(\mathbf{w} \times \mathbf{v}_h, \mathbf{u})| \leqslant \alpha^{\frac{1}{2}}\|\mathbf{v}_h\|(\alpha^{-1}\|\mathbf{w}\|_\infty^2\|\mathbf{u}\|^2)^{\frac{1}{2}}.$$

This, (6.12), (6.13), and trivial estimates for other terms in (6.11) proves (6.10). □

The following result gives an error estimate for scheme C which is not too far from that of scheme B apart from the last (consistency error) term in (6.17).

THEOREM 6.1 Assume the solution $U = \{\mathbf{u}, p\}$ to the problem (2.7) is sufficiently smooth. One has the following estimates for positive $\tau, \delta, \sigma, \gamma$, satisfying the conditions from Lemma 6.1 and with η and χ defined in Lemma 6.2.

Case C1.

$$\begin{aligned}|||U - U_h||| \leqslant c((\nu^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + \eta(\tau^{-\frac{1}{2}} + \alpha^{\frac{1}{2}} + \delta^{-\frac{1}{2}})h)h^l\|\mathbf{u}\|_{H^{l+1}} \\ + (\chi h + \sigma h + \delta^{\frac{1}{2}})h^k\|p\|_{H^{k+1}} + \delta\gamma^{\frac{1}{2}}\|\Delta p\|);\end{aligned} \tag{6.17}$$

Case C2.

$$\begin{aligned}|||U - U_h||| \leqslant c((\nu^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + (\tau^{\frac{1}{2}}\|\mathbf{w}\|_\infty + \alpha^{\frac{1}{2}} + \delta^{-\frac{1}{2}})h \\ + \|\mathbf{w}\|_\infty \min\{\nu^{-\frac{1}{2}}, \alpha^{-\frac{1}{2}}\}\, h\,)h^l\|\mathbf{u}\|_{H^{l+1}} \\ + (\chi h + \sigma h + \delta^{\frac{1}{2}})h^k\|p\|_{H^{k+1}} + \delta\gamma^{\frac{1}{2}}\|\Delta p\|).\end{aligned} \tag{6.18}$$

Proof. The results are again a consequence of stability, continuity, and approximation. Assume that $\hat{U}_h = \{\hat{\mathbf{u}}_h, \hat{p}\}$ is a proper interpolant for U. As an example let us check (6.18). Suppose that the minimum on the RHS of (6.10) is attained for the first argument. According to Lemma 6.1 there exists such $V_h \in W_h$ that

$$\begin{aligned}\tfrac{1}{12}|||U_h - \hat{U}_h|||\,|||V_h||| &\leqslant a_h(V_h, U_h - \hat{U}_h) = \tilde{a}_h(V_h, U - \hat{U}_h) - \delta\gamma(\operatorname{div}\mathbf{v}_h, \Delta p) \\ &\leqslant c|||V_h|||\left(\nu\|\nabla(\mathbf{u} - \hat{\mathbf{u}}_h)\|^2 + (\delta^{-1} + \alpha)\|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 + \frac{\|\mathbf{w}\|_\infty^2}{\nu}\|\mathbf{u} - \hat{\mathbf{u}}_h\|^2 \right. \\ &\quad \left. + \chi^2\|p - \hat{p}\|^2 + \delta^2\gamma\|\Delta p\|^2\right)^{\frac{1}{2}} \\ &\leqslant c((\nu^{\frac{1}{2}}h^l + (\delta^{-\frac{1}{2}} + \alpha^{\frac{1}{2}})h^{l+1} + \|\mathbf{w}\|_\infty\nu^{-\frac{1}{2}}h^{l+1})\|\mathbf{u}\|_{H^{l+1}} \\ &\quad + \chi h^{k+1}\|p\|_{H^{k+1}} + \delta\gamma^{\frac{1}{2}}\|\Delta p\|)|||V_h|||.\end{aligned}$$

On the other hand, from the approximation property we get

$$|||U - \hat{U}_h||| \leqslant c(\nu^{\frac{1}{2}} + \gamma^{\frac{1}{2}} + (\alpha^{\frac{1}{2}} + \tau^{\frac{1}{2}}\|\mathbf{w}\|_\infty)h)h^l\|\mathbf{u}\|_{H^{l+1}} + (\sigma^{\frac{1}{2}}h + \delta^{\frac{1}{2}})h^k\|p\|_{H^{k+1}}.$$

The triangle inequality leads to (6.18) after some upper bounds and reorganization of the terms. Estimate (6.17) is proved in the same manner. □

The choice of τ, δ, γ, subject to several conditions from Lemma 6.1, is still free. Below we try to optimize the convergence estimates. The choice of optimal parameters is somewhat dependent on the order of FE used, e.g. by setting $\delta = ch^r$, one can vary r to balance the convergence order in (6.17), (6.18) with a formal approximation order of (6.1) to (2.7). As an example we apply Lemma 6.1 to an equal-order linear (bilinear) pressure–velocity FE pair. The latter is a common choice in engineering applications due to a simple data structure and effective implementation on a parallel architecture. In the first example below we assume $\alpha = 0$.

EXAMPLE 6.1 Let the assumptions of Theorem 6.1 and (3.4) with $k = l = 1$ be valid and assume $\mathrm{Ek}_h^{-1} := h^2\|\mathbf{w}\|_\infty/\nu \geqslant h$ (this includes the convection-dominated case), then for $\delta = c\,h\|\mathbf{w}\|_\infty^{-1}$, $\sigma = c_0\kappa/16$ and $\gamma = c^{-1}h\|\mathbf{w}\|_\infty$ the following estimates hold.

Case C1

$$\begin{aligned} \|\nabla(\mathbf{u} - \mathbf{u}_h)\| &\leqslant c(1 + \eta \mathrm{Ek}_h^{-\frac{1}{2}} h^{-\frac{1}{2}})h\,\|\mathbf{u}\|_2 + \mathrm{Ek}_h^{-\frac{1}{2}} h^{\frac{1}{2}}\|p\|_2, \\ \|\nabla(p - p_h)\| + \|\mathrm{P_h}(\mathbf{w} \times (\mathbf{u} - \mathbf{u}_h))\| &\leqslant c\eta\|\mathbf{w}\|_\infty h\|\mathbf{u}\|_2 + h\|p\|_2; \end{aligned} \tag{6.19}$$

Case C2.

$$\begin{aligned} &\|\nabla(\mathbf{u} - \mathbf{u}_h)\| \leqslant c(1 + \mathrm{Ek}_h^{-\frac{1}{2}} h^{-\frac{1}{2}} + \mathrm{Ek}_h^{-1}h^{-1})h\,\|\mathbf{u}\|_2 + \mathrm{Ek}_h^{-\frac{1}{2}} h^{\frac{1}{2}}\|p\|_2, \\ &\|\nabla(p - p_h)\| + \|\mathrm{P_h}(\mathbf{w} \times (\mathbf{u} - \mathbf{u}_h))\| \leqslant c\|\mathbf{w}\|_\infty(1 + \mathrm{Ek}_h^{-\frac{1}{2}} h^{-\frac{1}{2}})h\,\|\mathbf{u}\|_2 + h\|p\|_2. \end{aligned} \tag{6.20}$$

Condition (6.2) in Lemma 6.1 permits the optimal choice $\tau = \delta/2$ if $\mathrm{Ek}_h^{-1} \geqslant c\,h$. With this choice the estimates (6.19), (6.20) are satisfied straightforwardly.

Although the convergence of velocity gradients depends on the relation between viscosity and mesh size, it seems to be better than the Galerkin approximation of the usual convection–diffusion problem. Compared to the Galerkin approximation of the usual convection–diffusion problem we also control the skew-symmetric terms. Moreover, we have optimal and almost ν-independent estimates for the pressure gradient. The latter can be of great importance regarding the physical meaning of dynamical pressure, since the dynamical pressure aggregates both velocity and kinematic pressure and turns out to be an invariant in the Euler limit.

EXAMPLE 6.2 Consider the next example with $\alpha > 0$ and the same choice of FE and parameters δ and γ. Typically α originates from an implicit treatment of an unsteady problem, so $\alpha^{-1} \sim (\delta t)$. We assume $\alpha \sim \|\mathbf{w}\|_\infty$. There are at least two reasons for such a choice of α. The first reason is that α and $\|\mathbf{w}\|_\infty$ have the same dimension. Another one is as follows: suppose that the time step is restricted by the CFL condition $\delta t \leqslant h/\|\mathbf{u}_h\|_\infty$. If we assume $\mathbf{w} = \operatorname{curl} \mathbf{u}_h$, then $\|\mathbf{w}\|_\infty \leqslant ch^{-1}\|\mathbf{u}_h\|_\infty$. Hence CFL implies $\|\mathbf{w}\|_\infty \leqslant c\alpha$. Still we assume $\alpha \sim \|\mathbf{w}\|_\infty$. Now we are likely to be in the case C1, but otherwise Theorem 6.1 for case C2 leads to the same results provided $\mathrm{Ek}_h \leqslant 1$ and thanks to the choice $\alpha^{-\frac{1}{2}}h$ in the last term of the $\mathbf{u}$-dependent part of (6.18). Hence for the gradients of the error in velocity and pressure the estimate (6.19) holds, furthermore

$$\|\mathbf{u} - \mathbf{u}_h\| \leqslant ch^{\frac{3}{2}}(\|\mathbf{u}\|_2 + \|\mathbf{w}\|_\infty^{-1}\|p\|_2).$$

7. Conclusions

The linearized Navier–Stokes equations in rotation form obey the same error estimates for a class of stabilized FE schemes as the ones for convection form of these equations. The role of mesh Reynolds number is played by Ek_h^{-1} defined locally as $\|\operatorname{curl}\mathbf{u}\|\, h^2\nu^{-1}$, therefore $\mathrm{Ek}_h^{-1} \leqslant c\ \mathrm{Re}_h$. Furthermore, if the solution is locally smooth, then $\mathrm{Ek}_h^{-1} \ll \mathrm{Re}_h$.

If the pressure is not stabilized, as is often the case for div-stable FE, then optimal estimates are not straightforward for equations with any form of convection. However, we proved such a result, if a reaction term (stemming from implicit time integration) is involved in the momentum equation and/or if lowest-order pressure elements are used.

Promising stability and convergence estimates were proved for a pressure-regularized Oseen problem with rotation form of convection. Hence being easier to solve, but only of first-order accuracy, this approximation is expected to serve as a good predictor, while accurate schemes studied in Sections 4 and 5 can serve as correctors in implicit calculations of steady or unsteady Navier–Stokes flows.

So as a conclusion of the paper, we believe that Newton-like iterations for the incompressible Navier–Stokes problem, studied e.g. in Turek (1999) for the convection form, can be extended to its rotation form. Moreover, preconditioners to the pressure-regularized Oseen problem being well-tuned to the case of large Reynolds numbers are at hand (Olshanskii, 1999). Numerical results are in preparation and will be reported elsewhere.

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REFERENCES

CIARLET, P. G. (1991) Basic error estimates for elliptic problems. *Handbook of Numerical Analysis*, vol. 2. (P. G. Ciarlet & J. L. Lions, eds). Amsterdam: North-Holland.

CLEMENT, PH. (1975) Approximation by finite-element functions using local regularization. *RAIRO Anal. Numer.*, **2**, 77–84.

CODINA, R. (1993) A finite element formulation for viscous incompressible flows, Centro Internac. Metodos Numer Ing., Politec. Cataluna, Barcelona.

CODINA, R. & SOTO, O. (1997) Finite element solution of the Stokes problem with dominating Coriolis force. *Comput. Meth. Appl. Mech. Eng.*, **142**, 215–234.

CODINA, R. (2001) A stabilized finite-element method for generalized incompressible flows. *Comput. Meth. Appl. Mech. Eng.*, **190**, 2681–2706.

DAUGE, M. (1989) Stationary Stokes and Navier–Stokes systems on two- or three-dimensional domains with corners. Part I: linearized equations. *SIAM J. Math. Anal.*, **20**, 74–97.

GIRAULT, V. & RAVIART, P. A. (1986) *Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms*. Berlin: Springer.

KNOBLOCH, P. & TOBISKA, L. (1999) A streamline diffusion method for nonconforming finite-element approximations applied to the linearized incompressible Navier–Stokes equations. *Proc. 4th Intern. Conf. Numer. Meths. Applic.* Sofia, 1998. (O. P. Iliev *et al.*, ed.). World Sc., pp. 530–538.

OLSHANSKII, M. A. (1999) An iterative solver for the Oseen problem and the numerical solution of incompressible Navier–Stokes equations. *Numer. Linear Alg. Appl.*, **6**, 353–378.

OLSHANSKII, M. A. & REUSKEN, A. (2000) Navier–Stokes equations in rotation form: a robust multigrid solver for the velocity problem, Inst. Geom. Prakt. Mathem., RWTH, Aachen (`www.igpm.rwth-aachen.de`), *Report* 193.

OLSHANSKII, M. A. (2001) A low order Galerkin finite-element method for the Navier–Stokes equations of steady incompressible flow: a stabilization issue and iterative methods, Inst. Geom. Prakt. Mathem., RWTH, Aachen, *Report* 202.

PIRONNEAU, O. (1989) *Finite Element Methods for Fluids*. New York: Wiley.

ROOS, H. G., STYNES, M. & TOBISKA, L. (1996) *Finite Element Methods for Fluids*. Berlin: Springer.

SCHÄFER, M. & TUREK, S. (1996) Benchmark computations of laminar flow around a cylinder. *Flow Simulations with High-performance Computers II*, Notes Numer. Fluid Mech. 2. (E. H. Hirschel, ed.). Braunschweig: Vieweg.

TEMAM, R. (1977) *Navier–Stokes Equations. Theory and Numerical Analysis*. Amsterdam: North-Holland.

TUREK, S. (1999) *Efficient Solvers for Incompressible Flow Problems: An Algorithmic Approach in View of Computational Aspects*. Berlin: Springer.

VERFÜRTH, R. (1999) Error estimates for some quasi-interpolation operators. *RAIRO Model. Math. Anal. Numer.*, **33**, 695–713.

ZANG, T. A. (1991) On the rotation and skew-symmetric forms for incompressible flow simulations. *Appl. Numer. Math.*, **7**, 27–40.